Cluster-Decomposition Properties of φ^3-Perturbation-Theory Amplitudes at High Energy*

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We establish that a cluster-decomposition technique, similar to that used in statistical mechanics, can be applied to the study of high-energy scattering processes. In particular, we examine in detail the multiperipheral amplitudes in a φ^3 field theory with one space and one time dimension. The cluster decomposition provides (1) a mathematically elegant and physically intuitive way of treating terms nonleading in $(\ln s)^n$; (2) a framework in which the Regge asymptotic behavior of the scattering amplitude emerges naturally; and (3) a direct means of calculating the one-particle and multiparticle spectra in inclusive reactions. In addition, we consider further possible applications of the technique, including its extension to a φ^3 theory with three space and one time dimension. Here we establish a simple criterion, based on the form of the cluster decomposition, to determine whether a set of amplitudes leads to a Regge pole or to a more complicated singularity structure.

I. INTRODUCTION

Much of the complexity of strong interactions at very high energy derives from the possible coupling of an initial two-particle state to multiparticle final states. The rapid growth of the number of allowed final states and the resulting inherently multichannel nature of a high-energy scattering process suggest that these processes should perhaps be treated by methods similar to those of statistical mechanics. Recently, an explicit proposal of this type was made independently by Feynman and by Wilson,[1] who argued that high-energy interactions should be analogous to the behavior of a real gas contained in a finite volume. Relying in part on this analogy, these authors speculated on the probable properties of hadronic interactions at high energy. The analog of surface effects (scattering events called fragmentation in the nomenclature of BCYY[2]) should depend strongly on the nature of the initial particles; the analog of volume effects (processes called pionization) should, however, be independent of such details. Further qualitative predictions based on this analogy – for example, the distribution dx/x [where $x = p_{\parallel}/(p_{\parallel})_{\text{inc}} \approx p^0/E_{\text{inc}}$] for final-state particles in pionization reactions – were also formulated.

Since many of these predictions are supported by present experimental results, it is clearly of interest to study two questions:

(1) To what extent can the present rather qualitative "gas analogy" be extended?

(2) Can one support this analogy by quantitative calculations in a quantum-field-theoretic framework?

It is our purpose in this paper to answer the latter question affirmatively and in some detail; in the process we shall discover several remarkable results which bear on the former question as well.

To motivate our choice of the specific field-theoretic model in which to study the gas analogy, we turn to the following experimental results[3]:

(a) $\langle n \rangle \propto \ln s$, where $\langle n \rangle$ is the average multiplicity in a high-energy interaction;

(b) the distribution of final-state particles with small momenta relative to the initial particles seems to follow a dx/x law, with x defined as above; and

(c) $\langle p_\perp \rangle \lesssim 350$ MeV/c, so that the transverse components of momentum are finite and limited, even as the center-of-mass energy squared ($\equiv s$) gets very large.

That the first two experimental results are also theoretical consequences of multiperipheral models suggests that we consider a model of this type. The third result supports the conclusion that the longitudinal components of momenta,

$$p_\pm \equiv p^0 \pm p^3 , \tag{1.1}$$

play a more crucial role in the dynamics at large energies than do the transverse components, given by

$$p_\perp = (0, p^1, p^2, 0) . \tag{1.2}$$

If this conclusion is correct, then a valid first approach to the problem might be simply to ignore the transverse momenta. Combining these two observations, we are led in the simplest case to a perturbation-theory calculation of the properties of the set of all ladder diagrams in a φ^3 field theory with one space and one time dimension.

With regard to this proposed model we should

mention two points. First, perturbation-theory diagrams have long been the theoreticians' laboratory for examining strong interactions; consequently, the literature abounds with discussions of the high-energy behavior of ladder diagrams in φ^3 theories.[4] Our reasons for presenting this further study are to introduce and apply the important concept, new in this context, of a "cluster decomposition" similar to those used in statistical mechanics[5] and to extend the knowledge of certain detailed properties – for instance, of one-particle and multiparticle spectra – of this model. Second, there are some uncertainties about the validity of scattering theory in a $(1+1)$-dimensional space; these arise basically because of the lack of asymptotic states. However, since we are interested only in the formal analogy to scattering in a $(3+1)$-dimensional space, we shall not worry about this subtlety. In any case, we shall be able to show that the general, important features of our results also apply in $(3+1)$-dimensional models. Thus our emphasis of the $(1+1)$-dimensional case is intended primarily to achieve calculational simplicity and pedagogical clarity.

In Sec. II we introduce the full details of the model considered and mention certain properties of the amplitudes. In addition we discuss the distinction between amplitudes corresponding to "fragmentation" and those corresponding to "pionization."

Section III deals specifically with pionization processes. Here we establish the important "cluster-decomposition" properties of the pionization amplitudes; as the nomenclature suggests, these properties form the basis for a more quantitative application of the principles of statistical mechanics than is directly indicated in the original gas analogy.[1] We illustrate that the cluster-decomposition approach permits a simple interpretation of the exponentiation of the nonleading terms in the ladder diagrams to the Regge form $s^{\alpha(\lambda)}$; in essence, the cluster decomposition provides a new way of grouping terms so that this exponentiation becomes manifest. In this regard we should note that our discussion extends earlier work of Yan, Yao, and Chang[6] on nonleading $(\ln s)^n$ terms of ladder diagrams in a φ^3 theory. These authors demonstrated that the first few nonleading $(\ln s)^n$ terms in ladder diagrams of $(3+1)$-dimensional model indeed exponentiate; this result suggested the possibility of a cluster decomposition and thus provided the impetus for our present study.

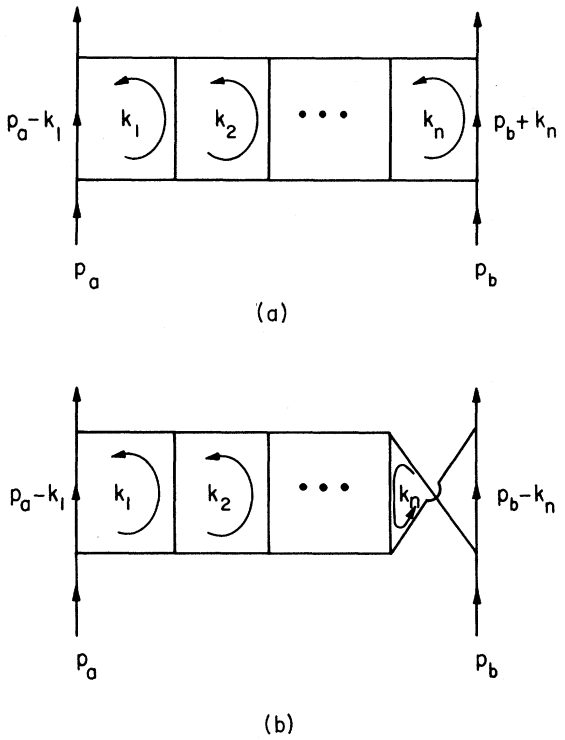

FIG. 1. (a) and (b) The general ladder and twisted-ladder diagrams.

In Sec. IV we apply the concept of cluster decompositions to fragmentation processes. Section V deals with the one-particle and multiparticle spectra which follow from our model; again we are able to establish the conceptual and calculational utility of the new approach.

Finally, we discuss in Sec. VI the applicability of the cluster decomposition to other, more general cases. In particular, we demonstrate its validity both for iterated cross diagrams in a $(1+1)$-dimensional theory and for ladder diagrams in a $(3+1)$-dimensional theory. In this case we are also able to establish a simple criterion, based on the properties of the cluster functions associated with particular diagrams, to distinguish between sets of diagrams producing Regge poles and those yielding Regge cuts or essential singularities.

II. MODEL

A. The Amplitudes A_n

For the purpose of detailed analysis we take as our model the infinite set of ladder and "cross" ladder (in the sense of Fig. 1) Feynman diagrams in a φ^3 quantum field theory with one space and one time dimension.[7] Later we shall demonstrate that our important results are common to classes of more complicated diagrams as well.[8]

In Fig. 1 the two external particles have masses m_a and m_b, respectively, whereas the common mass of the internal particles is μ; the various two-momenta are as defined by this figure. Initially, for ease of exposition, we assume that $m_a = m_b = 0$; we shall later discuss in detail this simplifying restriction and the consequences of relaxing it. We shall find that the qualitative features of our results are not sensitive to this simplification.

Let us first describe the approach briefly. Our intention to investigate the high-energy properties of the diagrams suggests that we introduce in place of the space and time components of the momenta the linear combinations

$$k_\pm \equiv k^0 \pm k^3,$$

where $k^\mu \equiv (k^0, k^3)$; these variables are known to be useful in the $s \to \infty$ limit.[9] Clearly, $k_+ k_- = k^2$. In addition we introduce

$$p_{a\pm} \equiv p_a^0 \pm p_a^3, \quad p_{b\pm} \equiv p_b^0 \pm p_b^3,$$

and note that our assumption of vanishing external mass implies $p_{a+}p_{a-} = p_{b+}p_{b-}$. We choose coordinates such that $p_{a+} \neq 0$ and $p_{b-} \neq 0$. In general, we shall work in the "center-of-mass frame" so that $p_{a+} = \sqrt{s} = p_{b-}$. Following Chang and Ma,[9] we write the amplitudes in terms of the $k_\pm$ and then integrate over all the internal k_- components; the result is equivalent to the infinite-momentum technique of Weinberg.[10]

As an explicit example, consider the diagram of Fig. 2(a).[11] The amplitude, $A_3(s)$, corresponding to this diagram is given by[12]

$$A_3(s) = i(ig)^8 \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2k_3}{(2\pi)^2} \frac{i}{(p_a - k_1)^2 - \mu^2 + i\epsilon} \frac{-1}{(k_1^2 - \mu^2 + i\epsilon)^2}$$
$$\times \frac{i}{(k_1 - k_2)^2 - \mu^2 + i\epsilon} \frac{-1}{(k_2^2 - \mu^2 + i\epsilon)^2} \frac{i}{(k_2 - k_3)^2 - \mu^2 + i\epsilon} \frac{-1}{(k_3^2 - \mu^2 + i\epsilon)^2} \frac{i}{(p_b + k_3)^2 - \mu^2 + i\epsilon}. \tag{2.1}$$

With $k_{i\pm} \equiv k_i^0 \pm k_i^3$ and $\frac{1}{2} dk_{i+} dk_{i-} = d^2k_i$, we may write this as

$$A_3(s) = \frac{i(ig)^8 i^{10}}{(2\pi)^6 2^3} \int dk_{1+} dk_{2+} dk_{3+} \int dk_{1-} dk_{2-} dk_{3-}$$
$$\times \frac{1}{(p_{a+} - k_{1+})(-k_{1-}) - \mu^2 + i\epsilon} \frac{1}{(k_{1+}k_{1-} - \mu^2 + i\epsilon)^2} \frac{1}{(k_{1+} - k_{2+})(k_{1-} - k_{2-}) - \mu^2 + i\epsilon} \frac{1}{(k_{2+}k_{2-} - \mu^2 + i\epsilon)^2}$$
$$\times \frac{1}{(k_{2+} - k_{3+})(k_{2-} - k_{3-}) - \mu^2 + i\epsilon} \frac{1}{(k_{3+}k_{3-} - \mu^2 + i\epsilon)^2} \frac{1}{k_{3+}(p_{b-} + k_{3-}) - \mu^2 + i\epsilon}. \tag{2.2}$$

Label the seven distinct factors in the integrand $d_1, d_2, \ldots, d_7$, sequentially, and consider the integral over k_{3-}. In the complex k_{3-} plane, the integration contour lies initially on the real axis. If this contour is deformed, contributions to the integral will arise from the poles caused by the vanishing of the denominators in d_5, d_6, and d_7. However, unless $k_{2+} - k_{3+}$ have the same sign, the three poles will all be on the same side of the real axis, and the integral over k_{3-} can be shown to vanish. For this integral not to vanish we thus require

$$\mathrm{sgn}(k_{2+} - k_{3+}) = \mathrm{sgn}(k_{3+})$$

or

$$|k_{2+}| > |k_{3+}|. \tag{2.3}$$

Similar arguments applied to the k_{2-} and k_{1-} integrals lead to the restrictions

$$|k_{1+}| > |k_{2+}|, \quad \mathrm{sgn}(k_{1+}) = \mathrm{sgn}(k_{2+}),$$
$$|p_{a+}| > |k_{1+}|, \quad \mathrm{sgn}(p_{a+}) = \mathrm{sgn}(k_{1+}), \tag{2.4}$$

and hence, since $p_{a+} > 0$, to the crucial ordering restriction

$$p_{a+} > k_{1+} > k_{2+} > k_{3+} > 0. \tag{2.5}$$

It is clear that this restriction generalizes to the case of n-rung ladders; the critical role played by the assumption of zero external mass will be discussed later.

If we now do the explicit k_{i-} integrations in sequence using Cauchy's theorem and remembering the restriction (2.5), we obtain

$$A_3(s)=-g^2\left(\frac{g^2}{4\pi}\right)^3\int_0^{p_{a+}}\frac{dk_{1+}}{k_{1+}}\int_0^{k_{1+}}\frac{dk_{2+}}{k_{2+}}\int_0^{k_{2+}}\frac{dk_{3+}}{k_{3+}}\frac{1}{p_{a+}-k_{1+}}\frac{1}{k_{1+}}\left(\frac{\mu^2}{p_{a+}-k_{1+}}+\frac{\mu^2}{k_{1+}}\right)^{-2}$$
$$\times\frac{1}{k_{1+}-k_{2+}}\frac{1}{k_{2+}}\left(\frac{\mu^2}{p_{a+}-k_{1+}}+\frac{\mu^2}{k_{1+}-k_{2+}}+\frac{\mu^2}{k_{2+}}\right)^{-2}\frac{1}{k_{2+}-k_{3+}}\frac{1}{k_{3+}}\left(\frac{\mu^2}{p_{a+}-k_{1+}}+\frac{\mu^2}{k_{1+}-k_{2+}}+\frac{\mu^2}{k_{2+}-k_{3+}}+\frac{\mu^2}{k_{3+}}\right)^{-2}$$
$$\times\frac{1}{k_{3+}}\left(-p_{b-}+\frac{\mu^2}{p_{a+}-k_{1+}}+\frac{\mu^2}{k_{1+}-k_{2+}}+\frac{\mu^2}{k_{2+}-k_{3+}}+\frac{\mu^2}{k_{3+}}-i\epsilon\right)^{-1}. \tag{2.6}$$

This result, as anticipated, agrees exactly with that obtained from Weinberg's infinite-momentum diagrammatic rules.[10]

From the form of (2.6) and from the arguments used to derive that equation the general result for A_n is apparent. For compactness of notation we introduce the scaled variables $x_i\equiv k_{i+}/p_{a+}$. Then

$$A_n(s)=-\frac{g^2}{\mu^2}\left(\frac{g^2}{4\pi\mu^4}\right)^n\int_0^1\frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{x_2}\cdots\int_0^{x_{n-1}}\frac{dx_n}{x_n}a^{(n)}(s), \tag{2.7}$$

where $s=p_{a+}p_{b-}$. We have suppressed the x dependence in $a^{(n)}(s)$, which in terms of x_i and s is given by

$$a^{(n)}(s)=\frac{1}{1-x_1}\frac{1}{x_1}\left(\frac{1}{1-x_1}+\frac{1}{x_1}\right)^{-2}\frac{1}{x_1-x_2}\frac{1}{x_2}\left[\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\frac{1}{x_2}\right]^{-2}\cdots\frac{1}{x_{n-1}-x_n}\frac{1}{x_n}\left[\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\cdots+\frac{1}{x_{n-1}-x_n}+\frac{1}{x_n}\right]^{-2}$$
$$\times\frac{1}{x_n}\left[-\frac{p_{a+}p_{b-}}{\mu^2}+\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\cdots+\frac{1}{x_n}-i\epsilon\right]^{-1}. \tag{2.8}$$

Note that the coupling constant g in a $(1+1)$-dimensional φ^3 theory has the dimension of (mass)^2. Thus, $g^2/4\pi\mu^4$ is a dimensionless constant.

