Proposed Test of Charge-Conjugation Invariance in Positronium Decay*

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If the electromagnetic interactions are invariant under charge conjugation, C, then singlet positronium (Ps) can only decay into an even number of photons, while triplet Ps can only decay into an odd number of photons. In this paper we consider a model Hamiltonian which violates C invariance but conserves parity. If λ is the coupling constant in the proposed C-noninvariant interaction, then the branching ratio of the forbidden decay, $1\,^3S_1 \to 4\gamma$, to the allowed decay, $1\,^3S_1 \to 3\gamma$, has been found to be $F_T^{4\gamma} = 1.2\times 10^{-5}\lambda^2$.

An experimental upper limit on $F_T^{4\gamma}$ would provide a novel check of C invariance in electromagnetic interactions, and would also yield new information on the $\bar{e}e\ \gamma\gamma\gamma\gamma$ interaction. At present, indirect limits on $F_T^{4\gamma}$ show it to be less than about 10^{-2}. We are now working on an experiment which is designed to reduce this upper limit to 10^{-5}–10^{-6}. If successful, it will subject the proposed interaction, or any similar interaction leading to $1\,^3S_1 \to 4\gamma$, to a stringent test. An outline of the experiment is included in this paper.

I. INTRODUCTION

We consider, in this note, a new test for charge-conjugation (C) invariance in electromagnetic interactions. If the electromagnetic interactions are invariant under charge conjugation, then singlet positronium (Ps), $^1S_0$, can only decay into an even number of photons, while triplet Ps, $^3S_1$, can only decay into an odd number of photons.[1]

We have studied the decay $^3S_1 \to 4\gamma$ for ground-state Ps, $1\,^3S_1$. Hereafter ^1S and ^3S will refer only to $1\,^1S_0$ and $1\,^3S_1$. As ^3S has spin 1, it cannot decay into two photons[2] even if C-noninvariant interactions are present. Thus the minimum number of photons that ^3S can decay into, via a C-noninvariant interaction, is four.

The main motivation for considering such an interaction is that very little is known about the $\bar{e}e\ \gamma\gamma\gamma\gamma$ interaction. In addition, past experience, at least with the weak interactions, shows that symmetries under parity (P), C, or CP cannot be taken for granted. It has been fruitful to check the invariances under these operations under widely different conditions. Current limits which can be set on $^3S \to 4\gamma$ are quite poor. They will be discussed in Sec. II.

A secondary consideration is that a large violation of C invariance in electromagnetic interactions is a possible cause of the CP-noninvariant decay mode of the K_2^0 meson, i.e., $K_2^0 \to \pi^+ + \pi^-$.

II. A PROPOSED C-NONINVARIANT INTERACTION

We consider here models in which an $\bar{e}e\ \gamma\gamma\gamma\gamma$ interaction violates C invariance. We could of course start with $\bar{e}e\ \gamma\gamma$ or $\bar{e}e\ \gamma\gamma\gamma$ violating C invariance. However, these C-noninvariant terms contain several derivative couplings and hence one would expect them to lead to a small decay rate. Further, there is already an upper bound on these interactions, from the experiments performed by Mills and Berko[3] who investigated the decay mode $^1S \to 3\gamma$. The only interaction one can write with the minimum number of derivatives is

$$H_i = \frac{\lambda}{m_e^{\,8}} e^4 \partial_\alpha \rho_\beta F^{\alpha\delta} F_\delta{}^\beta F^{\mu\nu} F_{\mu\nu}\,, \qquad (1)$$

where $\rho_\beta = \bar{\psi}\gamma_\beta\psi$ and λ is the coupling constant on which we hope to set new upper limits. The other symbols have their usual meanings and we have set $\hbar = c = 1$ and $e^2 = 4\pi\alpha$. For Ps decay it is sufficient to consider the nonrelativistic limit, and this leads to, after contracting the $\bar{e}e$ state,

$$H_i \approx \frac{\lambda e^4}{m_e^{\,8}} (-2iM)\vec{\sigma}\cdot(\vec{E}\times\vec{H})(E^2 - H^2)\,, \qquad (2)$$

where M, the mass of Ps, is about $2m_e$, $\vec{\sigma}$ is the Pauli spinor, and $\vec{E}$ and $\vec{H}$ represent the electric and magnetic fields in the interaction.

