Asymptotic Behavior and Complex Zeros of Veneziano and Physical $\pi\pi$ Partial Waves*

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We conjecture for $\pi\pi$ amplitudes with $I = 0$ and $I = 2$ that Veneziano partial waves contain essential singularities at infinity, and that both Veneziano and physical partial waves contain infinitely many complex zeros on the physical sheet.

I. INTRODUCTION AND CONJECTURES

The asymptotic behavior of Veneziano partial-wave amplitudes has been a subject of considerable interest, primarily because the Veneziano model may provide clues pertaining to the existence and, should they exist, nature of dispersion relations for physical partial waves.

In the early literature, it was made plausible by Drago and Matsuda[1] that all Veneziano partial waves grow faster than any power along the negative real axis of the energy-squared variable. However, Park and Desai[2] recently studied $I=1$ Veneziano $\pi\pi$ partial waves, and discovered that these partial waves tend asymptotically to zero in all directions (except along the ray at $\theta=0$, where resonance poles occur at equally spaced intervals). Thus the plausibility argument of Drago and Matsuda is not valid, and the asymptotic behavior of Veneziano $\pi\pi$ partial waves with $I=0$ and 2 remains to be determined.

In this paper we report an investigation which leads us to make the following conjectures:

(A) Veneziano $\pi\pi$ partial waves with $I=0$ and $I=2$ grow faster than any power along any ray into the left half-plane of the energy-squared variable.

(B) Veneziano $\pi\pi$ partial waves with $I=0$ and $I=2$ possess infinitely many complex zeros at angles near $\theta=\pm\frac{1}{2}\pi$ on the physical sheet, with a unique accumulation point at infinity.

(C) Physical $\pi\pi$ partial waves with $I=0$ and $I=2$ possess infinitely many complex zeros on the physical sheet.

The considerations which lead to the preceding conjectures are outlined below, together with some of the consequences.

II. BASES FOR CONJECTURES

Consider the following Veneziano representation for $\pi\pi$ elastic scattering amplitudes[3]:

$$\begin{aligned} A^0 &= \tfrac{1}{2}F(t,u) - \tfrac{3}{2}[F(s,t)+F(s,u)], \\ A^1 &= F(s,u) - F(s,t), \\ A^2 &= -F(t,u), \end{aligned} \tag{2.1}$$

where the superscript on A^I denotes the s-channel isospin,

$$\begin{aligned} F(x,y) &\equiv \beta \frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} + \text{secondary terms}, \\ \alpha(x) &\equiv a + bx, \end{aligned} \tag{2.2}$$

where a and b are real.

We shall denote the pion mass by μ, and we use the convenient energy-squared variable

$$\nu \equiv |\vec{k}_{\text{c.m.}}|^2 = \tfrac{1}{4}(s-4\mu^2).$$

We normalize the A^I such that if they were unitary, their partial waves would satisfy the representation

$$A^{(l)I}(\nu) = \mu^{-1}(1+\mu^2/\nu)^{1/2} R_l^I(\nu)\, e^{i\delta_l^I} \sin\delta_l^I \tag{2.3a}$$

for $\nu > 0$, where R_l^I is the ratio of elastic to total partial-wave cross sections, and the phase shifts δ_l^I are real. The representation (2.3a) exists if and only if

$$\operatorname{Im}(1/A^{(l)I}) = -\mu[(1+\mu^2/\nu)^{1/2} R_l^I]^{-1} \tag{2.3b}$$

for $\nu > 0$, which is a necessary and sufficient condition for $A^{(l)I}$ to be consistent with unitarity [except that $1/A^{(l)I}$ contains a pole wherever $\delta_l^I = n\pi$ above threshold, and each of these poles may be regarded as a δ function in $\operatorname{Im}(1/A^{(l)I})$]. Bose symmetry implies that partial waves with l even (odd) are different from zero only if I is even (odd). As a final remark on notation, we shall denote partial-wave projections of the Veneziano amplitudes (2.1) by $V^{(l)I}$.