B. Properties of the A_n and $a^{(n)}$

Two properties of $a^{(n)}$ are particularly important, for they are found in amplitudes corresponding to diagrams more complicated than the simple ladders. The first important characteristic is that the x_i are ordered in the sense that

$$1>x_1>x_2>x_3>\cdots>x_n>0. \tag{2.9}$$

In the present case this result is, as we shall now show, related to our assumption that $m_a=0=m_b$. However, we will establish that in the more realistic case, $m_a\neq0\neq m_b$, the x_i's – with the possible exception of those characterizing the ends of the diagram – will still be ordered at large s.

To begin let us demonstrate directly the relation between (2.9) and the assumption of zero-mass external particles. Consider the amplitude, A_3', corresponding to Fig. 2(a), in the case that $m_b\neq0$. Since $p_{b+}p_{b-}=m_b^2\neq0$, we have $p_{b+}\neq0$; in coordinates in which $p_{b-}=\sqrt{s}$, $p_{b+}=m_b^2/\sqrt{s}$. Thus A_3' will differ from A_3, but only in that the factor d_7 in the integrand becomes

$$d_7'=\frac{1}{(p_{b+}+k_{3+})(p_{b-}+k_{3-})-\mu^2+i\epsilon}. \tag{2.10}$$

However, this minor change is sufficient to remove the requirement that

$$|k_{2+}|>|k_{3+}|$$

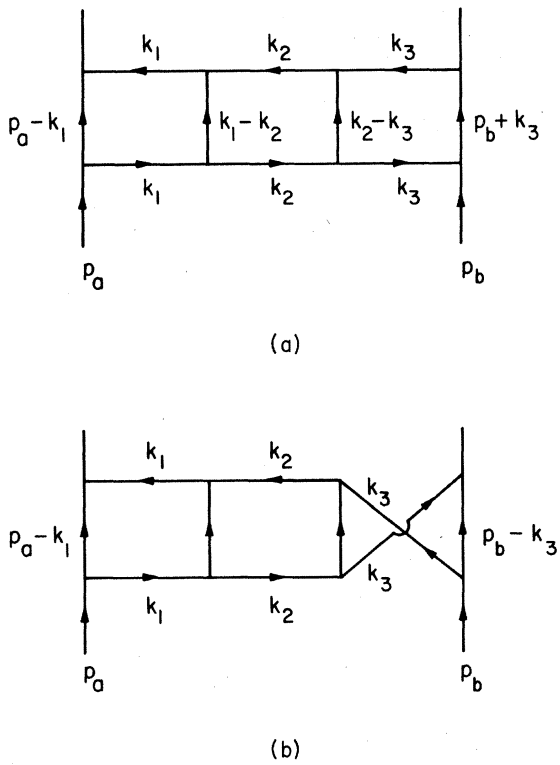

FIG. 2. (a) The ladder diagram corresponding to $A_3(s)$. (b) The twisted-ladder diagram corresponding to $A_3(-s)$.

for the integral over k_{3-} to be nonzero and thus to destroy the exact ordering of the k_{i+}. To establish this we note that if

$$|p_{b+}|>|k_{3+}| \quad \text{and} \quad k_{3+}<0, \tag{2.11}$$

then regardless of the magnitude or sign of k_{2+} there is a nonzero contribution to A_3' from the integral over k_{3-}.

To study in more detail the influence of the external mass on the ordering restriction, we should distinguish the three possible cases: (1) $m_a\neq 0$, $m_b=0$; (2) $m_a=0$, $m_b\neq 0$; and (3) $m_a\neq 0\neq m_b$. Considering case (1), in which we have $p_{a+}\neq 0\neq p_{a-}$, we observe that the ordering argument for the k_{i+} will proceed exactly as before, since none of the coefficients of the k_{i-} in the d_i has changed. Thus the exact ordering of k_{i+} will be preserved. Symmetry then suggests that in case (2) we consider a possible ordering of the k_{i-} components and do the explicit Cauchy integrals over the k_{i+} components; in this manner we can establish that for the amplitude describing Fig. 2(a) in case (2) the ordering

$$0<k_{1-}<k_{2-}<k_{3-}<p_{b-} \tag{2.12}$$

holds. Thus the only external mass configuration in which neither the k_{i+} nor the k_{i-} are ordered exactly is that of case (3). An extension of the argument preceding Eq. (2.11) to the amplitude A_n indicates that in case (3) a nonvanishing contribution to A_n can come from a region in which the $x_i=k_{i+}/p_{a+}$ are restricted to

$$1>x_1>x_2>\cdots>x_j$$

and (2.13)

$$0<|x_j|<|x_{j+1}|<\cdots<|x_{n-1}|<|x_n|<p_{b+}/p_{a+}.$$

Equation (2.13) indicates that all momenta obeying $x_i\geq p_{b+}/p_{a+}=m_b^2/s$ are ordered. Only those x's whose corresponding longitudinal momenta k_+ are comparable to that of particle b are no longer properly ordered. But as we shall see, these few x's can be identified as belonging to the fragmentation region of particle b and therefore are only finite in number. The contributions due to these regions are finite and do not increase with s. Hence at large s, except for possible deviations near the ends of the diagram, the x_i satisfy (2.9) even in the case of nonzero external mass. We remark that this ordering holds in more general diagrams than ladders; in fact, it is valid for any multiperipheral-type diagram.

The second crucial property of the $a^{(n)}$ is a type of factorizability; for $(x_1,\ldots,x_l)\gg(x_{l+1},\ldots,x_m)\gg(x_{m+1},\ldots,x_n)$ we find

$$a^{(n)}(s)\to\frac{\mu^2}{s}\left[f_L^{(l)}b^{(m-l)}f_R^{(n-m)}+O\left(\frac{x_{l+1}}{x_l},\frac{x_{m+1}}{x_m}\right)\right]. \tag{2.14}$$

In the case $m_a=m_b=0$ direct calculation from (2.8) yields

$$f_L^{(l)}(x_1,\ldots,x_l)=\frac{1}{(1-x_1)x_1(x_1-x_2)x_2(x_2-x_3)\cdots(x_{l-1}-x_l)x_l^{\,2}}\times\left(\frac{1}{1-x_1}+\frac{1}{x_1}\right)^{-2}\left(\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\frac{1}{x_2}\right)^{-2}\cdots\left(\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\cdots+\frac{1}{x_{l-1}-x_l}+\frac{1}{x_l}\right)^{-2}, \tag{2.15a}$$

$$b^{(k)}(x_1,\ldots,x_k)=\frac{x_1/x_k}{(x_1-x_2)x_2(x_2-x_3)x_3\cdots x_k}\left(\frac{1}{x_1-x_2}+\frac{1}{x_2}\right)^{-2}\times\left(\frac{1}{x_1-x_2}+\frac{1}{x_2-x_3}+\frac{1}{x_3}\right)^{-2}\cdots\left(\frac{1}{x_1-x_2}+\frac{1}{x_2-x_3}+\cdots+\frac{1}{x_{k-1}-x_k}+\frac{1}{x_k}\right)^{-2}, \tag{2.15b}$$

and

$$f_R^{(m)}(x_1,\ldots,x_m)=\frac{x_1/x_m}{(x_1-x_2)x_2(x_2-x_3)x_3\cdots x_m}\left(\frac{1}{x_1-x_2}+\frac{1}{x_2}\right)^{-2}\times\left(\frac{1}{x_1-x_2}+\frac{1}{x_2-x_3}+\frac{1}{x_3}\right)^{-2}\cdots\left(\frac{1}{x_1-x_2}+\frac{1}{x_2-x_3}+\cdots+\frac{1}{x_{m-1}-x_m}+\frac{1}{x_m}\right)^{-2}\times s\left(-s+\frac{\mu^2}{x_1-x_2}+\frac{\mu^2}{x_2-x_3}+\cdots+\frac{\mu^2}{x_m}-i\epsilon\right)^{-1}. \tag{2.15c}$$

In reducing (2.8) via (2.14) to the form of (2.15), we have used the result $p_{a+}p_{b-}=s$. We remark that in

(2.15) the functions are defined so that in general for large s they are $O(1)$.

By examining each of the factors in (2.15) separately we can simplify the detailed investigation of the expressions $a^{(n)}$ and thus of the amplitude A_n. More important than this mathematical simplicity, however, are the direct physical interpretations which we will establish for these factors. The expressions f_L and f_R will represent events classified as "fragmentation," while b will describe those called "pionization." The significance of these concepts rests on the results that the suitably normalized distribution properties of the "pionization" processes are independent of the masses and natures of the incoming particles, whereas those of the fragmentation of, say, the target particle are independent of the energy and nature of the projectile.

To clarify briefly further aspects of these two types of processes, let us first consider fragmentation events, that is, those interactions involving final-state particles which carry away momenta comparable in size to the large incident momentum. We note at the outset that the apparent asymmetry between f_L and f_R, which represent, respectively, the "left"- and the "right"-fragmentation amplitudes in our model, arises because of our emphasis of the variables $x_i \equiv k_{i+}/p_{a+}$. In terms of $y_i \equiv k_{i-}/p_{b-}$, we would find that $f_R(y_i)$ is similar to $f_L(x_i)$ and that $f_L(y_i)$ is related to $f_R(x_i)$. Further, we can see that these fragmentation amplitudes depend strongly on the values of the external masses: f_L on m_a and f_R on m_b. To illustrate this mass dependence, let us study the form f_L when we take $m_a \neq 0$. We begin by considering the form of $A_3(s)$, corresponding to the diagram of Fig. 2(a), in the case $p_{a+}p_{a-} = m_a^2 \neq 0$. The only modification from the form given by (2.2) occurs in the factor which we have called d_1:

$$d_1 \to d_1' = \frac{1}{(p_{a+}-k_{1+})(p_{a-}-k_{1-})-\mu^2+i\epsilon}. \tag{2.16}$$

Since the ordering restriction (2.5) on the k_{1+} components is not altered by this change, we may proceed exactly as in the previous case to obtain

$$\begin{aligned} A_3(s) = -g^2\left(\frac{g^2}{4\pi}\right)^3 \int_0^{p_{a+}} \frac{dk_{1+}}{k_{1+}} \int_0^{k_{1+}} \frac{dk_{2+}}{k_{2+}} \int_0^{k_{2+}} \frac{dk_{3+}}{k_{3+}} \frac{1}{p_{a+}-k_{1+}} \frac{1}{k_{1+}} \left(-\frac{m_a^2}{p_{a+}}+\frac{\mu^2}{p_{a+}-k_{1+}}+\frac{\mu^2}{k_{1+}}-i\epsilon\right)^{-2} \\ \times \frac{1}{k_{1+}-k_{2+}} \frac{1}{k_{2+}} \left(-\frac{m_a^2}{p_{a+}}+\frac{\mu^2}{p_{a+}-k_{1+}}+\frac{\mu^2}{k_{1+}-k_{2+}}+\frac{\mu^2}{k_{2+}}-i\epsilon\right)^{-2} \\ \times \frac{1}{k_{2+}-k_{3+}} \frac{1}{k_{3+}} \left(-\frac{m_a^2}{p_{a+}}+\frac{\mu^2}{p_{a+}-k_{1+}}+\frac{\mu^2}{k_{1+}-k_{2+}}+\frac{\mu^2}{k_{2+}-k_{3+}}+\frac{\mu^2}{k_{3+}}-i\epsilon\right)^{-2} \\ \times \frac{1}{k_{3+}} \left(-p_{b-}-\frac{m_a^2}{p_{a+}}+\frac{\mu^2}{p_{a+}-k_{1+}}+\frac{\mu^2}{k_{1+}-k_{2+}}+\frac{\mu^2}{k_{2+}-k_{3+}}+\frac{\mu^2}{k_{3+}}-i\epsilon\right)^{-1}. \end{aligned} \tag{2.17}$$

The $i\epsilon$ in the first three expressions can be ignored if the mass of the incident particle satisfies $m < 2\mu$; this is the requirement that particle a does not have enough mass to decay into two exchanged particles. In the following, we shall always assume that $m < 2\mu$ and thus ignore these $i\epsilon$'s.