The interaction given in Eq. (1) conserves P and violates C invariance. Using it, one can calculate the lifetime against $^3S \to 4\gamma$ in the usual manner. This gives

$$\frac{1}{\tau_T^{4\gamma}} = \frac{3}{4}\frac{|\psi(0)|^2}{4!(2\pi)^8} \int \frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{16\omega_1\omega_2\omega_3\omega_4} \times |y|^2 \delta^4(P - k_1 - k_2 - k_3 - k_4)\,, \qquad (3)$$

where $\psi(0)$ is the Ps wave function evaluated at $r=0$; $\omega_1, \ldots, \omega_4$ represent the γ energies; $k_1, \ldots, k_4$ their four-momenta; and P is the four-momentum of the Ps. Finally, $|y|^2$ is the square of the matrix element averaged over the initial polarization and summed over the final states. It is given by

$$|y|^2 = \frac{96\lambda^2 M^2 e^8}{m_e^{16}} \{5\omega_1^2\omega_2^2\omega_3^2\omega_4^2 + 2\omega_3^2\omega_4^2(\vec{k}_1\cdot\vec{k}_2)^2 - 8\omega_1\omega_2\omega_3^2\omega_4^2\vec{k}_1\cdot\vec{k}_2 - (\vec{k}_1\cdot\vec{k}_2)^2(\vec{k}_3\cdot\vec{k}_4)^2$$
$$+ 12\omega_4\omega_2\vec{k}_2\cdot\vec{k}_4(\vec{k}_1\cdot\vec{k}_3)^2 - 8\omega_1\omega_2\omega_3\omega_4\vec{k}_1\cdot\vec{k}_2\vec{k}_3\cdot\vec{k}_4 + 4\omega_1^2\vec{k}_2\cdot\vec{k}_3\vec{k}_3\cdot\vec{k}_4\vec{k}_4\cdot\vec{k}_2 + 4\omega_2\omega_4\omega_1^2\vec{k}_2\cdot\vec{k}_3\vec{k}_3\cdot\vec{k}_4$$
$$- 2\vec{k}_1\cdot\vec{k}_2\vec{k}_2\cdot\vec{k}_4\vec{k}_4\cdot\vec{k}_3\vec{k}_3\cdot\vec{k}_1 - 4\omega_2\omega_3[(\vec{k}_1\times\vec{k}_4)\cdot\vec{k}_2][(\vec{k}_1\times\vec{k}_4)\cdot\vec{k}_3] - 8\omega_2\omega_4\vec{k}_1\cdot\vec{k}_4\vec{k}_2\cdot\vec{k}_3\vec{k}_3\cdot\vec{k}_1\}. \quad (4)$$

On doing the integration in Eq. (3), one obtains

$$R_T^{4\gamma} = \frac{1}{\tau_T^{4\gamma}} = 88\lambda^2 \sec^{-1}. \quad (5)$$

Thus, since $R_T^{3\gamma}$, the allowed $^3S \to 3\gamma$ decay rate, is about $7\times10^6 \sec^{-1}$, the C-noninvariant to C-invariant branching ratio, defined as $F_T^{4\gamma}$, is

$$F_T^{4\gamma} = \frac{R_T^{4\gamma}}{R_T^{3\gamma}} = 1.2\times10^{-5}\lambda^2. \quad (6)$$

A direct upper limit on λ results from comparison of the theoretical and measured values of $R_T^{3\gamma}$, which are[4(a)] $R_T^{3\gamma} = 7.21\times10^6$ and[4(b)] $R_{T\,\text{expt}}^{3\gamma} = (7.34 \pm 0.11)\times10^6$. Even if all the 4γ events in the experiment were interpreted as 3γ events, a λ of about 50 would be necessary before the increased decay rate would be detected. Finally, we note that the C-noninvariant process $^1S \to 3\gamma$ has a branching ratio[3] of less than 2.8×10^{-6}. This experimental result sets an upper limit on λ, which a very rough estimate gives as $\lambda \approx \alpha^{-1} \approx 10^2$.