The simplest way to obtain information about the large-ν behavior of $V^{(l)I}$ in all directions is to compute $V^{(l)I}$ over a large region of the ν plane. Keeping only the leading term in the Veneziano series (2.2) and using the values[3] $a=0.483$, $b=0.017\mu^{-2}$, we have computed the $V^{(l)I}$ for $l=0$, 1, and 2 at a set of closely spaced mesh points spanning the region $-250\mu^2 \le \operatorname{Re}\nu \le 250\mu^2$, $0 \le \operatorname{Im}\nu \le 250\mu^2$. (Since $V^{(l)I}(\nu^*) = [V^{(l)I}(\nu)]^*$, the values for negative $\operatorname{Im}\nu$ follow immediately.)

Our results for $V^{(1)1}$ support the conclusions of

Park and Desai about $I=1$ partial waves. However, we find that the partial waves with $I=0$ and 2 have a markedly different behavior in the left half-plane. Specifically, we find that $V^{(0)I}$ and $V^{(2)I}$ for $I=0$ and 2 can be well approximated over the aforementioned region (except for a narrow strip along the real axis) by functions $G^{(l)I}$ of the form

$$G^{(l)I} = C_1^{(l)I} e^{-0.0471(\nu/\mu^2)} + \frac{C_2^{(l)I}(\nu/\mu^2)^{-0.517}}{C_3^{(l)I} + \ln(\nu/\mu^2)}, \quad (2.4)$$

where the $C_n^{(l)I}$ are constants whose values are given in Table I.

Along any ray with $|\theta| < \frac{1}{2}\pi$, $G^{(l)I}$ tends to zero like $\nu^{\beta-1}/\ln\nu$, which agrees with the known asymptotic behavior of $V^{(l)I}$ along a ray displaced slightly from the positive real axis.[4] However, along any ray with $\frac{1}{2}\pi < |\theta| < \pi$, $G^{(l)I}$ grows exponentially, reaching values of the order of $10^5\mu^{-1}$ within the aforementioned region over which we have computed the $V^{(l)I}$.

We also note that for any $\epsilon > 0$, $G^{(l)I}$ possesses infinitely many zeros inside the wedge $(\frac{1}{2}\pi - \epsilon) < \theta < \frac{1}{2}\pi$, and at most finitely many zeros in the upper half-plane outside this wedge. Since $G^{(l)I}(\nu^*) = [G^{(l)I}(\nu)]^*$, similar remarks hold for the conjugate wedge and the lower half-plane. In both cases, the zeros have a unique accumulation point at infinity.

Note in Table I that $C_1^{(l)2}/C_1^{(l)0} = -2.00$ for $l=0$ and 2. Since $A^2 = -F(t,u)$ while A^0 contains $F(t,u)$ with a coefficient of $\frac{1}{2}$, the exponential term in $G^{(l)I}$ is due entirely to the $F(t,u)$ term in A^I with $I=0$ or 2, at least within the accuracy of our computations (three significant figures).

To facilitate our discussion of the accuracy of the approximations $V^{(l)I}(\nu) \cong G^{(l)I}(\nu)$ for $I=0$ and 2, we introduce discrepancy functions $\Delta^{(l)I}(\nu)$ defined by

$$\Delta^{(l)I}(\nu) \equiv \frac{|V^{(l)I}(\nu) - G^{(l)I}(\nu)|}{\langle |V^{(l)I}(\nu)| \rangle_{\text{local}}}, \quad (2.5)$$

where the denominator is the average value of $|V^{(l)I}(\nu)|$ taken over a disk of radius $\Delta\nu = 20\mu^2$ centered about the point ν. (A local average is used in the denominator because of the aforementioned zeros.)

TABLE I. Values for $C_n^{(l)I}$ with $\beta = 0.50\,\mu^{-1}$, which corresponds to $\Gamma(\rho) \cong 125$ MeV.