Again the general form of the nth-order contribution is apparent from this third-order term. Omitting the intermediate steps, we note simply that in terms of the x_i

$$A_n(s) = \frac{-g^2}{\mu^2}\left(\frac{g^2}{4\pi\mu^4}\right)^n \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \cdots \int_0^{x_{n-1}} \frac{dx_n}{x_n} a^{(n)}(s), \tag{2.18}$$

where

$$\begin{aligned} a^{(n)}(s) = \frac{1}{1-x_1}\frac{1}{x_1}\left(-\frac{m_a^2}{\mu^2}+\frac{1}{1-x_1}+\frac{1}{x_1}\right)^{-2} \frac{1}{x_1-x_2}\frac{1}{x_2}\left(-\frac{m_a^2}{\mu^2}+\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\frac{1}{x_2}\right)^{-2} \cdots \\ \times \frac{1}{x_{n-1}-x_n}\frac{1}{x_n}\left(-\frac{m_a^2}{\mu^2}+\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\cdots+\frac{1}{x_{n-1}-x_n}+\frac{1}{x_n}\right)^{-2} \frac{1}{x_n}\left(-\frac{p_{a+}p_{b-}}{\mu^2}-\frac{m_a^2}{\mu^2}+\frac{1}{1-x_1}-+\cdots+\frac{1}{x_n}-i\epsilon\right)^{-1}. \end{aligned}$$

The form of the factorization property of $a^{(n)}(s)$ in the limit $(x_1, \ldots, x_l) \gg (x_{l+1}, \ldots, x_m) \gg (x_{m+1}, \ldots, x_n)$ remains unchanged; in addition, the functions $b^{(k)}(x_1, \ldots, x_k)$ and $f_R^{(m)}(x_1, \ldots, x_m)$ are given exactly by (2.15). Only the function $f_L^{(l)}(x_1, \ldots, x_l)$ is modified; we find

$$\begin{aligned} f_L^{(l)}(x_1, \ldots, x_l) = \frac{1}{(1-x_1)x_1(x_1-x_2)x_2 \cdots (x_{l-1}-x_l)x_l^2} \\ \times \left(-\frac{m_a^2}{\mu^2}+\frac{1}{1-x_1}+\frac{1}{x_1}\right)^{-2}\left(-\frac{m_a^2}{\mu^2}+\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\frac{1}{x_2}\right)^{-2} \cdots \left(-\frac{m_a^2}{\mu^2}+\frac{1}{1-x_1}+\frac{1}{x_1-x_2}+\cdots+\frac{1}{x_l}\right)^{-2}. \end{aligned} \tag{2.19}$$

In addition to verifying the dependence of the fragmentation functions on the external mass, these considerations establish the equally important result that the expressions $b^{(n)}(x_1, \ldots, x_n)$ do not depend on the external masses at all; this is one typical characteristic of the "pionization" events, which involve final-state particles having momenta which are small compared to those of the incident particles. Further properties of the $b^{(n)}(x_1, \ldots, x_n)$ are examined in Sec. III.

To conclude the present discussion, we remark that although we have chosen to distinguish "fragmentation" from "pionization," there is no clear, natural distinction between the particles produced in these two processes. When we examine the one-particle spectrum in the final state, for example, we expect to observe a smooth transition from the region of large longitudinal momentum (called fragmentation) to that of small longitudinal momentum (pionization). Therefore, any classification differentiating these two types of particles will be artificial. However, for the sake of definiteness, we shall call a particle a "fragment" if its longitudinal momentum k_+ is larger than a small, but fixed and s-independent, fraction (η) of the large incident longitudinal momentum p_+. We shall define a particle to be a "pionization product" if $k_+ < \eta p_+$. To keep a scale in mind, we may take $\eta \simeq 0.01$. Our previous arguments lead us to anticipate that the distribution properties for very soft fragments having $k_+ \sim \eta p_+$ should be similar to those of the pionization particles. Further, the overall contributions and various distribution properties of the soft fragments and the pionization should be insensitive to – actually, independent of – the value of η.

III. PIONIZATION

A. The Pionization Amplitude

The nth-order contribution to the pionization amplitude in our model is, in the notation of Sec. II, determined by the function

$$B_n(s) \equiv \int_{1/s}^{1} \frac{dx_1}{x_1} \int_{1/s}^{x_1} \frac{dx_1}{x_2} \cdots \int_{1/s}^{x_{n-1}} \frac{dx_n}{x_n} b^{(n)}(x_1, x_2, \ldots, x_n), \tag{3.1}$$

where $b^{(n)}$ is given by (2.15b) and the ordering restriction (2.9) applies. The variable $s = k_{\max}/k_{\min}$ is the ratio of the maximum-to-minimum longitudinal momentum k_i allowed in the pionization region. It is a measure of the available longitudinal-momentum phase space. The limits $k_{\max}$ and $k_{\min}$ can be taken as ηp_{a+} and $m^2/\eta' p_{b-}$, respectively, with η, η' being some small positive but energy-independent numbers and m being a typical mass. Then,

$$s = k_{\max}/k_{\min} = (\eta\eta'/m^2)(p_{a+}p_{b-}) = (\eta\eta'/m^2) \times (\text{invariant energy squared}).$$

Thus, to within a multiplicative factor $\eta\eta'/m^2$, s is the conventional invariant energy of the system. As we shall discuss in Sec. IV, after combining the contributions from fragmentation regions and pionization regions, the η and η' dependence in the final amplitude will cancel out. This will lead to a final amplitude which has the same power dependence on the actual (energy)^2 as the pionization amplitude does on our variable s. In the following discussion of the pionization amplitude, we shall, therefore, concentrate on s dependence and will refer to the variable s simply as the "energy".

We define the full pionization amplitude, $B(s, \lambda)$, to be

$$B(s, \lambda) \equiv \sum_{n=1}^{\infty} \lambda^n B_n(s), \quad \lambda = g^2/4\pi\mu^4. \tag{3.2}$$

We refer to $B_n(s)$, or sometimes to $b^{(n)}(x_1, \ldots, x_n)$, as a partial pionization amplitude. To remind ourselves that the pionization amplitude describes only the central portion of the Feynman diagram of Fig. 1, we associate with the amplitude $B_n(s)$ the diagram of Fig. 3.

From the form of $b^{(n)}$, we see directly that whenever $(x_1, x_2, \ldots, x_m) \gg (x_{m+1}, x_{m+2}, \ldots, x_n)$, we have

$$b^{(n)}(x_1, \ldots, x_n) \to b^{(m)}(x_1, \ldots, x_m) b^{(n-m)}(x_{m+1}, \ldots, x_n) + O(x_{m+1}/x_m). \tag{3.3}$$

Thus, when sets of the scaled momentum variables x_i differ greatly in magnitude and hence are widely separated in x space, the pionization amplitude breaks into independent pionization amplitudes involving only those x_i which remain "close" to each other. This remarkable analogy to the "cluster-decomposition property"[13] of scattering events (in real space-time) suggests that a similar cluster decomposition be applied here. Let us consider this possibility further; we begin with a qualitative description of our approach. Suppose we think of the x_i as representing the "coordinates" of molecules in a dilute gas; this analogy is at least *prima facie* applicable, since the x_i are independent – in the sense described above – if they are widely separated in the scaled longitudinal-momentum space.[14] As we shall establish, however,

the analogy has more than superficial validity, for we shall be able to characterize the detailed dynamics of our model – just as one can in the case of real gases – by introducing "correlation functions," $c^{(n)}(x_1, \ldots, x_n)$, which describe the (irreducible) n-particle correlations that can exist when $x_1 \approx x_2 \approx \cdots \approx x_n$. These irreducible correlation functions can be defined in terms of the amplitudes $b^{(n)}(x_1, \ldots, x_n)$ in the standard way. From the $c^{(n)}(x_1, \ldots, x_n)$ we can calculate the integrated correlation functions, $C_n(s)$, given by

$$C_n(s) = \int_{1/s}^{1} \frac{dx_1}{x_1} \int_{1/s}^{x_1} \frac{dx_2}{x_2} \cdots \int_{1/s}^{x_{n-1}} c^{(n)}(x_1, \ldots, x_n) . \tag{3.4}$$

The set of $C_n(s)$ then contains dynamical details of the pionization process. Further since the $C_n(s)$ are irreducible and since each is associated with a unique power of g^2, we anticipate that the full dependence on the coupling constant of dynamical quantities – cross section, one-particle spectra – to order g^{2k} can be obtained from knowledge of $C_1(s), \ldots, C_k(s)$. As we shall discuss in Sec. III D this is the important way in which the cluster-decomposition techique represents a conceptual improvement over standard techniques for evaluation of the asymptotic behavior of sets of Feynman diagrams. In the intervening sections we establish in a quantitative manner the validity of the cluster-decomposition approach.

B. The Cluster Decomposition

In terms of the pionization functions $b^{(n)}(x_1, \ldots, x_n)$, we define the irreducible correlation functions, $c^{(n)}(x_1, \ldots, x_n)$ describing the cluster decomposition by

$$\begin{aligned} &c^{(1)}(x) = b^{(1)}(x) = 1 , \\ &c^{(2)}(x_1, x_2) = b^{(2)}(x_1, x_2) - c^{(1)}(x_1) c^{(1)}(x_2) , \\ &c^{(3)}(x_1, x_2, x_3) = b^{(3)}(x_1, x_2, x_3) - c^{(1)}(x_1) c^{(1)}(x_2) c^{(1)}(x_3) - c^{(1)}(x_1) c^{(2)}(x_2, x_3) - c^{(1)}(x_2) c^{(2)}(x_1, x_3) - c^{(1)}(x_3) c^{(2)}(x_1, x_2) , \end{aligned} \tag{3.5}$$

and similarly for the higher $c^{(n)}$. Hence we have in the general case

$$b^{(n)}(x_1, x_2, \ldots, x_n) = \sum \left\{ \prod [c^{(n_i)}(x_1, \ldots) \cdots c^{(n_j)}(\ldots, x_n)] \right\} , \tag{3.6}$$

where the summation is such that $n = \sum_{i=1}^{j} n_i$ and where the arguments of the $c^{(n_i)}$ are distributed in all possible ways consistent with the ordering restriction that $x_1 > x_2 > \cdots > x_n$. We may also write (3.6) in the equivalent form

$$\begin{aligned} b^{(n)}(x_1, x_2, \ldots, x_n) = {} & c^{(1)}(x_1) b^{(n-1)}(x_2, \ldots, x_n) + \sum_{i=2}^{n} c^{(2)}(x_1, x_i) b^{(n-2)}(\cdots) \\ & + \sum_{i>j} c^{(3)}(x_1, x_i, x_j) b^{(n-3)}(\cdots) + \cdots + c^{(n)}(x_1, \ldots, x_n) . \end{aligned} \tag{3.7}$$

From (3.5) we observe that the first few $c^{(n)}(x_1, \ldots, x_n)$ have the crucial property that if $x_i \gg x_{i+1}$ for any i then

$$c^{(n)}(x_1, \ldots, x_n) \leq O(x_{i+1}/x_i) \to 0 . \tag{3.8}$$

The proof of this result in the general case follows by induction from (3.6) and the factorization property of $b^{(n)}(x_1, \ldots, x_n)$. Hence, as we anticipated, the $c^{(n)}(x_1, \ldots, x_n)$ describe correlation effects which can exist only when the x_i are clustered together in scaled momentum space.

One important consequence of (3.6) and (3.8) is that the integrated cluster function, $C_n(s)$, is at most linear in $\ln s$; that is,

$$\begin{aligned} C_n(s) &\equiv \int_{1/s}^{1} \frac{dx_1}{x_1} \int_{1/s}^{x_1} \frac{dx_2}{x_2} \cdots \int_{1/s}^{x_{n-1}} \frac{dx_n}{x_n} c^{(n)}(x_1, \ldots, x_n) \\ &= \alpha_n \ln s + \beta_n + O(1/s) . \end{aligned} \tag{3.9}$$

The significance of this property will become apparent when we establish the relation between $C_n(s)$ and the full pionization amplitude, $B_n(s)$. Although this type of result is well known in statistical-mechanical applications of cluster decompositions,[13] we present a brief argument here.

By introducing $z_i = \ln x_i$, we can transform (3.4) to

$$C_n(s) = \prod_{i=1}^{n} \int_{-\ln s}^{0} dz_i \, \theta(z_i - z_{i+1}) \bar{c}^{(n)}(z_1, \ldots, z_n) , \tag{3.10}$$

where $z_{n+1} \equiv \ln s$ and where $\tilde{c}^{(n)}$ is the appropriate function of z_i as determined by $c^{(n)}(x_1, \ldots, x_n)$. In particular, from (3.8) it follows that for $x_i \gg x_{i+1}$ and thus $z_i \gg z_{i+1}$

$$\tilde{c}^{(n)}(z_1, \ldots, z_n) \to \exp[-O(|z_i - z_{i+1}|)]. \tag{3.11}$$

We note that the z_i correspond to the "rapidity" variables which have recently been suggested as appropriate variables to apply in the study of high-energy interactions.[1,15]

Equation (3.11) indicates that we should make a further change of variables to $\omega_i = z_i - z_{i+1}$, $\omega_n \equiv -z_n$. Then (3.10) becomes

$$C_n(s) = \int_0^{\ln s} d\omega_n \prod_{i=1}^{n-1} \int_0^{\ln s} d\omega_i \tilde{c}^{(n)}(\omega_1, \ldots, \omega_n), \tag{3.12}$$

where now $\tilde{c}^{(n)}$ is the appropriate function of the ω_i as determined by $c^{(n)}(x_1, \ldots, x_n)$. Each of the ω_i integrations for $1 \leq i \leq n-1$ converges to some finite value as $\ln s \to \infty$; this follows directly from (3.11). Thus any possible factors of $\ln s$ in $C_n(s)$ can come only from the ω_n integration. We shall now show that $\tilde{c}^{(n)}(\omega_1, \ldots, \omega_n)$ is in fact independent of ω_n and thus that the ω_n integration does lead to

$$C_n(s) = \alpha_n \ln s + \beta_n + O(1/s). \tag{3.9}$$

To see this result we return to the original pionization expression $b^{(n)}(x_1, \ldots, x_n)$ given by (2.15b) and remark that it is invariant under the scale transformation $x_i \to c x_i$. This indicates that $b^{(n)}$ depends on only $(n-1)$ of the x_i or, equivalently, that $\tilde{b}^{(n)}(\omega_1, \ldots, \omega_n)$ depends only on $\omega_1, \ldots, \omega_{n-1}$. We verify this explicitly by rewriting (2.15b) as

$$\begin{aligned} b^{(n)}(x_1, \ldots, x_n) = {} & \left[\frac{x_1}{x_2}\frac{x_2}{x_3}\cdots\frac{x_{n-1}}{x_n}\Big/\left(\frac{x_1}{x_2}-1\right)\left(\frac{x_2}{x_3}-1\right)\cdots\left(\frac{x_{n-1}}{x_n}-1\right)\right] \\ & \times\left[\left(\frac{x_1}{x_2}-1\right)^{-1}+1\right]^{-2}\left\{\left[\frac{x_2}{x_3}\left(\frac{x_1}{x_2}-1\right)\right]^{-1}+\left(\frac{x_2}{x_3}-1\right)^{-1}+1\right\}^{-2}\cdots \\ & \times\left\{\left[\frac{x_2}{x_3}\cdots\frac{x_{n-1}}{x_n}\left(\frac{x_1}{x_2}-1\right)\right]^{-1}+\left[\frac{x_3}{x_1}\cdots\frac{x_{n-1}}{x_n}\left(\frac{x_2}{x_3}-1\right)\right]^{-1}+\cdots+\left(\frac{x_{n-1}}{x_n}-1\right)^{-1}+1\right\}^{-2}. \end{aligned} \tag{3.13}$$

Since the $c^{(n)}(x_i)$ are defined completely in terms of the $b^{(n)}(x_i)$, they too depend only on x_i/x_{i+1}, $1 \leq i \leq n-1$, and thus (3.9) is established.