It is seen that a substantial reduction in the current upper limit on $F_T^{4\gamma}$ is necessary if the C noninvariance postulated in Eq. (1), or any similar interaction leading to $^3S \to 4\gamma$, is to be subjected to a stringent test.

We will now show that, by *direct observation* of the 4γ events, it should be possible to detect λ, even if λ is only of the order of unity. The basis of this conclusion will be discussed in the context of a specific experiment currently underway at our laboratory.

III. THE EXPERIMENTAL METHOD

1. General Outline

We propose to detect the decay $^3S \to 4\gamma$ by forming Ps in a small spatial region, and locating four identical γ detectors at the corners of a tetrahedron centered on the Ps source (Figs. 1 and 2). If simultaneous counts are recorded in all counters, we have a possible C-noninvariant event. Four simultaneous counts can occur in many ways other than through $^3S \to 4\gamma$. We call all such events "noise." In order to suppress noise and to positively identify the forbidden decay, we must introduce several auxiliary systems. They are:

(a) A Pb collimator on each detector to minimize counter-counter backscattering.

(b) A γ detection system, DS. This system detects the emitted γ's and rejects all events in which any one of the four γ's does not appear within a preset singles energy window, ΔE_s, or in which the sum of the four energies is not within an add window, ΔE_A, centered at 1022 keV, the mass of Ps.

(c) A γ anticoincidence arrangement, AC, which also serves to determine the time between Ps formation and triplet decay (T_T). It will henceforth be called the anticoincidence-timing or ACT system.

We now discuss the experiment in a more specific manner.

(a) *Positron source, 0.1 mCi of Na^{22}.* The prompt 1270-keV nuclear γ emitted with each positron will be used for both anticoincidence and timing purposes. The source strength is limited by timing effects in the ACT system and by pulse pileup in the DS system.

(b) *Ps formation.* The source is at the center of a 2-cm diameter thin-walled hollow plastic sphere. The sphere is filled with MgO or Al_2O_3 in the form of a fine-grained powder of bulk density about 0.1. Roughly 30% of the positrons which en-

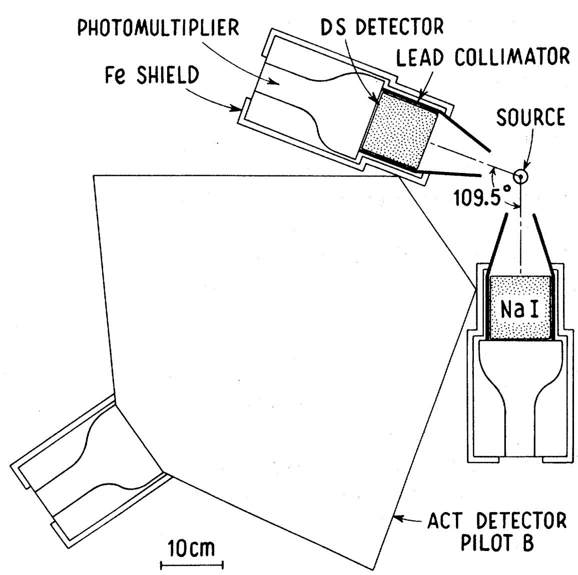

FIG. 1. A schematic diagram of the proposed experiment as cut through one plane.

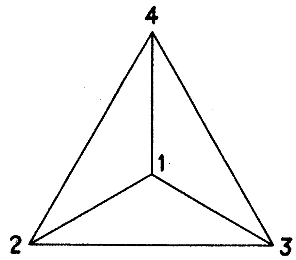

FIG. 2. The plan of the counter arrangement proposed to detect the decay $^3S \to 4\gamma$.

ter the powder form Ps. It has been demonstrated[5(a),5(b)] that if the individual grains are less than about 300 Å in size, up to 95% of the triplet Ps formed escapes from the grain before being picked off, i.e., before the positron annihilates with a lattice electron of opposite spin. The Ps then remains within the intergrain space, annihilating with almost its vacuum lifetime if the sphere is evacuated or flooded with an inert gas such as argon. This technique has the obvious advantages of simplicity, and an approximation to point geometry as compared to the usual method of Ps formation in gas at high pressure.