$(l)I$	$\mu C_1^{(l)I}$	$\mu C_2^{(l)I}$	$C_3^{(l)I}$
(0)0	$0.768i$	$+28.5 + 10.4i$	$-4.44 - 19.8i$
(0)2	$-1.54i$	$-5.23 - 0.45i$	$-2.52 + 0.20i$
(2)0	$-0.049 - 0.336i$	$-356 + 378i$	$+298 + 156i$
(2)2	$0.098 + 0.673i$	$-6.45 + 1.68i$	$-1.32 - 1.28i$

In Fig. 1, we present upper bounds on the values of $\Delta^{(0)I}$ and $\Delta^{(2)I}$ for $I=0$ and 2 over the region $250\mu^2 \leq |\text{Re}\,\nu|$, $25\mu^2 \leq \text{Im}\,\nu \leq 250\mu^2$. For $|\nu| \geq 100\mu^2$, the upper bounds on $\Delta^{(0)I}$ and $\Delta^{(2)I}$ displayed in Fig. 1 are sufficiently small to suggest strongly that $V^{(0)I}$ and $V^{(2)I}$ with $I=0$ and 2 satisfy conjecture (A), and also the hypotheses of the following theorem.

Theorem. Let $F(\nu)$ denote a function which is analytic on the physical sheet except for possible cuts and poles along the real axis, subject to the restrictions that at least one interval along the real axis is free of singularities, and that $F(\nu)$ is real on every such interval. Suppose there exists

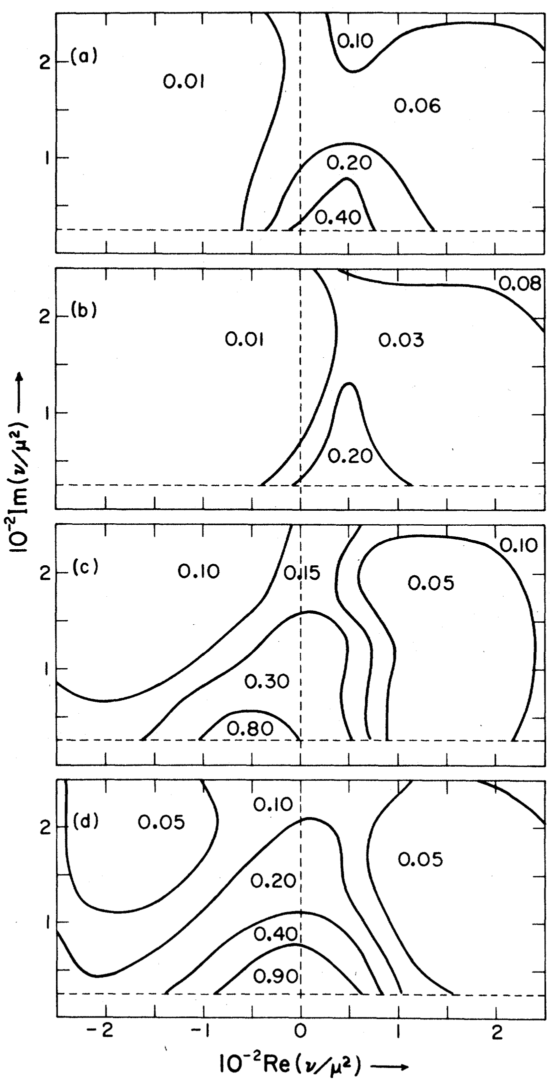

FIG. 1. Upper bounds on (a) $\Delta^{(0)0}$, (b) $\Delta^{(0)2}$, (c) $\Delta^{(2)0}$, and (d) $\Delta^{(2)2}$.

an $\epsilon > 0$ such that $F(\nu)$ does not tend to zero as rapidly as $\nu^{-1+\epsilon}$ along any path to ∞ on the physical sheet. Suppose also that $F(\nu)$ grows as rapidly as ν^{ϵ} along each of its cuts. If there exists a path to ∞ on the physical sheet along which $F(\nu)$ tends to zero, then $F(\nu)$ contains infinitely many zeros on the physical sheet.