The invariance of $b^{(n)}(x_i)$ under the transformation $x_i \to c x_i$ has an interesting interpretation in terms of the gas analogy. Let us digress briefly on this point. The x_i are defined in terms of k_{i+}; thus Lorentz transformations will act on the x_i via

$$x_i \to x_i' = e^{\lambda} x_i. \tag{3.14}$$

Hence the scale invariance simply asserts that the structure of pionization is the same in all Lorentz frames related by (finite) Lorentz transformations to the center-of-mass frame. If we write (3.14) in terms of $z_i = \ln x_i$, we find that scale invariance is just translational invariance in the z_i space[15]:

$$z_i \to z_i' = z_i + \lambda. \tag{3.15}$$

In terms of the analogy to molecules in a gas, this means that interactions which take place in the interior of a container – and which, therefore, correspond to pionization processes – are invariant under translations provided these do not move the molecules so close to the surface of the container that edge effects are important.[16] This argument suggests that $z_i = \ln x_i$ should be taken as the variable corresponding to length in our gas analogy.

To conclude this general discussion of the cluster-decomposition approach, we present the proof that the correlation functions $C_n(s)$ determine the full pionization amplitude $B(s, \lambda)$. To simplify this proof we begin by extending the definitions of the pionization and cluster functions, $b^{(n)}(x_1, \ldots, x_n)$ and $c^{(n)}(x_1, \ldots, x_n)$, so that they are completely symmetric in their arguments; for example, the function $c^{(2)}(x_1, x_2)$, which for $x_1 > x_2$ is given by

$$c^{(2)}(x_1, x_2) = -\frac{x_2}{x_1}, \tag{3.16}$$

is replaced by the symmetric function

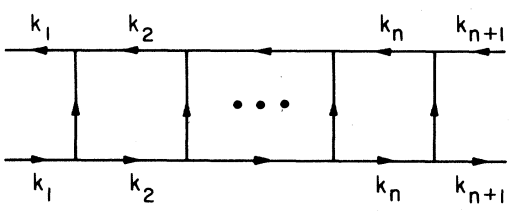

FIG. 3. The nth-order ladder corresponding to the pionization amplitude $B_n(s)$.

$$c_s^{(2)}(x_1, x_2) = -\frac{x_2}{x_1}\theta(x_1 - x_2) - \frac{x_1}{x_2}\theta(x_2 - x_1).$$

We define the symmetric, integrated pionization and cluster functions by

$$B_n^s(s) \equiv \prod_{i=1}^{n} \int_{1/s}^{1} \frac{dx_i}{x_i} b_s^{(n)}(x_1, \ldots, x_n)$$

and (3.17)

$$C_n^s(s) \equiv \prod_{i=1}^{n} \int_{1/s}^{1} \frac{dx_i}{x_i} c_s^{(n)}(x_1, \ldots, x_n).$$

These functions have the property that

$$B_n^s(s) = n! B_n(s)$$

and (3.18)

$$C_n^s(s) = n! C_n(s).$$

This is clear in the case of $c^{(2)}(x_1, x_2)$ and follows simply in the general case.

In terms of these symmetric functions, (3.7) remains unchanged. Thus multiplying it by

$$\prod_{i=1}^{n} \int_{1/s}^{1} \frac{dx_i}{x_i}$$

and integrating yields

$$B_n^s(s) = C_1^s(s) B_{n-1}^s(s) + \binom{n-1}{1} C_2^s(s) B_{n-2}^s(s) + \binom{n-1}{2} C_3^s(s) B_{n-3}^s(s) + \cdots . \tag{3.19}$$

The binomial coefficients arise because of the number of terms in the summations in (3.7). Dividing by $(n-1)!$, multiplying by $\lambda^n \equiv (g^2/4\pi\mu^4)^n$, and summing over n yields

$$\sum \frac{\lambda^n B_n^s(s)}{(n-1)!} = \sum \lambda^n C_1^s(s) \frac{B_{n-1}^s(s)}{(n-1)!} + \sum \lambda^n \frac{C_2^s(s)}{1!} \frac{B_{n-2}^s(s)}{(n-2)!} + \sum \lambda^n \frac{C_3^s(s)}{2!} \frac{B_{n-3}^s(s)}{(n-3)!} + \cdots . \tag{3.20}$$

Now

$$\lambda \frac{\partial B(s, \lambda)}{\partial \lambda} \equiv \sum \lambda^n n B_n(s) = \sum \lambda^n \frac{B_n^s(s)}{(n-1)!}. \tag{3.21}$$

Rearranging the terms on the right-hand side of (3.20) yields

$$\lambda \frac{\partial B}{\partial \lambda}(s, \lambda) = \left(\lambda C_1^s(s) + \frac{\lambda^2 C_2^s(s)}{1!} + \frac{\lambda^3 C_3^s(s)}{2!} + \cdots \right) B(s, \lambda). \tag{3.22}$$

But the term in brackets is just

$$\sum \frac{\lambda^n C_n^s(s)}{(n-1)!} = \lambda \frac{\partial C(s, \lambda)}{\partial \lambda}$$

and thus

$$\lambda \frac{\partial B(\lambda, s)}{\partial \lambda} = \lambda \frac{\partial C(s, \lambda)}{\partial \lambda} B(s, \lambda), \tag{3.23}$$

which has the solution

$$B(\lambda, s) = B(0, s) \exp[C(\lambda, s) - C(0, s)]. \tag{3.24}$$

The importance of (3.9) now becomes evident, as it is essential that $C(\lambda, s)$ be linear in $\ln s$ for $B(\lambda, s)$ to exhibit Regge behavior. Together, Eqs. (3.9) and (3.24) provide a simple general proof to all orders in λ of the Regge behavior of the pionization amplitude. If we define

$$\alpha(\lambda) = \sum \alpha_n \lambda^n$$

and (3.25)

$$\beta(\lambda) = \sum \beta_n \lambda^n,$$

then

$$B(\lambda, s) = B(0, s) e^{\beta(\lambda)} s^{\alpha(\lambda)}. \tag{3.26}$$

Note that our $\alpha(\lambda) \to 0$ as $\lambda \to 0$. This definition of $\alpha(\lambda)$ is related to the conventional Regge-trajectory function $\alpha_c(\lambda)$ by $\alpha(\lambda) = \alpha_c(\lambda) + 1$. We have chosen the normalization such that our trajectory function contains only dynamical effects. The extra kinematic factor $(1/s)$ corresponding to two-particle exchange has been extracted explicitly in Eq. (2.14).

C. The Nearest-Neighbor Approximation

We have been able to illustrate in a rather general manner the validity of the cluster-decomposition approach to evaluating sums of sets of Feynman diagrams; we have also been able to establish in this way the well-known Regge behavior of the ladder graphs comprising our model. However, to solve explicitly for the nth-order integrated cluster function describing the pionization amplitude even in this simple model remains too difficult. Since, in addition to presenting the new conceptual approach of cluster decomposition, we wish to study the detailed physical properties predicted by the model, we choose to define an approximation to the full amplitude in which we can solve explicitly for the C_n. An added advantage of making this approximation is that we can then calculate the full pionization amplitude, $B(s, \lambda)$, by a method independent of the cluster decomposition; the agreement between this direct result and that obtained via the $C_n(s)$ will provide a detailed confirmation, complementary to the general arguments of Sec. III B, of our technique.

In choosing this approximation we are again influenced by the results of statistical mechanics; returning to the analogy between the x_i corresponding to lines in the Feynman diagram of Fig. 3

and the particles of a gas contained in a closed volume, we suggest that the influence of "nearest neighbors" should be most important. We implement this idea explicitly by retaining only the expressions $1/(x_{m-1}-x_m)+1/x_m$ in the mth term in (2.15), the equation for $b^{(n)}(x_1, \ldots, x_n)$. After some rearrangement of factors, we can then write this "nearest-neighbor" approximation to $b^{(n)}(x_1, \ldots, x_n)$ as

$$b^{(n)}_{\text{n.n.a.}}(x_1, \ldots, x_n) = (1-x_2/x_1)(1-x_3/x_2)\cdots(1-x_n/x_{n-1}). \tag{3.27}$$

It is important to note that $b^{(n)}_{\text{n.n.a.}}$ obeys the same factorization property (3.3) as does $b^{(n)}$. Hence the cluster-decomposition properties remain valid in this approximation. For the remainder of this section we shall deal only with $b^{(n)}_{\text{n.n.a.}}$; thus we can drop the subscript on both the $b^{(n)}$ and $c^{(n)}$ without confusion.

By direct calculation from (3.27) and (3.6) we find that the first few correlation functions are

$$\begin{aligned}
&c^{(1)}(x)=1,\\
&c^{(2)}(x_1,x_2)=-x_2/x_1,\\
&c^{(3)}(x_1,x_2,x_3)=2x_3/x_1,\\
&c^{(4)}(x_1,x_2,x_3,x_4)=-4x_4/x_1-2x_3x_4/x_1x_2,\\
&c^{(5)}(x_1,x_2,x_3,x_4,x_5)=8\frac{x_5}{x_1}+8\frac{x_4x_5}{x_1x_2}+4\frac{x_3x_5}{x_1x_2}+4\frac{x_4x_5}{x_1x_2},\\
&\text{etc.}
\end{aligned} \tag{3.28}$$

The corresponding integrated cluster functions, defined by (3.9), are in leading order[17]

$$\begin{aligned}
&C_1(s)=\ln s,\\
&C_2(s)=-\ln s+1+O(1/s),\\
&C_3(s)=2\ln s-4+O(1/s),\\
&C_4(s)=-5\ln s+\tfrac{29}{2}+O(1/s),\\
&C_5(s)=14\ln s+\text{const}+O(1/s),\\
&\text{etc.}
\end{aligned} \tag{3.29}$$

Notice that, as our general arguments indicated, each of the C_n is indeed linear in $\ln s$. From our discussions of Sec. III B we expect the full pionization amplitude to be given by

$$B(s,\lambda)=Ce^{\beta(\lambda)}s^{\alpha(\lambda)}, \tag{3.30}$$

where from (3.9) and (3.25) we deduce that

$$\alpha(\lambda)=\lambda-\lambda^2+2\lambda^3-5\lambda^4+14\lambda^5+\cdots$$

and

$$\beta(\lambda)=\lambda^2-4\lambda^3+\tfrac{29}{2}\lambda^4+\cdots.$$

There are now two questions of interest. First, can we establish the general form of the C_n and thus obtain the full power series for $\alpha(\lambda)$ and $\beta(\lambda)$? Second, can we find an alternative, direct approach to evaluating $B(\lambda)$, in order to verify in this explicit example the cluster-decomposition technique? In the Appendix we give a detailed answer to the first question; let us continue here by responding to the second.

From (3.1) we see that in the nearest-neighbor approximation the nth-order pionization amplitude is given by

$$B_n(s)=\int_{1/s}^1\frac{dx_1}{x_1}\int_{1/s}^{x_1}\frac{dx_2}{x_2}\cdots\int_{1/s}^{x_{n-1}}\frac{dx_n}{x_n}\left(1-\frac{x_2}{x_1}\right)\cdots\left(1-\frac{x_n}{x_{n-1}}\right). \tag{3.31}$$

Define $y_i=x_i/x_{i-1}$ with $x_0=1$. Then (3.31) may be rewritten as

$$\begin{aligned}
B_n(s)&=\int_{1/s}^1\frac{dy_1}{y_1}\int_{1/x_1s}^1\frac{dy_2}{y_2}\cdots\int_{1/x_{n-1}s}^1\frac{dy_n}{y_n}(1-y_2)\cdots(1-y_n)\\
&=\int_0^1\frac{dy_1}{y_1}\int_0^1\frac{dy_2}{y_2}\cdots\int_0^1\frac{dy_n}{y_n}\theta(y_1\cdots y_n-1/s)(1-y_2)\cdots(1-y_n),
\end{aligned} \tag{3.32}$$

where we have used $x_n=y_1\cdots y_n$ and where one can verify explicitly that the function $\theta(y_1\cdots y_n-1/s)$ assures that the lower limits of integration are correct. A further modification enables us to write $B_n(s)$ as

$$B_n(s)=\int_0^1\frac{dy_1}{y_1}D_{n-1}\left(\frac{1}{y_1s}\right),$$

where[18]

$$D_{n-1}\left(\frac{1}{y_1s}\right)\equiv\left(\prod_{i=2}^n\int_0^1\frac{dy_i}{y_i}(1-y_i)\right)\theta(y_1\cdots y_n-1/s). \tag{3.33}$$

We introduce a Laplace-type transform of D_m with respect to its argument via

$$\tilde{D}_m(\nu)\equiv\int_0^1 D_m(t)t^{\nu-1}dt,\quad \nu>0. \tag{3.34}$$

Applying this transform to D_{n-1} with $1/y_1s\equiv t$ yields

$$D_{n-1}(\nu)=\left(\prod_{i=2}^{n}\int_0^1\frac{dy_i}{y_i}(1-y_i)\right)\int_0^{y_2\cdots y_n}t^{\nu-1}dt$$
$$=\left(\prod_{i=2}^{n}\int_0^1 dy_i(1-y_i)y_i^{\nu-1}\right)\frac{1}{\nu}$$
$$=\frac{1}{\nu}\left(\frac{1}{\nu(\nu+1)}\right)^{n-1},\quad n\geq 1. \tag{3.35}$$

The full pionization amplitude is given by

$$B(\lambda, s)=\sum_{n=1}^{\infty}\lambda^n B_n(s),$$

where $\lambda=g^2/4\pi\mu^4$. Thus we wish to consider

$$D(\lambda, t)\equiv\sum_{n=0}^{\infty}\lambda^n D_n(t) \tag{3.36}$$

or, equivalently, its Laplace transform

$$\tilde{D}(\lambda, \nu)\equiv\sum_{n=0}^{\infty}\lambda^n\tilde{D}_n(\nu)=\frac{\nu+1}{\nu(\nu+1)-\lambda}, \tag{3.37}$$

where the final equality follows from (3.35). To obtain $D(\lambda, t)$ from (3.37) we need the inverse of the transform (3.34). By changing variables in this equation to $y=-\ln t$, we see that (3.34) is equivalent to

$$\tilde{D}_{n-1}(\nu)=\int_0^{\infty}e^{-\nu y}dy\,D_{n-1}(e^{-y}),$$

and thus its inversion formula is

$$D_{n-1}(e^{-y})=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\nu y}d\nu\,\tilde{D}_{n-1}(\nu), \tag{3.38}$$

where the integration contour lies to the right of all the poles of $\tilde{D}_{n-1}(\nu)$ in the ν plane. Applying this inverse transform to (3.37) and keeping the contribution of the leading singularity in the λ plane only, we find

$$D\left(\lambda,\frac{1}{y_1 s}\right)=(y_1 s)^{[-1+(1+4\lambda)^{1/2}]/2}\frac{1+(1+4\lambda)^{1/2}}{2(1+4\lambda)^{1/2}}. \tag{3.39}$$

From this result the form of $B(\lambda, s)$ follows simply. Multiplying (3.33) by λ^n and summing over all gives

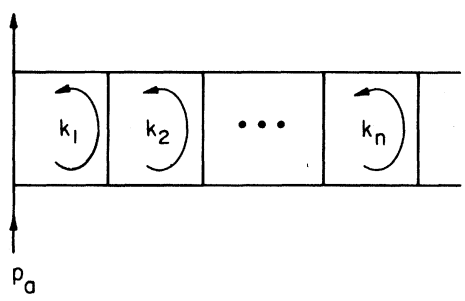

FIG. 4. The nth-order ladder corresponding to the fragmentation amplitude F_n.

$$B(\lambda, s)=\lambda\int_0^1\frac{dy_1}{y_1}D\left(\lambda,\frac{1}{y_1 s}\right)$$
$$=\lambda\int_0^1\frac{dy_1}{y_1}(y_1 s)^{[-1+(1+4\lambda)^{1/2}]/2}\frac{1+(1+4\lambda)^{1/2}}{2(1+4\lambda)^{1/2}}$$
$$=\frac{1}{4}\frac{[1+(1+4\lambda)^{1/2}]^2}{(1+4\lambda)^{1/2}}s^{[-1+(1+4\lambda)^{1/2}]/2}. \tag{3.40}$$

Comparing this result with (3.30), we observe that this direct calculation gives

$$\alpha(\lambda)=[-1+(1+4\lambda)^{1/2}]/2$$
$$=\lambda-\lambda^2+2\lambda^3-5\lambda^4+14\lambda^5+\cdots$$

and

$$\beta(\lambda)=\ln\left(\frac{[1+(1+4\lambda)^{1/2}]^2}{4(1+4\lambda)^{1/2}}\right)$$
$$=\lambda^2-4\lambda^3+\tfrac{29}{2}\lambda^4+\cdots, \tag{3.41}$$

which agree exactly with the results obtained from the cluster-decomposition technique. The proof to all orders in λ is relegated to the Appendix.