(c) *DS system*. The γ detectors are 4-in.×4-in. NaI crystals, each subtending a solid angle of $\Delta\Omega/4\pi \approx 0.025$ at the source. This upper limit on $\Delta\Omega$ arises because conservation of momentum requires that the three γ's from triplet decay lie in a plane containing the source. Such a plane cannot be passed through any three counters if $\Delta\Omega/4\pi \lesssim 0.025$. The probability of three simultaneous counts, a major source of noise when coupled to a fourth random pulse, is therefore much reduced. The kinematically allowed singles energies are 50–375 keV. The singles window will be set to about this opening, while the add window will be about 50 keV wide. We anticipate a fourfold-coincidence time resolution, T_r, of 10–20 nsec at the above singles energies. A fourfold coincidence satisfying ΔE_S and ΔE_A criteria will be called a DS "event." The singles γ rate, R_S, will be 2×10^5 cps; high, but manageable.[6]

(d) *Collimator*. A 5-mm thick Pb collimator placed as in Fig. 1.

(e) *ACT system*. This system consists of six large blocks of plastic scintillator positioned as in Fig. 1. The T part of ACT is triggered, with about 80% probability, by the prompt Na^{22} γ. The AC system vetoes all DS events if it is activated after the $T=0$ trigger but before reset at about $T_T = \tau_T$, the triplet lifetime.

2. Absolute Signal Rate, $R_T^{4\gamma}$

To within a factor of 2, $R_T^{4\gamma}$ is given by the expression

$$R_T^{4\gamma} = F_{Ps}F_{DE}F_{DC}F_T^{4\gamma}R\,, \tag{7}$$

where

F_{Ps} = fraction of emitted positrons which form ^3S.

F_{DE} = DS γ-detection efficiency = $(\gamma \text{ efficiency/counter})^4 \approx 0.5$.

F_{DC} = fraction of $^3S \to 4\gamma$ decays which enter our counters. It has been computed on the assumption that the matrix elements are constant, i.e., on the basis of pure phase-space restrictions. The result (Appendix A) is that $F_{DC} = 4.4(\Delta\Omega/4\pi)^3$. This use of pure phase space should give a conservative estimate of F_{DC}.

R = source decay rate into positrons.

Several small losses which reduce $R_T^{4\gamma}$ by about a factor of 2 have not been included in Eq. (7). Finally then, we have, for the parameters listed,

$$R_T^{4\gamma} \approx 10^6 F_T^{4\gamma}/\text{day}\,. \tag{8}$$

This may be taken as a conservative estimate of the absolute counting rate, which would be 12 counts/day for $\lambda \approx 1$.

3. Sources of Noise

The noise rate R_N is defined as the total rate of DS events not due to $^3S \to 4\gamma$. The signal-to-noise ratio, $R_T^{4\gamma}/R_N$, is designated as Q. Contributions to R_N are of two general types: (a) events due to four simultaneous correlated pulses; (b) events due to four simultaneous pulses in the DS of which two or more are uncorrelated. The most serious of the noise sources are

(a) *Correlated pulses*:

(1) $^1S \to 4\gamma$, $^3S \to 5\gamma$, $e^+ + e^- \to 4\gamma, 5\gamma$.

(2) $^3S \to {}^1S$ followed by $^1S \to 4\gamma$. Triplet-singlet conversion could be induced by Ps-powder collisions, or by the presence of a paramagnetic gas such as O_2 in the intergrain space.

(3) $^3S + e^- \to 4\gamma$. This is "pickoff" (Sec. I).

(4) $^3S \to 3\gamma$, $\gamma + e^- \to e^- + \gamma + \gamma$. One of the triplet γ's undergoes double Compton scattering (DCS), most likely in the powder.

(b) *Uncorrelated or partially correlated pulses*:

(1) Four simultaneous, but uncorrelated, pulses due to γ's, phototube dark noise, etc.

(2) Processes involving two correlated γ's from either $e^+ + e^- \to 2\gamma$ or $^1S \to 2\gamma$, together with two uncorrelated γ's.

(3) $^3S \to 3\gamma$ plus one uncorrelated γ from either Na^{22} recoil (1270 keV), direct annihilation (511 keV), or singlet Ps decay (511 keV).

(4) $^3S \to 3\gamma$, $^3S \to 3\gamma$. Two triplets decaying within $2T_r$ of each other, with four of the six γ's striking detectors.