Proof. If $F(\nu)$ satisfies the preceding hypotheses, then $1/F(\nu)$ satisfies a once-subtracted dispersion relation of the following form:

$$\frac{1}{F(\nu)} = \frac{1}{F(\nu_s)} + \frac{\nu - \nu_s}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{\mathrm{Im}[1/F(\nu')]}{(\nu' - \nu_s)(\nu' - \nu)} + (\nu - \nu_s)\sum_i \frac{1}{\bar{\nu}_i - \nu_s}\frac{r_i}{\nu - \bar{\nu}_i}, \qquad (2.6)$$

where ν_s denotes the point at which the subtraction has been made, and where $\bar{\nu}_i$ and r_i denote the positions and residues, respectively, of any poles which may lie on the physical sheet. The assumed growth of F along each of its cuts implies that $\mathrm{Im}[1/F]$ tends to zero as rapidly as $\nu^{-\epsilon}$ along each cut, so the term involving the integral on the right-hand side of Eq. (2.6) tends to a constant as $\nu \to \infty$. Since F is assumed to tend to zero along some path to ∞, there exists a path to ∞ along which $1/F$ grows without limit. Such growth is inconsistent with Eq. (2.6) unless *infinitely* many poles are present in the sum. Q.E.D.

All $V^{(l)I}$ are real analytic functions with no singularities except for left cuts and, if $I=0$ or 1, poles along the positive real axis. All $V^{(l)I}$ tend to zero like $\nu^{a-1}/\ln\nu$ along a ray displaced slightly from the real axis.[4] The aforementioned results of our computation of the $V^{(l)I}$ with $l=0$, 1, and 2 suggest strongly that the $V^{(0)I}$ and $V^{(2)I}$ with $I=0$ and 2 also satisfy the remaining hypotheses of the preceding theorem. Since the $V^{(l)2}$ do not oscillate along the real axis, it then follows that $V^{(0)2}$ and $V^{(2)2}$ possess infinitely many zeros at *complex* points on the physical sheet. Because of the infinitely many resonance poles in $V^{(l)0}$, there are infinitely many zeros in $V^{(l)0}$ along the positive real axis, and these zeros are sufficient to satisfy the conclusion of the theorem. However, the approximating functions $G^{(l)I}$ possess infinitely many zeros at angles near $\theta = \pm\frac{1}{2}\pi$, with a unique accumulation point at infinity, and we conjecture that the $V^{(0)I}$ and $V^{(2)I}$ with $I=0$ and 2 share these properties.

Although we have not computed any $V^{(l)I}$ with $l>2$, the fact that the $V^{(0)I}$ and $V^{(2)I}$ with $I=0$ and 2 can all be approximated by functions $G^{(l)I}$ of the form (2.4) leads us to extend the preceding conjectures about $I=0$ and 2 partial waves to hold for arbitrary l.

Thus far, our discussion has concentrated on the leading term of the Veneziano series (2.2). However, the inclusion of secondary terms could not invalidate conjecture (A) unless the secondary terms contained an essential singularity which precisely canceled the one in the leading term. It is not presently known whether there exists a combination of secondary terms which would result in such a cancellation. As for conjecture (B), the existence of infinitely many zeros in $V^{(l)I}$ follows from the preceding theorem, with quite modest hypotheses about asymptotic behavior. If these hypotheses are satisfied by the leading term of the Veneziano series, then they will be satisfied when secondary terms are included, unless such extreme cancellations occur as to modify the asymptotic behavior in one or more specific and dramatic ways. However, that part of conjecture (B) which places the zeros near $\theta = \pm\frac{1}{2}\pi$ has been made plausible only for the leading term, and might require modification when secondary terms are included.

Let us now consider physical $A^{(l)I}$, concerning which we state and prove the following theorem.

Theorem. If there exists an $\epsilon > 0$ such that $A^{(l)I}$ does not tend to zero more rapidly than $\nu^{-1+\epsilon}$ along any path to ∞ on the physical sheet, and R_l^I does not tend to zero more rapidly than $\nu^{-1+\epsilon}$ as $\nu \to +\infty$, and if $A^{(l)I}$ grows as rapidly as ν^{ϵ} for some $\epsilon > 0$ as $\nu \to -\infty$, then $A^{(l)I}$ contains infinitely many zeros on the physical sheet.