D. Nonleading Terms in $\lambda \ln s$ and the Cluster Decomposition

To underscore the significance of the cluster-decomposition approach, let us recall briefly the manner in which the standard perturbation-theory technique proceeds.

In this approach one begins by analyzing the leading asymptotic behavior of the nth-order ladder diagram. One would find in our model

$$B_n(\lambda, s)\propto\frac{(\lambda\ln s)^n}{n!}. \tag{3.42}$$

Thus

$$B(\lambda, s)\propto e^{\lambda s}. \tag{3.43}$$

This represents the leading term in $\lambda\ln s$; however, the nth-order graph also contains a term of the form

$$B_n(\lambda, s)\propto\lambda^n(\ln s)^{n-1}. \tag{3.44}$$

Although this term is nonleading in $\lambda\ln s$, its s dependence is as large as that of the leading term of the $(n-1)$-first-order ladder; thus one feels obliged to study terms of this type in order to understand the details of the asymptotic behavior. Polkinghorne[19] has given a general discussion of this problem, demonstrating that Regge behavior still arises when all orders are included and deriving an implicit equation for the trajectory function, $\alpha(\lambda)$. In this approach, however, the simple physical meaning of the nonleading terms remains obscured.

In contrast, in the cluster-decomposition approach, the emergence of Regge behavior even when all nonleading terms are considered is clear

from a simple general argument. Further, to calculate the kth coefficients in the power-series expansion

$$\alpha(\lambda)=\sum \lambda^n \alpha_n ,$$

we need only consider diagrams up to order k. Thus, in addition to providing a simpler intuitive picture, the cluster decomposition offers a more direct calculational method for determining the asymptotic behavior of a perturbation-theory amplitude.

IV. FRAGMENTATION

A. The Fragmentation Amplitude

In this section we shall compute the contribution to the total amplitude of ladder diagrams with loop momenta comparable to those of the incident particles; these diagrams correspond to fragmentation processes.

From the discussion of Sec. II, we know that the nth-order contribution to the fragmentation amplitude in the case of nonzero external mass is determined by

$$F_n(\eta)=\int_\eta^1 \frac{dx_1}{x_1}\int_\eta^{x_1}\frac{dx_2}{x_2}\cdots\int_\eta^{x_{n-1}}\frac{dx_n}{x_n} f_L^{(n)}(x_1, x_2, \ldots, x_n), \tag{4.1}$$

where $f_L^{(n)}(x_1, x_2, \ldots, x_n)$ is given by (2.19). The full fragmentation amplitude is then given by

$$F(\lambda, \eta)=\sum_n \lambda^n F_n(\eta), \quad \lambda = g^2/4\pi\mu^4 . \tag{4.2}$$

In analogy to the functions $b^{(n)}(x_1, x_2, \ldots, x_n)$ describing the pionization amplitude, the $f^{(n)}(x_1, x_2, \ldots, x_n)$ play a central role in exhibiting various distribution properties in the fragmentation region. In particular, the $f_L^{(n)}(x_1, x_2, \ldots, x_n)$ satisfy a crucial factorization property, similar to that exhibited by the $b^{(n)}(x_1, x_2, \ldots, x_n)$, whenever the x_i satisfy $(x_1, \ldots, x_m) \gg (x_{m+1}, \ldots, x_n)$. In this case in the denominators of (2.19) we can ignore the factors m_a^2/μ^2 and $1/(x_i - x_{i+1})$, for $i > m$, in comparison with the factors $1/(x_j - x_{j+1})$, for $j > m$, since the latter involve only small x's. A possible objection to this simplification is that in regions $x_i - x_{i+1} \lesssim x_{m+1}$ $(i < m)$, the factor $1/(x_i - x_{i+1})$ is comparable to, or perhaps even larger than, $1/x_m$. However, the contributions from these regions are rendered unimportant by the small phase-space factors and thus can be ignored in the limit of $x_m \gg x_{m+1}$. The reader is invited to work out a few simple examples to verify this point. With this simplification, for

$$(x_1, x_2, \ldots, x_m) \gg (x_{m+1}, x_{m+2}, \ldots, x_n),$$

we have

$$f_L^{(n)}(x_1, x_2, \ldots, x_n)=f_L^{(m)}(x_1, \ldots, x_m) b^{(n-m)}(x_{m+1}, \ldots, x_n)+O(x_{m+1}/x_m), \tag{4.3}$$

where $b^{(n)}(x_1, \ldots, x_n)$ is the corresponding integrand in the pionization region. Thus (4.3) represents a simple generalization of (3.3). Equation (4.3) can also be evaluated in the region where all x_i are small and satisfy $1 \gg x_i$ for all i. Then we have

$$f_L^{(n)}(x_1, x_2, \ldots, x_n)=b^{(n)}(x_1, \ldots, x_n)+O(x_i) . \tag{4.4}$$

In terms of the rapidity variables, $z_i = \ln x_i$, the inequality $x_m \gg x_{m+1}$ implies that $z_m - z_{m+1} \gg 1$. Equation (4.3) can be expressed in terms of z_i's, for $z_m - z_{m+1} \gg 1$, as

$$f_L^{(n)}(z_1, \ldots, z_n)=f_L^{(m)}(z_1, z_2, \ldots, z_m) b^{(n-m)}(z_{m+1}, \ldots, z_n)+O(e^{-|z_m - z_{m+1}|}) . \tag{4.5}$$

B. The Cluster Decomposition

The factorization property of $f_L^{(n)}(x_1, \ldots, x_n)$ suggests that a cluster decomposition similar to that found in the case of pionization should also hold in the fragmentation region. Indeed, we can introduce clusters in the fragmentation region in exactly the same manner as in the pionization region:

$$\begin{aligned} &g^{(1)}(x)=f_L^{(1)}(x), \\ &g^{(2)}(x_1, x_2)=f_L^{(2)}(x_1, x_2)-g^{(1)}(x_1) g^{(2)}(x_2), \\ &\cdots \\ &g^{(n)}(x_1, \ldots, x_n)=f^{(n)}(x_1, \ldots, x_n)-\sum_{\text{all partitions}} \left[\prod g^{(n_i)}(\cdots)\cdots g^{(n_j)}(\cdots)\right]. \end{aligned} \tag{4.6}$$

By the use of Eqs. (4.3), (4.4), and (4.6), it is straightforward to show that $g^{(n)}(x_1, \ldots, x_n)$ reduces to

$c^{(n)}(x_1, x_2, \ldots, x_n)$ if all x's are small and vanishes as $O(x_{m+1}/x_m)$ if $x_m \gg x_{m+1}$. These are well known properties of Mayers' clusters. As in the case of pionization, these results guarantee that the integral of $g^{(n)}(x_1, x_2, \ldots, x_n)$ over $\prod_{i=1}^{n} dx_i/x_i$ with the appropriate limits contains only one $\ln(1/\eta)$ factor:

$$G_n(\eta) \equiv \int_\eta^1 \frac{dx_1}{x_1} \int_\eta^{x_1} \frac{dx_2}{x_2} \cdots \int_\eta^{x_{n-1}} \frac{dx_n}{x_n} g^{(n)}(x_1, \ldots, x_n)$$

$$= \alpha_n \ln(1/\eta) + \beta_n' + O(\eta). \tag{4.7}$$

Notice the analogy between the variable η in (4.7) and the factor $1/s$ in the corresponding Eq. (3.9) for the pionization region. It is important to note that, since $g^{(n)}(x_1, x_2, \ldots, x_n) - c^{(n)}(x_1, x_2, \ldots, x_n)$ vanishes as any $x \to 0$, the dx/x integrations converge at the lower limit and thus do not contribute to a $\ln\eta$ term. Hence, clusters $g^{(n)}$ and $c^{(n)}$ give rise to the same α_n. This has the consequence that, by cluster-decomposition theorem, the total fragmentation amplitude

$$F(\eta, \lambda) \equiv \sum \lambda^n F_n(\eta) = e^{\beta'(\lambda)} (1/\eta)^{\alpha(\lambda)}, \tag{4.8}$$

with

$$\alpha(\lambda) = \sum \lambda^n \alpha_n, \quad \beta'(\lambda) = \sum \lambda^n \beta_n', \tag{4.9}$$

should have the same power dependence on $1/\eta$ [namely, $(1/\eta)^{\alpha(\lambda)}$] as the pionization amplitude has on s.

The match of power dependences in the fragmentation region and in the pionization region is important to guarantee a smooth transition between these two regions and to obtain an over-all amplitude that is independent of the cutoff η. To clarify the latter point, let us consider a ladder amplitude which covers both the fragmentation region and some of the pionization region. A typical diagram is Fig. 4, but with k_{n+} extended below to $(k_{n+})_{\min} = (\mu^2/s)p_+$. The quantity s is a measure of energy squared of the ladder and is assumed to be much larger than μ^2/η; that is, $\mu^2/s \ll \eta$. We can break up this ladder amplitude into two regions. The first region consists of a part of the ladder such that $k_{m+}/p_+ > \eta$, and the second region consists of the remaining part of the ladder with $\eta > k_+/p_+ > \mu^2/s$. These two regions can be identified obviously as corresponding to fragmentation and pionization, respectively. The power dependences of these two regions are, by the above arguments,

$$\begin{aligned} &(1/\eta)^{\alpha(\lambda)} \text{ for fragmentation} \\ &\text{and} \\ &(\eta s/\mu^2)^{\alpha(\lambda)} \text{ for pionization.} \end{aligned} \tag{4.10}$$

Therefore, since the over-all amplitude has a power dependence which is the product of these two factors, we obtain for its power dependence

$$(1/\eta)^{\alpha(\lambda)} (\eta s/\mu^2)^{\alpha(\lambda)} = (s/\mu^2)^{\alpha(\lambda)}. \tag{4.11}$$

The η dependence indeed cancels, as we anticipated.

C. The Nearest-Neighbor Approximation

To give an explicit example of the cluster decomposition for fragmentation we return to the nearest-neighbor approximation described in detail in Sec. III C. We again implement this approximation by retaining only the "dominant" term $1/(x_{m-1} - x_m) + 1/x_m$ in the factor

$$-\frac{m_a^2}{\mu^2} + \frac{1}{1-x_1} + \frac{1}{x_1 - x_2} + \cdots + \frac{1}{x_{m-1} - x_m} + \frac{1}{x_m}$$

that appears in the denominator of (2.19). Then the function $f_L^{(n)}(x_1, x_2, \ldots, x_n)$ reduces to

$$f^{(n)}_{\text{n.n.a.}}(x_1, x_2, \ldots, x_n) = \frac{(1-x_1)(1-x_2/x_1)\cdots(1-x_n/x_{n-1})}{[1-(m_a^2/\mu^2)x_1(1-x_1)]^2}. \tag{4.12}$$

A particularly simple form of $f^{(n)}_{\text{n.n.a.}}$ can be obtained in the limit of $m_a/\mu \ll 1$,

$$f_0^{(n)}(x_1, \ldots, x_n) \equiv f^{(n)}_{\text{n.n.a.}}(x_1, \ldots, x_n)|_{m=0} = (1-x_1)(1-x_2/x_1)\cdots(1-x_n/x_{n-1}). \tag{4.13}$$

Note that this simplified $f_0^{(n)}(x\text{'s})$ still satisfies the factorization property Eq. (4.3), with function $b^{(n)}(x\text{'s})$ being modified to the corresponding function in the nearest-neighbor approximation,

$$b^{(n)}_{\text{n.n.a.}}(x_1, \ldots, x_n) = (1-x_2/x_1)(1-x_3/x_2)\cdots(1-x_n/x_{n-1}). \tag{3.27}$$

Hence, we can make a cluster decomposition of the expression $f_0^{(n)}(x_1, \ldots, x_n)$ just as of $f_L^{(n)}(x_1, x_2, \ldots, x_n)$. Moreover, for this simple $f_0^{(n)}$, which corresponds to fragmentation in the limit of zero external mass, we can also compute

$$F^0(\eta, \lambda) \equiv \sum \lambda^n \int_\eta^1 \frac{dx_1}{x_1} \cdots \int_\eta^{x_{n-1}} \frac{dx_n}{x_n} f_0^{(n)}(x_1, \ldots, x_n) \tag{4.14}$$

exactly. This exact calculation will be used as a separate check on the validity of our cluster decomposition.