The results of our investigations of these noise sources are summarized below.

(a) Correlated pulses:

(1a)–(3a) The over-all branching ratios for all decays into four or five γ's, are estimated to be lower than 10^{-7} (Appendix B). In addition, all of the 4γ processes have vanishing matrix elements if the decay is required to have tetrahedral symmetry (Appendix C).

(4a) The cross section[7] for DCS has been evaluated for our arrangement.[8] About 3 in 10^3 γ's undergo ordinary small-angle Compton scattering (CS) in the powder, while 5 in 10^8 undergo a DCS consistent with geometric and DS energy constraints. Using this ratio, it is straightforward to show that $Q(4a) \approx 8\times10^4(\Delta\Omega/4\pi)^{-1}F_T^{4\gamma}$ or $Q(4a) \approx 3$ at $F_T^{4\gamma}=10^{-6}$.

Many other sources of correlated pulses have been considered, however, only (4a) is important at $F_T^{4\gamma}\approx 10^{-6}$.

(b) *Uncorrelated or partially correlated pulses*:

(1b) and (2b) These events are proportional to $(R_S T_r)^3$ and $(R_S T_r)^2$, respectively, and are completely negligible.

(3b) If the fourth pulse is source-related, consideration of the probable energy loss of CS γ's, photopeak fractions of the various γ's, etc., shows that roughly 3% of the events satisfy DS requirements. Note, however, that almost every γ not originating in triplet decay has an energy of 511 or 1270 keV and has two prompt γ's associated with it. These extra γ's are recorded by ACT with (90–95)% probability[9] each, and either or both cause a veto. Thus $Q(3b)$ is increased by an average factor of 200, so that $Q(3b)\approx 9\times10^6\, F_T^{4\gamma}$.

(4b) The rate of γ pairs from ^3S entering two specific counters, with energies in the range ΔE_S, is $R_T^{2\gamma}=6\,(\Delta\Omega/4\pi)^2\, F_{\rm Ps}R$. The rate of fourfold triplet-triplet coincidence is therefore $(3R_T^{2\gamma})(2T_r R_T^{2\gamma})$. This is reduced a factor of 30 by the DS add requirement, and a factor of 2000 by the AC conditions on the two extra triplet γ's and the extra γ at 1270 keV. The final result is $Q(4b)=4\times10^6 F_T^{4\gamma}$, making it the largest source of partially correlated noise.

The total signal-to-noise ratio is

$$\frac{1}{Q}=\frac{1}{Q(1a)}+\cdots\frac{1}{Q(4b)}\approx\frac{5\times10^{-7}}{F_T^{4\gamma}}.$$

The above calculations have of necessity been approximate; however, the results are quite conservative.

4. Systematic Tests

If an effect is present at such a level that Q is somewhat larger than unity, systematic checks are not of crucial importance, since Q can be predicted to within a factor of 2 or 3 for a given λ.

For Q consistent with $\lambda=0$, however, it is of importance to set the best possible upper limit on λ. Many systematic noise tests are possible. Some of these include (a) increasing the counter solid angles to test the effect of geometry in suppressing 3γ coincidences, (b) increasing the powder thickness to evaluate the contribution of DCS and CS, (c) a 1–2-kG magnetic field at the source to quench the $m=0$ triplet decay without changing other parameters, and (d) a delayed-coincidence requirement for DS events based on the ACT trigger. This permits the monitoring of prompt correlated decays such as $^1S\to4\gamma$. These and other systematic tests should prove ample to identify and check the various components of R_N as well as any observed positive effect.

IV. CONCLUSION

We have calculated the 4γ to 3γ branching ratio of ground-state triplet Ps, on the assumption of a specific C-noninvariant interaction to drive the 4γ decay. The branching ratio, defined as $F_T^{4\gamma}$, is given by $1.2\times10^{-5}\lambda^2$, where λ is the coupling constant of the interaction.

We have also discussed the feasibility of a stringent experimental test of the theory. We conclude that if $F_T^{4\gamma}$ is greater than a few parts in 10^6, it should be observable. Work on the experiment described is now underway.