Proof. If $A^{(l)I}$ satisfies the preceding hypotheses, then $1/A^{(l)I}$ satisfies a once-subtracted dispersion relation of the following form:

$$\frac{1}{A^{(l)I}(\nu)} = \frac{1}{A^{(l)I}(\nu_s)} + \frac{\nu - \nu_s}{\pi}\int_{-\infty}^{-\mu^2} d\nu' \frac{\mathrm{Im}[1/A^{(l)I}(\nu')]}{(\nu' - \nu_s)(\nu' - \nu)} - \frac{\nu - \nu_s}{\pi}\int_0^{\infty} d\nu' \frac{(1+\mu^2/\nu')^{-1/2}}{R_l^I(\nu')(\nu' - \nu_s)(\nu' - \nu)} + (\nu - \nu_s)\sum_i \frac{1}{\bar{\nu}_i - \nu_s}\frac{r_i}{\nu - \bar{\nu}_i}, \qquad (2.7)$$

where ν_s denotes the point at which the subtraction has been made, and where $\bar{\nu}_i$ and r_i denote the positions and residues, respectively, of any poles which may lie on the physical sheet. The assumed growth of $A^{(l)I}$ as $\nu \to -\infty$ implies that $\mathrm{Im}(1/A^{(l)I})$ tends to zero with sufficient rapidity for the term involving the integral over the left cut to tend to a constant as $\nu \to -\infty$. However, the term involving the integral over the right cut grows logarithmically as $\nu \to -\infty$ if R_l^I tends to a nonzero constant as $\nu \to +\infty$. If $R_l^I \to 0$ as $\nu \to +\infty$, then the term involving the integral over the right cut grows more rapidly than logarithmically as $\nu \to -\infty$. Since R_l^I satisfies the rigorous bound $0 \leq R_l^I \leq 1$, we see that the integral over the right cut contributes a term to $1/A^{(l)I}$ which grows without limit as $\nu \to -\infty$, whereas $1/A^{(l)I}$ was assumed to tend to zero as $\nu \to -\infty$. Consequently, there must be infinitely many poles present in the sum over poles in order to cancel the growth of the term generated by the right cut. Q.E.D.

The present author believes it reasonable to conjecture that physical $A^{(l)I}$ with $I=0$ and 2 satisfy the hypotheses of the preceding theorem, and furthermore that infinitely many zeros occur at *complex* ν on the physical sheet.[5]

The theorem concerning physical $A^{(l)I}$ could, of course, be generalized to apply to functions other than $A^{(l)I}$, but we shall not bother with such a generalization here.

III. CONSEQUENCES

Having presented our conjectures and the motivations for them, we shall now discuss briefly their significance. The primary significance of conjecture (A) is that $V^{(l)I}$ with $I=0$ and 2 do not satisfy dispersion relations (with any finite number of subtractions). This suggests (but does not prove) that physical $A^{(l)I}$ with $I=0$ and 2 do not satisfy dispersion relations.

Notwithstanding the likelihood that $V^{(l)I}$ with $I=0$ and 2 do not satisfy dispersion relations, it is straightforward to establish that if one writes twice-subtracted partial-wave dispersion relations[6] (henceforth PWDR's) for the S waves $A^{(0)I}$, adjusts the subtraction constants to agree with the $V^{(0)I}$, and substitutes the Veneziano δ-function absorptive parts of the first two towers[7] of resonances into the integrals over the left and right cuts, then the resulting functions agree with the exact $V^{(0)I}$ within 10% below $E_{\text{c.m.}} = 700$ MeV, and within 25% below 900 MeV.[8,9] A similar result holds for the D waves.[9] Thus even if physical $A^{(l)I}$ contain essential singularities at infinity like those in $V^{(l)I}$ for $I=0$ and 2, it is reasonable to suppose that $A^{(l)I}$ can be well approximated below 700 MeV by solutions to twice-subtracted PWDR's.[6]

The primary significance of conjecture (C) is that $1/A^{(l)I}$ contains infinitely many poles at complex ν on the physical sheet for $I=0$ and 2. Since unitarity can be expressed in a very simple way for inverse amplitudes, namely, by Eq. (2.3b) together with a pole in $1/A^{(l)I}$ wherever $\delta_l^I = n\pi$ above threshold, many authors have obtained unitary $A^{(l)I}$ from models for $1/A^{(l)I}$.[10,11] However, almost none of these models for $1/A^{(l)I}$ have contained any poles at complex ν. The only exceptions of which the present author is aware are those models wherein $A^{(l)I}$ is obtained from $V^{(l)I}$ by an equation of the form[11]