D. Fragmentation with Zero External Mass in the Nearest-Neighbor Approximation

In the following we shall compute the cluster-decomposition functions $g_0^{(n)}(x_1, x_2, \ldots, x_n)$ determined by the $f_0^{(n)}(x_1, x_2, \ldots, x_n)$. Since we have already presented, in Sec. III C and the Appendix, a verification to all orders of the equivalence of the cluster-decomposition approach and an exact method for the case of pionization, for brevity we shall concentrate solely on the first few cluster functions here. These are given by

$$\begin{aligned} &g_0^{(1)}(x) = 1 - x, \\ &g_0^{(2)}(x_1, x_2) = -\frac{x_2}{x_1}(1 - x_1)^2, \\ &g_0^{(3)}(x_1, x_2, x_3) = 2\frac{x_3}{x_1}(1 - x_1)^2(1 - x_2), \\ &\text{etc.} \end{aligned} \tag{4.15}$$

The cluster function $g^{(n)}(x_1, \ldots, x_n)$ indeed reduces to the cluster function $c^{(n)}(x\text{'s})$ characterizing the pionization region in the limit $1 \gg x_i$ for all i; in the first few cases this follows by direct comparison of (4.15) and (3.28). The integrations over the cluster functions give

$$\begin{aligned} G_1^0 &\equiv \int_\eta^1 \frac{dx}{x} g_0^{(1)}(x) = \ln\frac{1}{\eta} - 1 + O(\eta), \\ G_2^0 &\equiv \int_\eta^1 \frac{dx_1}{x_1} \int_\eta^{x_1} \frac{dx_2}{x_2} g^{(2)}(x_1, x_2) \\ &= -\ln\frac{1}{\eta} + \frac{5}{2} + O(\eta), \\ G_3^0 &\equiv \int_\eta^1 \frac{dx_1}{x_1} \int_\eta^{x_1} \frac{dx_2}{x_2} \int_\eta^{x_2} \frac{dx_3}{x_3} g^{(3)}(x_1, x_2, x_3) \\ &= 2\ln\frac{1}{\eta} - \frac{22}{3} + O(\eta), \\ &\text{etc.} \end{aligned} \tag{4.16}$$

These contributions are, as anticipated, linear in $\ln(1/\eta)$; further, the coefficients of $\ln(1/\eta)$ in G_n^0 agree with the corresponding coefficients of lns in the pionization region. By the results of Secs. II B and III C, the total fragmentation amplitude is thus

$$F^0(\eta, \lambda) = e^{G^0(\eta, \lambda)} = e^{\beta_0(\lambda)}(1/\eta)^{\alpha(\lambda)}, \tag{4.17}$$

with

$$\begin{aligned} &\alpha(\lambda) = \lambda - \lambda^2 + 2\lambda^3 + \cdots, \\ &\beta_0(\lambda) = -\lambda + \tfrac{5}{2}\lambda^2 - \tfrac{22}{3}\lambda^3 + \cdots, \end{aligned} \tag{4.18}$$

and

$$e^{\beta_0(\lambda)} = 1 - \lambda + 2\lambda^2 - 10\lambda^3 + \cdots. \tag{4.19}$$

To compare these results with the exact form of $F^0(\lambda)$, we note that, using (4.13) to give $f_0^{(n)}(x_1, x_2, \ldots, x_n)$ and changing variables to the y_i as in Sec. III C,

$$\begin{aligned} F_n^0(\eta) = &\int_0^1 \frac{dy_1}{y_1} \int_0^1 \frac{dy_2}{y_2} \cdots \int_0^1 \frac{dy_n}{y_n} \\ &\times \theta(y_1 \cdots y_n - \eta)(1 - y_1) \cdots (1 - y_n). \end{aligned} \tag{4.20}$$

Recalling the definition of the amplitude $D_{n-1}(1/y_1 s)$ as given by (3.33), we see immediately that

$$F_n^0(\eta) = D_n(\eta). \tag{4.21}$$

Hence we may apply directly the results of Sec. III C to obtain the full zero-mass fragmentation amplitude

$$F^0(\eta, \lambda) = \sum_{n=1}^{\infty} \lambda^n F_n^0(\eta). \tag{4.22}$$

We find

$$\begin{aligned} F^0(\eta, \lambda) &= D(\eta, \lambda) \\ &= \left(\frac{1}{\eta}\right)^{[-1+(1+4\lambda)^{1/2}]/2} \left(\frac{1 + (1 + 4\lambda)^{1/2}}{2(1 + 4\lambda)^{1/2}}\right), \end{aligned} \tag{4.23}$$

which confirms Eqs. (4.17)–(4.19).

E. Fragmentation with Nonzero External Mass

Giving the external particle a finite mass ($m_a^2 \neq 0$) does not change our conclusions, although it does complicate the algebra considerably. For example, we can compute the explicit forms of the nonzero external mass fragmentation functions, $\tilde{F}(\nu, \lambda)$ and $F(\eta, \lambda)$, from (4.14). Since the procedure is similar to the $m_a = 0$ case, we shall not go to the details of the calculation, but shall simply quote the final results:

$$\tilde{F}(\nu,\lambda)=\frac{\nu(\nu+1)^2}{\nu(\nu+1)-\lambda}\int_0^1\frac{dy(1-y)y^{\nu-1}}{[1-(m_a^2/\mu^2)y(1-y)]^2};\tag{4.24}$$

in addition, the asymptotic expression of $F(\eta,\lambda)$ at large $1/\eta$ is

$$F(\eta,\lambda)\simeq\frac{\lambda[\alpha(\lambda)+1]}{2\alpha(\lambda)+1}\int_0^1\frac{dy(1-y)y^{\alpha(\lambda)-1}}{[1-(m_a^2/\mu^2)y(1-y)]^2}\left(\frac{1}{\eta}\right)^{\alpha(\lambda)}$$
$$=e^{\beta'(\lambda)}(1/\eta)^{\alpha(\lambda)}.\tag{4.25}$$

The coefficients β_n' of the cluster decomposition of the η-independent term can be obtained by expanding Eq. (4.25) as a function of λ; i.e.,

$$\beta'(\lambda)=\sum\lambda^n\beta_n'.$$

We have computed a few β_n' by this method and compared them with those calculated from cluster decomposition; the two methods do indeed agree with each other. Interested readers are invited to verify this explicitly.

F. The Full Amplitude

We now know how to calculate the amplitudes associated with the pionization region and the left-fragmentation region. The amplitude associated with the right-fragmentation region can be worked out analogously. The full amplitude, according to Eq. (2.7), can be expressed as

$$A(\lambda,s)=-\frac{g^2}{\mu^2}\sum\lambda^n\int\Pi\left(\frac{dx_i}{x_i}\right)a^{(n)}(s).\tag{4.26}$$

Let us define the left-fragmentation, pionization, and right-fragmentation regions according to $k_{i+}>\eta\sqrt{s}$, $\eta\sqrt{s}>k_{i+}>\mu^2/(\eta'\sqrt{s})$, and $k_{i+}<\mu^2/(\eta'\sqrt{s})$, where η and η' are small positive, but s-independent, numbers. In the case that none of the k_{i+}'s is in the vicinity of the dividing points $\eta\sqrt{s}$ and $\mu^2/(\eta'\sqrt{s})$, the factorization property (2.14) follows and $A(\lambda,s)$ can be evaluated in these three regions separately, giving

$$A(\lambda,s)=-\frac{g^2}{s}F_L(\lambda,\eta)B(\lambda,\eta\eta's/\mu^2)F_R(\lambda,\eta').\tag{4.27}$$

$B(\lambda,\eta\eta's/\mu^2)$ and $F_L(\lambda,\eta)$ are defined in Eqs. (3.2) and (4.2), and have the power dependence $(\eta\eta's/\mu^2)^{\alpha(\lambda)}$ and $(1/\eta)^{\alpha(\lambda)}$, respectively. The right-fragmentation amplitude $F_R(\lambda,\eta')$ has a similar power dependence $(1/\eta')^{\alpha(\lambda)}$. Thus, the η and η' dependences in $A(\lambda,s)$ are cancelled out, giving

$$A(\lambda,s)\propto(s/\mu^2)^{\alpha(\lambda)-1},\tag{4.28}$$

as expected. In actual calculation, k_{i+} may take values in the vicinity of the dividing points. However, it is well known from the cluster-decomposition theory in statistical mechanics that contributions from the boundary regions do not affect the power dependence in $A(\lambda,s)$. Hence, to within an s-independent multiplicative factor, Eqs. (4.27) and (4.28) give the correct form of $A(\lambda,s)$.

V. THE ONE-PARTICLE SPECTRUM

In this section we wish to apply our model to production processes. The inelastic channels are the usual multiperipheral states (Fig. 5), and the inelastic amplitudes and cross sections can be computed in a straightforward way. The partial cross section of an n-particle multiperipheral production process, for example, can be obtained from the corresponding n-runged ladder amplitude by the Cutkosky prescription, just as in the case of a φ^3 theory in $(3+1)$ dimensions. Therefore, we expect that the s dependence and the concept of cluster decomposition will apply to the inelastic partial cross sections as well.

We have demonstrated in Secs. III and IV that the over-all amplitude of the ladder diagrams exponentiates to a Regge form, with the exponents given by the sum over all integrated cluster functions. Even though the nearby rungs are kinematically related, the contributions arising from the various clusters are completely decoupled. Thus one can compute the contribution arising from each type of cluster *independently* and obtain the full amplitude by taking the product of all individual terms. In the following, we shall show further that this decoupling of contributions due to different clusters allows us to compute the one-particle spectrum by evaluating each individual cluster's contribution independently and then adding the results.

Consider a particular cluster consisting of n particles (i.e., of n rungs). Let $\alpha_n(\lambda)\equiv\lambda^n\alpha_n$ be its corresponding partial contribution to the Regge exponent. The over-all contribution to the Regge pole $\alpha(\lambda)$ is simply the sum of the contributions due to all kinds of clusters; that is, $\alpha(\lambda)=\sum_n\alpha_n(\lambda)$. To within a multiplicative constant, the partial cross section due to N clusters of order n is just

$$\sigma_I\propto\frac{1}{s^2}\frac{1}{N!}[\alpha_n(\lambda)\ln s+\beta_n(\lambda)]^N.\tag{5.1}$$

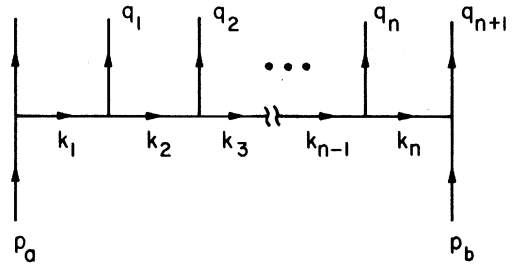

FIG. 5. The multiperipheral amplitude for the production process $2\to n+2$.

Recall that one of the $1/s$ factors comes from the optical theorem while the second comes from our definition of the amplitude as given by Eqs. (2.7) and (2.14). Equation (5.1) sums to $(1/s^2)e^{\beta_n(\lambda)}s^{\alpha_n(\lambda)}$ as expected. The one-cluster spectrum in the pionization region, in the case of N identical clusters, can be obtained by fixing one cluster and integrating the remaining N-1 clusters over the available phase space, giving

$$d\sigma \propto \frac{1}{s^2}\frac{1}{(N-1)!}[\alpha_n(\lambda)\ln s+\beta_n(\lambda)]^{N-1}\alpha_n(\lambda)\frac{dx_c}{x_c}, \tag{5.2}$$

where x_c is the fractional longitudinal momentum associated with the observed cluster. Notice the crucial result that the coefficient of dx_c/x_c in (5.2) is independent of x_c; this is a consequence of the translational invariance in rapidity space of the clusters corresponding to pionization. Since such translational invariance does not hold for the fragmentation clusters, we anticipate that the analog of (5.2) in the fragmentation region will be more complicated.

One can understand the form of (5.2) by noting that when this equation is integrated over x_c from $1/s$ to 1, it must yield (5.1), modulo a counting factor. Here again we can see the potential value of the analogy to a system of gas molecules; by the principles of statistical mechanics, we should be able to derive (5.2) simply by differentiating (5.1) with respect to the analog of volume. Since in our case this analog is $\ln s$, (5.2) follows immediately from (5.1) by this argument.

Summing (5.2) over N, we find that the one-cluster spectrum for a particular kind of cluster (in this case, nth-order) is

$$d\sigma \propto s^{\alpha_n(\lambda)-2}\alpha_n(\lambda)\frac{dx_c}{x_c} \tag{5.3}$$

or equivalently

$$\left(\frac{d\sigma}{\sigma_T}\right)_{\text{one-cluster}} = \alpha_n(\lambda)\frac{dx_c}{x_c}. \tag{5.4}$$

Since different kinds of clusters are independent, Eq. (5.4) will not be affected even if contributions due to different kinds of clusters are included.

Although the one-*cluster* distribution is very interesting theoretically, what one observes experimentally are *one-particle* or *multiparticle* spectra. It is not very difficult to compute these spectra from Eq. (5.4). For one-particle spectra, one need only remember that a cluster with n rungs will eventually decay into n final particles in our model. Thus, the one-particle spectrum in the pionization region is simply a weighted sum of (5.4) over all kinds of clusters,

$$\left(\frac{d\sigma}{\sigma_T}\right)_{\text{one particle}} = \sum n\alpha_n(\lambda)\frac{dx}{x} = h(\lambda)\frac{dx}{x}, \tag{5.5}$$

with

$$h(\lambda) = \sum n\alpha_n(\lambda) = \sum n\alpha_n\lambda^n = \lambda\frac{d\alpha(\lambda)}{d\lambda} \tag{5.6}$$

$$= \frac{\lambda}{(1+4\lambda)^{1/2}}. \tag{5.7}$$

Note that $h(\lambda)$ is both s- and x-independent. Again we remark that this is due to the translational invariance of the amplitude in the rapidity space. Equation (5.6) depends only on the fact that an nth-order cluster decays into n final particles. Thus we expect that this equation should be valid in both the exact (1+1)-dimensional and the exact (3+1)-dimensional multiperipheral theories as well; that is, it should be independent of the nearest-neighbor approximation and of the existence of transverse momenta. Of course, the explicit form (5.7) is model dependent.

It is essential to note, however, that Eq. (5.5) is expressed in terms of the longitudinal-momentum variable $x = k_+/p_{a+}$ of a cluster, rather than the proper-momentum variable q of the final-state particle (see Fig. 5). Later we shall comment on the importance of this distinction for general multiparticle spectra. In the present case, however, it is easy to see that $dx/x = dq_+/q_+$ because of the invariance of rapidity phase space under translations. Hence

$$\left(\frac{d\sigma}{\sigma_T}\right)_{\text{one particle}} = h(\lambda)\frac{dq_+}{q_+} = \frac{\lambda}{(1+4\lambda)^{1/2}}\frac{dq_+}{q_+}. \tag{5.8}$$

It is probably important to remark that, even though Eq. (5.8) has no explicit s dependence, the one-particle spectrum

$$d\sigma = \sigma_T(\lambda, s)h(\lambda)\frac{dq_+}{q_+}$$

$$\propto e^{\beta(\lambda)}s^{\alpha(\lambda)-2}h(\lambda)\frac{dq_+}{q_+}$$

does have an explicit power dependence $s^{\alpha(\lambda)-2}$. This s dependence is divided out in $d\sigma/\sigma_T$. In other words, the Feynman scaling law does not hold in this simple model except for a particular value of the coupling constant λ for which $\alpha(\lambda)-2 \equiv \alpha_c(\lambda)-1 = 0$. This is a typical result of multiperipheral models,[20] which do not predict scaling unless the amplitude is dominated by the Pomeranchuk exchange.