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APPENDIX A: CALCULATION OF $F_{\rm DC}$

The quantity $F_{\rm DC}$ is defined as the probability that each of the four photons from an arbitrary $^3S\to4\gamma$ decay enters a different detector. A conservative estimate of $F_{\rm DC}$ may be obtained by evaluating the invariant phase space open to the decay, under the assumption that the C-noninvariant matrix element is constant. Our detectors are taken to have equal solid angles $\Delta\Omega$, and their centers are to be placed at each of the vertices of a tetrahedron centered on the Ps source.

The fraction of $^3S\to4\gamma$ decays into this configuration is equal to the ratio of the probability for 4γ decays constrained to hit the detectors to the total probability for 4γ decay. When the matrix element for this decay is taken to be constant, this ratio of probabilities becomes simply the ratio of the phase spaces available in each case.[10] In other words,

$$F_{DC}=C\int_{\Delta\Omega}\frac{d^3k_1}{\omega_1}\frac{d^3k_2}{\omega_2}\frac{d^3k_3}{\omega_3}\frac{d^3k_4}{\omega_4}\times\delta^4(P-k_1-k_2-k_3-k_4)\,. \qquad (A1)$$

The integration is executed over the solid angle and momentum range consistent with hitting all four detectors. The constant C is the inverse of the total available phase space for 4γ decay, i.e.,

$$C\int_{4\pi}\frac{d^3k_1}{\omega_1}\frac{d^3k_2}{\omega_2}\frac{d^3k_3}{\omega_3}\frac{d^3k_4}{\omega_4}\delta^4(P-k_1-k_2-k_3-k_4)=1\,. \qquad (A2)$$

The integral in (A2) is over all allowed solid angles and momenta; it gives the result $C=6/\pi^3M^4$. After numerically evaluating the integral in (A1), F_{DC} turns out to be $4.4(\Delta\Omega/4\pi)^3$ for $\Delta\Omega/4\pi\ll1$.

The approximation of taking constant matrix elements should give a conservative result here, since it weights all decay configurations equally. In fact, many such configurations are not allowed if momentum and energy are conserved.

APPENDIX B: ESTIMATES OF THE $^1S\to4\gamma$, $^3S\to5\gamma$ BRANCHING RATIOS

Let us consider the branching ratio

$$\text{Rate}(^1S\to4\gamma)/\text{Rate}(^1S\to2\gamma)\,.$$

Since Ps is neutral, each emission of a photon involves $F_{\mu\nu}$ and gives a phase-space factor $(\bar{k}_n a)^2$ for each photon in the rate. Here $a\approx\lambda_c=\hbar/m_0c$ and $\bar{k}_n$ is the average wave number of the decay γ's. For an n-photon process, $\bar{k}_n=2/n\lambda\!\!\!^{-}_c$ so $\bar{k}_n a\approx2/n$. Thus the ratio of the matrix elements is $(\alpha/\pi)(\bar{k}_4a)^4/(\bar{k}_2a)^2$, or[11]

$$\frac{R(^1S\to4\gamma)}{R(^1S\to2\gamma)}\approx\left(\frac{\alpha}{\pi}\right)^2\frac{(\bar{k}_4a)^8}{(\bar{k}_2a)^4}\approx2\times10^{-8}\,. \qquad (B1)$$

Similarly $R(^3S\to5\gamma)/R(^3S\to3\gamma)\approx4\times10^{-9}$.

APPENDIX C: PROOF THAT $^1S\to4\gamma$ IS FORBIDDEN FOR A DECAY WITH TETRAHEDRAL SYMMETRY

We show here that the $^1S\to4\gamma$ decay is forbidden for a final state in which the four photon momenta come out towards the four corners of a tetrahedron. The Ps is taken to be at the center of the tetrahedron. Since ^1S is completely rotationally invariant, the four γ's must be also. The photons can be designated by their polarization. We will designate the right-circularly polarized and left-circularly polarized photons by R and L, respectively. In Fig. 2 we show the plan of a tetrahedron. The corners are labeled 1, 2, 3, and 4 and we shall refer to the final state by the polarizations of the various photons. Thus $RRLR$ means that the photons in counters 1, 2, and 4 are right-circularly polarized and the photon in counter 3 is left-circularly polarized.