$$A^{(l)I} = \frac{V^{(l)I}}{1+\rho_l^I V^{(l)I}}. \tag{3.1}$$

Upon solving Eq. (3.1) for ρ_l^I, one finds that ρ_l^I is precisely the difference function

$$\rho_l^I = \frac{1}{A^{(l)I}} - \frac{1}{V^{(l)I}}. \tag{3.2}$$

Since $1/V^{(l)I}$ is real for $\nu > 0$, unitarity is satisfied if and only if the absorptive part of ρ_l^I is given by the right-hand side of Eq. (2.3b) for $\nu > 0$ (except that ρ_l^I may contain poles for $\nu > 0$, which would correspond to δ functions in $\text{Im}\rho_l^I$). The dynamics of Eq. (3.1) is contained in the real part of ρ_l^I; it is obvious from Eq. (3.2) that an arbitrary $A^{(l)I}$ can be generated from Eq. (3.1) by a suitable choice of ρ_l^I. The inverse-amplitude method for unitarizing Veneziano amplitudes (the original designation "K-matrix method" is clearly a misnomer) consists of analyzing and proposing models for the functions ρ_l^I defined by Eq. (3.2). Since none of the models yet proposed[11] for ρ_l^I have contained any poles at complex ν, these models for $1/A^{(l)I}$ have contained the same poles as $1/V^{(l)I}$ at complex ν.

In order to gain some insight into the potential significance of the complex poles in $1/A^{(l)I}$ for the low-energy region, let us consider the positions and residues of the nearest poles in $1/V^{(l)I}$. For the sake of definiteness, we keep only the leading term in the Veneziano series (2.2), and we again use $a = 0.483\mu^2$, $b = 0.017\mu^{-2}$. In Table II, we present the positions $\tilde{\nu}$ and residues r of the two poles in $1/V^{(0)0}$ which lie in the upper half-plane and are

TABLE II. Positions and residues of nearest poles in $1/V^{(l)I}$, together with values of the expansion parameters ξ_0 and ξ_1 defined in text. We have used $\beta = 0.50\ \mu^{-1}$, which corresponds to $\Gamma(\rho) \cong 125$ MeV.

$(l)I$	$\tilde{\nu}/\mu^2$	r/μ^3	ξ_0/μ	$\xi_1\mu$
(0)0	$29+67i$	$-26-136i$	3.70	0.0305
(0)0	$41+211i$	$76-176i$	1.47	0.0059
(0)2	$49+122i$	$-102-106i$	2.07	0.0000
(0)2	$59+256i$	$-84-228i$	1.83	0.0007
(2)0	$25+135i$	$154-364i$	4.80	0.0291
(2)2	$38+182i$	$-262-370i$	4.47	-0.0053

nearest the origin, the two such poles in $1/V^{(0)2}$, the nearest such pole in $1/V^{(2)0}$, and the nearest such pole in $1/V^{(2)2}$. Of course there are also poles with conjugate residues at the conjugate points.

Near threshold, the contribution of each pair of poles at conjugate points can be approximated by

$$\frac{r}{\nu-\tilde{\nu}}+\frac{r^*}{\nu-\tilde{\nu}^*}\cong\xi_0+\xi_1\nu. \quad (3.3)$$

We present in Table II the values of ξ_0 and ξ_1 corresponding to each of the aforementioned pairs of poles. The values of ξ_0 are quite substantial, but one subtraction is typically performed in a dispersion relation for the inverse of a partial wave, and ξ_0 can be absorbed into the subtraction constant. However, the effect of ξ_1 cannot be absorbed into the subtraction constant when only one subtraction is performed. The significance of the values of ξ_1 in Table II can be judged from the fact that ν takes on the values $3\mu^2$, $6\mu^2$, $10\mu^2$, and $20\mu^2$ for $E_{\text{c.m.}}$ =500, 730, 910, and 1260 MeV, respectively. Thus, for example, the two nearest pairs of poles in $1/V^{(0)0}$ make a contribution which changes by about 0.44μ between threshold and 1 GeV, while the two nearest pairs of poles in $1/V^{(0)2}$ make a contribution which changes by only 0.007μ over the same range of energies.