Let us now consider the one-cluster distribution function, $d\sigma/\sigma_T$, in the fragmentation region. We have already remarked that the lack of translational invariance exhibited by the fragmentation cluster, $g^{(n)}(x_1, \ldots, x_n)$, leads us to anticipate an x dependence in $d\sigma/\sigma_T$ in the fragmentation region more complicated than the simple dx/x phase-

space factor. This observation, when considered together with the result that a cluster function has a finite size in x space, implies that the one-cluster distribution function will depend on the definition of its "coordinate", x_c. For this reason, we define specifically the coordinate of a cluster $g^{(n)}(x_1, x_2, \ldots, x_n)$ to be $x_c = x_1$. This definition is natural because x_1 is the total scaled longitudinal momentum carried by the cluster.

It is easy to see that the contributions of individual clusters in the fragmentation region are still independent of each other. Hence, we can compute the one-cluster distribution function just as in the pionization region. For an nth-order cluster at x_c in the presence of $(N-1)$ identical clusters, this is

$$d\sigma \propto \frac{1}{s^2}\frac{1}{(N-1)!}[\alpha_n(\lambda)\ln(1/\eta)+\beta_n'(\lambda)]^{N-1}\left(\lambda^n\int_\eta^{x_c}\frac{dx_2}{x_2}\cdots\int_\eta^{x_{n-1}}\frac{dx_n}{x_n}g^{(n)}(x_c, x_2, \ldots, x_n)\right)\frac{dx_c}{x_c}. \tag{5.9}$$

The factor

$$\alpha_n(x_c, \lambda) \equiv \lambda^n\int_{\eta\approx 0}^{x_c}\frac{dx_2}{x_2}\cdots\int_{\eta\approx 0}^{x_{n-1}}\frac{dx_n}{x_n}g^{(n)}(x_c, x_2, \ldots, x_n) \tag{5.10}$$

is well defined at $\eta = 0$ and represents the one-cluster distribution function in the *absence* of other clusters. At small x_c ($1 \gg x_c \gg \eta \approx 0$), $g^{(n)}$ can be replaced by the translationally invariant function $c^{(n)}$; in this limit,

$$\begin{aligned}\alpha_n(x_c, \lambda) &\to \lambda^n\int_0^{x_c}\frac{dx_2}{x_2}\cdots\int_0^{x_{n-1}}\frac{dx_n}{x_n}c^{(n)}(x_c, x_2, \ldots, x_n)\\ &= \lambda^n\alpha_n \equiv \alpha_n(\lambda),\end{aligned} \tag{5.11}$$

as anticipated. Summing Eq. (5.9) over N and dividing by

$$\frac{1}{s^2}\exp[\alpha_n(\lambda)\ln(1/\eta)+\beta_n'(\lambda)],$$

we have the final one-cluster distribution

$$\frac{d\sigma}{\sigma_T} = \alpha_n(x_c, \lambda)\frac{dx_c}{x_c}, \tag{5.12}$$

which is a straightforward generalization of Eq. (5.2). Using the fragmentation cluster functions previously calculated in our model, we find that the first few $\alpha_n(x, \lambda)$ are

$$\begin{aligned}&\alpha_1(x, \lambda) = (1-x)\lambda,\\ &\alpha_2(x, \lambda) = -(1-x)^2\lambda^2,\\ &\alpha_3(x, \lambda) = (1-x)^2(2-x)\lambda^3,\\ &\text{etc.}\end{aligned} \tag{5.13}$$

These functions indeed reduce to $\alpha_n(\lambda)$ at small x.

To provide further support for the cluster approach and additional detail about the transition between the pionization and fragmentation regions, we shall now calculate the one-particle spectrum directly in the nearest-neighbor approximation. In the first instance we deal with the pionization region. We recall that in this case the full integrand is, in terms of the simplified notation $k_i \equiv k_{i+}$,

$$b^{(n)}(k_1, \ldots, k_n) = \left(1-\frac{k_2}{k_1}\right)\left(1-\frac{k_3}{k_2}\right)\cdots\left(1-\frac{k_n}{k_{n-1}}\right). \tag{5.14}$$

We are concerned with the process in which one of the rungs is detected with definite momentum $q = k_m - k_{m+1}$, where k_m and k_{m+1} are the momenta of the adjacent legs. The integration variables k_i are restricted according to

$$s > k_1 > k_2 > \cdots > k_{m-1} > k_m = q + k_{m+1} > k_{m+1} > \cdots > k_n > 1. \tag{5.15}$$

We shall integrate $b^{(n)}(k_1, k_2, \ldots, k_n)$ over the integration region available with fixed q and then sum over all n. We must also sum over m because any of the rungs in the ladder may be realized and detected as the final particle. The differential phase-space factor for fixed m and n can be written as

$$\prod_{i=1}^{n}\frac{dk_i}{k_i} = \left(\prod_{i=1}^{m-1}\frac{dk_i}{k_i}\right)\frac{dq}{k_{m+1}+q}\left(\prod_{j=1}^{n-m}\frac{dk_{m+j}}{k_{m+j}}\right). \tag{5.16}$$

We find that it is convenient to rescale the k_i for $i < m$ by

$$k_i' = \frac{k_i}{k_{m+1}+q} \tag{5.17}$$

such that the new variables obey

$$\frac{s}{k_{m+1}+q} > k_1' > k_2' > \cdots > k_{m-1}' > 1 . \tag{5.18}$$

In terms of these new variables, we have

$$\begin{aligned} dB^{(n)}(s,q) &= \int \prod_i \frac{dk_i}{k_i} b^{(n)}(k_1, k_2, \ldots, k_n)\Big|_{\text{fixed } q = k_m - k_{m+1}} \\ &= \int_1^{s-q} \frac{dk_{m+1}}{k_{m+1}} \frac{dq}{k_{m+1}+q}\left(1 - \frac{k_{m+1}}{k_{m+1}+q}\right) B_1^{(m-1)}\left(\frac{s}{k_{m+1}+q}\right) B_2^{(n-m-1)}(k_{m+1}), \end{aligned} \tag{5.19}$$

where

$$B_1^{(m-1)}\left(\frac{s}{k_{m+1}+q}\right) \int_1^{s/(k_{m+1}+q)} \frac{dk_1'}{k_1'} \cdots \int_1^{k_{m-2}'} \frac{dk_{m-1}'}{k_{m-1}'}\left(1 - \frac{k_2'}{k_1'}\right)\left(1 - \frac{k_3'}{k_2'}\right)\cdots\left(1 - \frac{1}{k_{m-1}'}\right) \tag{5.20}$$

and

$$B_2^{(n-m-1)}(k_{m+1}) = \int_1^{k_{m+1}} \frac{dk_{m+2}}{k_{m+2}} \cdots \int_1^{k_{n-1}} \frac{dk_n}{k_n}\left(1 - \frac{k_{m+2}}{k_{m+1}}\right)\left(1 - \frac{k_{m+3}}{k_{m+2}}\right)\cdots\left(1 - \frac{k_n}{k_{n-1}}\right). \tag{5.21}$$

Multiplying (5.19) by λ^n and summing over m, n, we have

$$\begin{aligned} dB(s,q) &= d[\textstyle\sum \lambda^n B^{(n)}(s)]_{\text{fixed } q} \\ &= \lambda^2 \int_1^{s-q} \frac{dk}{k} \frac{dq}{k+q}\left(1 - \frac{k}{k+q}\right) B_1\left(\frac{s}{k+q}, \lambda\right) B_2(k, \lambda), \end{aligned} \tag{5.22}$$

where

$$B_{1,2} = \textstyle\sum \lambda^n B_{1,2}^{(n)} . \tag{5.23}$$

Using the methods introduced in Sec. III to solve for $B(s,\lambda)$, $B_{1,2}(s)$ can be worked out straightforwardly; we find

$$B_1(s,\lambda) = B_2(s,\lambda) \underset{\text{large } s}{\approx} \frac{(1+4\lambda)^{1/2}+1}{2(1+4\lambda)^{1/2}} s^{\alpha(\lambda)} . \tag{5.24}$$

That B_1 and B_2 are the same function can be seen easily through the transformation $k_i \to k_i' = s/k_i$.

Given $B_{1,2}$, we can compute $dB(s,q)$ from (5.22). For $s \gg q_+ \gg 1$ – that is, for particle momentum $q(=q_+)$ much smaller than the incident particle momenta, but much larger than the momentum of the last particle in the ladder – we can extend the integration limits to $0 < k_{m+1} < \infty$. Then, we have

$$\begin{aligned} dB(s,q) &= \frac{\lambda^2[(1+4\lambda)^{1/2}+1]^2}{4(1+4\lambda)} \int_0^\infty \frac{dk}{k} \frac{dq}{k+q}\left(1 - \frac{k}{k+q}\right)\left(\frac{s}{k+q}\right)^{\alpha(\lambda)} k^{\alpha(\lambda)} \\ &= \frac{\lambda^2[(1+4\lambda)^{1/2}+1]^2}{4(1+4\lambda)} s^{\alpha(\lambda)} \frac{dq}{q} \int_0^\infty du \frac{u^{\alpha-1}}{(1+u)^{\alpha+2}} \\ &= \frac{\lambda[(1+4\lambda)^{1/2}+1]^2}{4(1+4\lambda)} s^{\alpha(\lambda)} \frac{dq}{q}, \end{aligned} \tag{5.25}$$

where $u = k/q$, and use has been made of

$$\alpha(\lambda) = \tfrac{1}{2}[(1+4\lambda)^{1/2} - 1] .$$

With the help of (3.40), we have

$$\left(\frac{d\sigma}{\sigma_T}\right)_{\text{one particle}} = \frac{dB(s,q)}{B(s,q)} = \frac{\lambda}{(1+4\lambda)^{1/2}} \frac{dq}{q}, \tag{5.26}$$

which is precisely Eq. (5.8).

Let us now examine both the structure of $(d\sigma/\sigma_T)_{\text{one particle}}$ in the fragmentation region and the manner in

which the distribution of soft fragments joins smoothly to that of pionization. Since we are interested only in the general features of the theory, we reduce the problem of fragmentation to the simplest case by putting $m=0$ and again make use of the nearest-neighbor approximation. As we shall see, many of the observed features of high-energy scattering can still be understood through this simplified model.

By analogy to Eq. (5.19), the one-particle spectrum in the fragmentation region is given by

$$dF(s,q)=\lambda^2\int_1^{s-q}\frac{dk}{k}\frac{dq}{k+q}\left(1-\frac{k}{k+q}\right)F_1\left(\frac{s}{k+q},\lambda\right)B_2(k,\lambda), \tag{5.27}$$

with

$$F_1(s_1,\lambda)\equiv\sum\lambda^n F_1^{(n)}(s_1) \tag{5.28}$$

and

$$F_1^{(n)}(s_1)\equiv\int_1^{s_1}\frac{dk_1}{k_1}\cdots\int_1^{k_{n-1}}\frac{dk_n}{k_n}\left(1-\frac{k_1}{s_1}\right)\left(1-\frac{k_2}{k_1}\right)\cdots\left(1-\frac{k_n}{k_{n-1}}\right)\left(1-\frac{1}{k_n}\right). \tag{5.29}$$

Since fragments can carry momenta comparable to that of incident particles, we cannot make the approximation that $s_1=s/(k+q)\gg 1$. Thus, we have to compute (5.29) exactly. It turns out that this is not very difficult in our model. Making a transformation of variables similar to that made in the case of pionization,

$$y_1=\frac{k_1}{s_1},\quad y_2=\frac{k_2}{k_1},\quad \ldots,\quad y_i=\frac{k_i}{k_{i-1}},\quad \ldots$$

and performing a Mellin transform on the variable $t=1/s_1$, we obtain

$$\int_0^1 dt\,F_1(t,\lambda)t^{\nu-1}=\frac{1}{\nu(\nu+1)-\lambda}=\frac{1}{(1+4\lambda)^{1/2}}\left(\frac{1}{\nu-\alpha}-\frac{1}{\nu+\alpha+1}\right). \tag{5.30}$$

$F_1(s_1,\lambda)$ can then be obtained by an inverse Mellin transform, giving ($s_1=1/t$)

$$F_1(s_1,\lambda)=\frac{1}{(1+4\lambda)^{1/2}}\left(s_1^{\alpha(\lambda)}-s_1^{-1-\alpha(\lambda)}\right). \tag{5.31}$$

Substituting (5.31) into (5.27), we have for $q\gg 1$,

$$dF(s,q)_{\text{one particle}}=\lambda^2 q\,dq\int_1^{s-q}\frac{dk}{k(k+q)^2}\frac{1}{(1+4\lambda)^{1/2}}\left[\left(\frac{s}{k+q}\right)^{\alpha}-\left(\frac{s}{k+q}\right)^{-1-\alpha}\right]\frac{(1+4\lambda)^{1/2}+1}{2(1+4\lambda)^{1/2}}k^{\alpha} \tag{5.32}$$

$$=\frac{\lambda^2[(1+4\lambda)^{1/2}+1]}{2(1+4\lambda)}\frac{dq}{q}s^{\alpha}\int_0^{s/q-1}du\frac{u^{\alpha-1}}{(u+1)^2}\left[\left(\frac{1}{u+1}\right)^{\alpha}-\frac{(u+1)^{\alpha+1}}{(s/q)^{1+2\alpha}}\right], \tag{5.33}$$

with $u=k/q$ as before. Dividing (5.32) by $F(s)$, and expressing the final results in terms of the fractional longitudinal momentum $x=q/s$, we have

$$\left(\frac{dF(s,x)}{F(s)}\right)_{\text{one particle}}=\frac{\lambda}{(1+4\lambda)}M(x)\frac{dx}{x}, \tag{5.34}$$

with

$$M(x)\equiv\lambda\int_0^{1/x-1}du\,u^{\alpha-1}\left(\frac{1}{(u+1)^{\alpha+2}}-(u+1)^{\alpha-1}x^{1+2\alpha}\right). \tag{5.35}$$

We shall call $M(x)$ the "scaled fragmentation distribution function." The difference between $M(x)$ and one, $M(x)-1$, is a measure of the deviation of the fragmentation particle distribution from the dx/x law which holds for pionization. It is easy to see that $M(x)\approx 1$ for $x\ll 1$; thus the "soft" fragments indeed have the same distribution law, $dF/F=[\lambda/(1+4\lambda)^{1/2}](dx/x)$, as do pionization particles. For $x\approx 1$, on the other hand, $M(x)$ vanishes as a power of $(x-1)$, $M(x)\simeq(1+4\lambda)^{1/2}(1-x)^{\alpha+1}$. Hence, the high-energy end of the one-particle spectrum vanishes as the available energy vanishes; this is a consequence of simple kinematic requirements. For the intermediate region, $M(x)$ can be expressed in terms of incomplete β functions in the form

$$M(x)=1-I_x(2,\alpha)-\lambda x^{1+2\alpha}B_{1-x}(\alpha,1-2\alpha), \tag{5.36}$$

with

$$I_x(a,b) \equiv B_x(a,b)/B(a,b)$$

and (5.37)

$$B_x(a,b) = \int_0^x dt\, t^{a-1}(1-t)^{b-1}.$$

Some typical $M(x)$'s are plotted in Fig. 6 for a few values of α (or λ). In terms of the gas model of Feynman and Wilson, $M(x)$ represents the boundary effect. These curves agree qualitatively with the observed one-particle spectra in the fragmentation region, such as analyzed by Bali *et al.*[21] Given $M(x)$, the scaled fragmentation distribution function, we can compute the distribution of the subenergies between any two adjacent particles in p_+ space. The average spacings between adjacent particles remain constant for particles in the pionization region and have different values for the first few fragmentation particles. The average scaled longitudinal momentum of the first few fragments as a function of n is plotted in Fig. 7. These curves agree qualitatively with the Echo Lake experiment[22] carried out by the University of Michigan group.