First consider the state $RRRR$. Take a line joining the center of the tetrahedron and the corner 1, and then rotate through 120°. This gives the same configuration, but the state is multiplied by $e^{2\pi i/3}$, because of photon 1 (photon $2\to3\to4\to2$). The initial state, however, remains the same, and hence such a configuration is forbidden. Exactly the same argument can be applied to $LRRR$, which under rotation picks up a factor of $e^{-2\pi i/3}$. Thus we have accounted for $RLRR$, $RRLR$, and $RRRL$. Lastly consider $RRLL$. Let lines join counters 1 and 2 and counters 3 and 4. Take the midpoints of these lines and draw a line connecting them. Rotating through 90° about this axis, and reflecting it about a plane perpendicular to the axis, brings it back to the state $RRLL$. One can show by explicitly performing these operations, using a definite set of polarization vectors, that the state goes into itself with a phase of +1. But the initial state is of odd parity and changes sign. Thus this state is not allowed. This result is an extension of a theorem of Fumi and Wolfenstein.[12]

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[11]A more detailed calculation of this branching ratio has been made by McCoyd [G. C. McCoyd, Ph.D. thesis, St. Johns University, N. Y., 1965 (unpublished)]. His result was $R(^1S\to 4\gamma)/R(^1S\to 2\gamma)\approx 3\times10^{-7}$ as compared to our estimate of 2×10^{-8}. In either case this contribution to the noise is negligible.

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Numerical Computations in the Inverse-Scattering Problem at Fixed Energy

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Constructing potentials from the phase shifts at a given energy yields an infinity of equivalent solutions. The deviations of these solutions from each other can, however, be analyzed according to *a priori* limitations on the derivatives and other features of "acceptable" potentials. A sketch of this analysis is given together with a numerical comparison of usual potential forms with the equivalent potentials obtained through Newton's method. The observed deviation gives an appraisal of the deviations from each other of all the equivalent potentials with similar bounds on the derivatives. The deviation is small when there are many phase shifts available, all of them definitely smaller than $\pi/2$. For a static potential these conditions can be met for high energies.

We study the elastic scattering of a particle obeying the Schrödinger equation with a spherically symmetric potential, at an energy $E=\hbar^2k^2/2m$, m being the reduced mass and k being the linear momentum. The "inverse problem" deals with the construction of the potential from the phase shifts. We therefore assume that the phase shifts have already been derived from the cross section – not a trivial assumption.[1] Once the phase shifts are known, many theoretical papers give us formal ways of obtaining the potential.

The seemingly simplest method is to use the JWKB formula for the phase shift, which yields δ_l as a Riemann-Liouville transform of a function associated with the potential and therefore reduces this step of the inverse problem to solving an Abel integral equation. This situation has been encountered a long time ago in other inverse problems, going from the Wiechert-Herglotz-Bateman method in seismology[2] to well-known results in spectrometric measurements.[3] In quantum mechanics it has been used by several authors.[4] However, some steps of the method are questionable as regards the problem studied in the present paper. Actually, the interest of such a method does not really reside in solving the inverse problem, but in reducing the computing time for obtaining a potential which fits the phase shifts; it does not give any information on how far from this potential may be other potentials fitting the same set of phase shifts. In short, it extracts from the phase shifts very much biased information. So does the computer[5] when, working by trial and error, it fits the phase shifts by matching three parameters in a Woods-Saxon potential.

However, the main interest of solving the inverse problem by inverse methods (viz., by methods which are not trial and error ones), is to obtain an evaluation of the amount of information contained in the scattering amplitude. Now we know, from the formal methods of Regge,[6] Newton,[7] Sabatier,[8] and Loeffel,[9] that an infinity of potentials corresponds to a given set of phase shifts. The "Regge-Loeffel" methods[9] are not suitable for computation, nor is the Martin-Targonski method,[10] which is of physical interest because it deals with generalized Yukawa potentials.

On the other hand, the so-called[11] Newton-Sabatier methods are easy to handle on a computer, but one has first to answer the following fundamental question: Let us take for granted that the "physical properties" of the potential can be mathematically expressed through bounds on the derivatives. Then, if a potential is constructed from its phase shifts by one of the above methods, how different can the result be from the original potential?

There are two complementary ways of answering this question. The first is to define for the problem