The pole parameters in Table II provide a model for the nearest poles in $1/A^{(l)I}$ for $l=0$ and 2, $I=0$ and 2. However, we have conjectured the existence of infinitely many poles, and their net effect must be considered. Toward that end, let us recall that as $\nu\to-\infty$, the net pole contribution to the right side of Eq. (2.7) for $1/A^{(l)I}$ changes in just such a way as to cancel the growth of the term involving the integral over the right cut. If one considers the simple case where $R_l^I(\nu)=1$ and performs the subtraction at $\nu_s=-\frac{1}{2}\mu^2$, then the term generated by the right cut is precisely

$$-\frac{\nu+\frac{1}{2}\mu^2}{\pi}\int_0^\infty d\nu'\,\frac{(1+\mu^2/\nu')^{-1/2}}{(\nu'+\frac{1}{2}\mu^2)(\nu'-\nu)}$$
$$=(2/\pi)(1+\mu^2/\nu)^{-1/2}\ln|\mu^{-1}[\nu^{1/2}+(\nu+\mu^2)^{1/2}]|-\tfrac{1}{2} \quad (3.4)$$

for $\nu<-\mu^2$. [For $\nu>0$, the right-hand side of Eq. (3.4) gives the *real* part of the expression on the left-hand side.] If R_l^I tends to zero as $\nu\to\infty$, then the term generated by the right cut grows more rapidly as $\nu\to-\infty$ than is indicated by Eq. (3.4). For example, if R_l^I tends to zero like $\nu^{-1/2}$ as $\nu\to\infty$, then the term generated by the right cut grows like $\nu^{1/2}$ as $\nu\to-\infty$. However, the first derivative of $\nu^{1/2}$ is quite small for large negative ν, where the poles must cancel the growth of the term generated by the right cut. Thus it may be that the net contribution of the infinitely many poles in $1/A^{(l)I}$ does not vary much more rapidly near threshold than the contributions of the poles presented in Table II. However, it could be otherwise, and we presently have no means at our disposal for deciding the matter.

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[1]F. Drago and S. Matsuda, Phys. Rev. 181, 2095 (1969).

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[4]K. M. Carpenter and S. M. Negrine, Nuovo Cimento 1A, 13 (1971).

[5]The subject of complex zeros in $A^{(l)I}$ has also been studied by Y. S. Jin and K. Kang, Phys. Rev. 152, 1227 (1966). However, it is tacitly assumed in their work that only finitely many zeros exist on the physical sheet, and their method fails when infinitely many zeros are present.

[6]E. P. Tryon, Phys. Rev. Letters 20, 769 (1968); also, in *Proceedings of a Conference on ππ and Kπ Interactions at Argonne National Laboratory, 1969*, edited by F. Loeffler and E. Malamud (Argonne National Laboratory, Argonne, Ill., 1969).

[7]By the first tower we mean all resonances with the mass of the ρ meson, and by the second tower we mean all resonances with the mass of the f_0.

[8]With only one subtraction and substitution of only the first tower of resonances into left and right cuts, the resulting function for $I=2$ disagrees with the exact $V^{(0)2}$ by 26% at 500 MeV, and by 37% at 700 MeV.

[9]E. P. Tryon, following paper, Phys. Rev. D 4, 1221 (1971).

[10]For an early study of the inverses of ππ partial waves, see J. W. Moffat, Phys. Rev. 121, 926 (1961). For a recent study of the general properties of these inverses, see E. P. Tryon, preceding paper, Phys. Rev. D 4, 1202 (1971).

[11]An equation of the form (8) was first proposed by C. Lovelace, CERN Report No. CERN-TH-1041, 1969 (unpublished), as a device for unitarizing Veneziano amplitudes. The first systematic study of the properties of the functions ρ_l^I is that by E. P. Tryon, Ref. 10, wherein a dispersive model is proposed for ρ_l^I. Simple models have been proposed for ρ_l^I by H. M. Lipinski, University of Wisconsin report, 1969 (unpublished) and revised version 1970 (unpublished), and also by David M. Scott, K. Tanaka, and R. Torgerson, Phys. Rev. D 2, 1301 (1970).