The idea of clusters can be used to understand qualitatively the multiparticle spectra as well. Let us discuss this application briefly, concentrating on the pionization region. The two-particle spectrum with observed momenta q_1, q_2 can be expressed as

$$\frac{d^2\sigma}{\sigma_T} = h(q_1, q_2)\frac{dq_{1+}}{q_{1+}}\frac{dq_{2+}}{q_{2+}} \tag{5.38}$$

$$= h(z_1, z_2)dz_1 dz_2, \tag{5.39}$$

with $z = \ln q_+$. In the pionization region, $h(q_1, q_2)$ will be s independent. For $q_{1+} \gg q_{2+}$, $h(q_1, q_2) = h(q_1)h(q_2)$. This is the important factorization property mentioned earlier. Physically, it implies the independent emissions of particles when $q_{1+} \gg q_{2+}$. This property indicates that final particles have only a limited range of correlation in the rapidity space.

In the general case of an n-particle spectrum, the reducible (uncorrelated) part of a distribution function $h^{(n)}(q_1, q_2, \ldots, q_n)$ is defined as the sum over products of the distribution functions for all possible uncorrelated emission of n particles. Hence this part of $h^{(n)}$ is determined by $h^{(m)}$, $m < n$, and is not of primary interest in studies of the n-particle spectrum. To explore the dynamics of any model, one should study the irreducible (correlated) parts of a multiparticle distribution. This irreducible part of a general distribution function, $h(q_1, q_2, \ldots, q_n)$, is analogous to the cluster function introduced earlier. In particular, the irreducible part of a two-particle distribution function is defined as

$$h_c(q_1, q_2) \equiv h(q_1, q_2) - h(q_1)h(q_2). \tag{5.40}$$

In analogy to the cluster function, $c^{(2)}(k_1, k_2)$, $h_c(q_1, q_2)$ has only a finite correlation length.

We would like to remind the readers at this point that our discussion of the multiparticle spectra is based on analysis of ladder diagrams, which represent a very specific type of multiperipheral model. It is not clear that even a more general multiperipheral-type model will lead to the correct

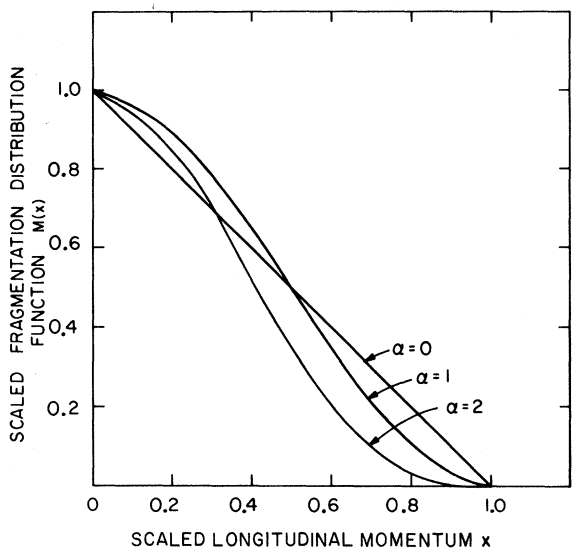

FIG. 6. $M(x)$ versus x, the scaled fragmentation distribution function versus the scaled longitudinal momentum.

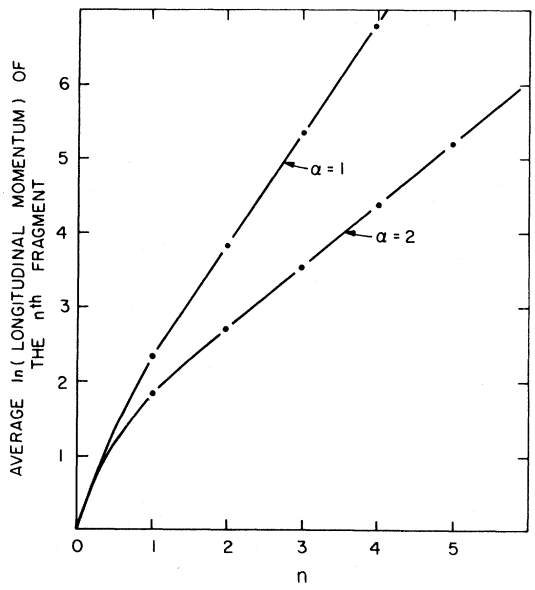

FIG. 7. The logarithm of the average longitudinal momentum of the nth fragment as a function of n.

high-energy s dependence and multiparticle spectra. This point will be discussed further in Sec. VI. For the present we note that even if we accept the multiperipheral model as an accurate description of strong interactions at high energy, the usefulness of the multiparticle distribution function, $h^{(n)}(q_1, q_2, \ldots, q_n)$, is still limited. This is because the scattering amplitude of a multiperipheral model is simple in terms of the momentum variables, k_i, carried by the sides of the generalized ladder. These variables are related to the final-state momenta q_i through $q_i = k_i - k_{i-1}$ (see Fig. 5) and $k_i = \sum_{j>i} q_j - p_b$. Although we can determine q_i from k_i unambiguously, the inverse is not true. We need to know the sequence of the final-state particles in order to determine k_i. All the nice features of our model, such as the ordering of k_{i+}, the cluster-decomposition property of the amplitude, and the relation between the cluster function $c^{(n)}$ and the trajectory function $\alpha(\lambda)$, rely heavily on the use of the k_i. Hence, if one takes the multiperipheral model seriously and wishes to extract the dynamics phenomenologically, one should first convert the multiparticle distribution function $h^{(n)}(q_1, \ldots, q_n)$ into the absorptive part of the ladder amplitude $b^{(n)}(k_1, \ldots, k_n)$, through a change of variables from q_i to k_i. Following the prescriptions given in Sec. III, one can then compute the cluster functions $c^{(n)}(k_1, \ldots, k_n)$, the trajectory function $\alpha(\lambda)$, and various correlations among the final particles from the amplitude $b^{(n)}(k_1, \ldots, k_n)$. Obviously, one of the most uncertain parts of such a phenomenological analysis is to determine the sequence of final particles in order to express the k's in terms of the q's. The interesting possibility of carrying out the cluster decomposition *directly* in terms of the q's is presently under investigation.

VI. DISCUSSION

The method developed in this paper to extract the asymptotic behavior of ladder diagrams is actually quite general and is applicable to other types of diagrams as well. In particular, we have studied the "iterated-cross" diagrams[23] [Fig. 8(a)] in a $(1+1)$-dimensional φ^3 field theory. In the pionization region, we are able to show that the longitudinal momenta, $x_i \equiv |k_{i+}|$, passing through the lines which join the crosses are ordered just as in the ladder diagram,

$$x_1 > x_2 > \cdots > x_n > 0 .$$

Further, in this region whenever one set of x's, say $(x_1, x_2, \ldots, x_m)$, is much larger than the remaining set, the integrand – call it $\rho^{(n)}(x_1, x_2, \ldots, x_n)$ – corresponding to the amplitude factors into the product of two similar functions, $\rho^{(m)}(x_1, x_2, \ldots, x_m) \times \rho^{(n-m)}(x_{m+1}, x_{m+2}, \ldots, x_n)$. This is the property

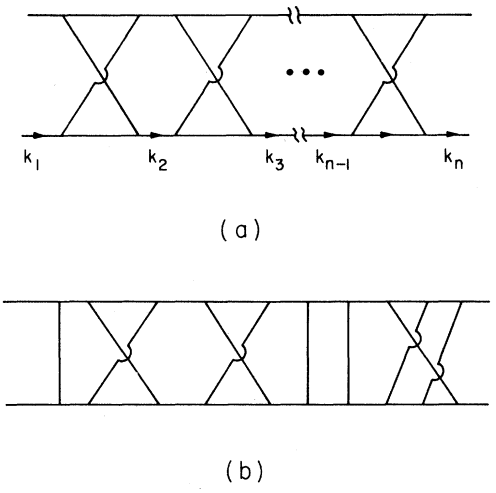

FIG. 8. (a) The nth-order iterated cross diagram. (b) A more general multiperipheral diagram.

which ensures the validity of the cluster decomposition. It is interesting to see in this case that the simplest cluster constructed from an iterated cross diagram is a single cross. A higher-order cluster is a combination of crosses. For a more complicated diagram such as that of Fig. 8(b), we can decompose the amplitude into contributions due to various clusters. The clusters, listed according to increasing powers of the coupling constant, are shown in Fig. 9. The resultant power dependence on s in the amplitude is determined by the sum over all possible contributions due to various clusters. It is important to see that, at least in the $(1+1)$-dimensional case, the inclusion of more complicated rungs and crosses in a ladder only modifies the exponent $\alpha(\lambda)$ of the s dependence in the amplitude, but does not alter its Regge form. This may explain why some of the nice features of the naive multiperipheral model

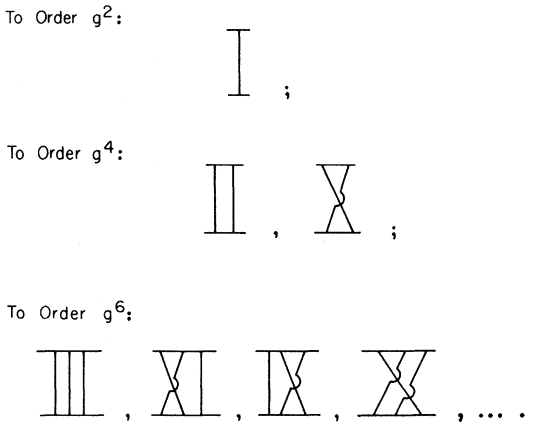

FIG. 9. The first few clusters corresponding to the diagram of Fig. 8(b).

persist even in a more realistic theory.

The technique studied here can also be generalized straightforwardly to a $(3+1)$-dimensional calculation. As pointed out in the Introduction, this paper is actually stimulated by an earlier calculation on nonleading terms in the $(3+1)$-ladder amplitude.[6] Let us describe briefly how the concept of a cluster decomposition can be applied to calculation of ladder diagrams in a $(3+1)$-dimensional field theory. For simplicity, we work only in the pionization region. A typical pionization diagram is shown in Fig. 10. The external momenta are labeled by k_1, k_1+k, and k_{n+1}, $k_{n+1}+k$, respectively, and chosen to satisfy $k_{1-} \equiv k_1^0 - k_1^3 = 0$, $k_{(n+1)+} \equiv k_{n+1}^0 + k_{n+1}^3 = 0$, and $k_+ = k_- = 0$. Thus, the momentum transfer $k_\mu \equiv (k_+, \vec{k}, k_-) = (0, \vec{k}, 0)$ is purely transverse. The momenta of the first and the last (k_+) integrals are restricted by $k_{1+} \leq (p_+)_{\max}$ and $k_{n+} \geq (p_+)_{\min}$. As in the $(1+1)$-dimensional case, $r = (p_+)_{\max}/(p_+)_{\min}$ is a measure of the invariant energy squared of the system. As we demonstrated earlier, it is more convenient to express the power dependence of the amplitude in the pionization region in terms of the variable r than in terms of the actual invariant energy. Of course, the amplitude in the pionization region has the same power dependence on r as the exact ladder amplitude does on the Mandelstam variable s. Thus if we are interested only in the power dependence [i.e., the trajectory function $\alpha(\lambda, k^2)$] or in the particle distribution properties *away* from the fragmentation region, we need only carry out our analysis in the pionization region.

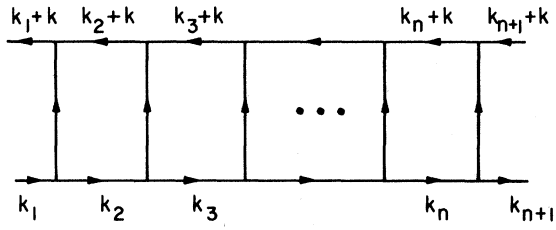

FIG. 10. The nth-order ladder diagram corresponding to pionization in a $(3+1)$-dimensional theory.

As in the $(1+1)$-dimensional case, we find that the n-rung amplitude $T^{(n)}(k_1, k_2, \ldots, k_n)$ in the pionization region has the following properties:

(1) After k_{i-} integrations, $T^{(n)}(k_1, k_2, \ldots, k_n)$ can be expressed as

$$T^{(n)}(k_1, k_2, \ldots, k_n) = -i\left(\frac{g^2}{4\pi}\right)^n \int \prod_i \frac{dx_i}{x_i} \frac{d^2\vec{k}_i}{(2\pi)^2} b_{\vec{k}}^{(n)}(k_1, k_2, \ldots, k_n). \tag{6.1}$$

The plus components of the k_i's are ordered just as in the $(1+1)$-dimensional case:

$$(p_+)_{\max} > x_1 > x_2 > \cdots > (p_+)_{\min}, \tag{6.2}$$

where $x_i = k_{i+}$.

(2) $b_{\vec{k}}^{(n)}(k_1, k_2, \ldots, k_n)$ satisfies the crucial factorization property that, when

$$(x_1, x_2, \ldots, x_m) \gg (x_{m+1}, x_{m+2}, \ldots, x_n), \tag{6.3}$$

we have

$$b_{\vec{k}}^{(n)}(k_1, k_2, \ldots, k_n) = b_{\vec{k}}^{(m)}(k_1, k_2, \ldots, k_n) b_{\vec{k}}^{(n-m)}(k_{m+1}, \ldots, k_n) + O(x_{m+1}/x_m). \tag{6.4}$$

That Eq. (6.4) is true can be verified explicitly. In deriving (6.4), we have made use of the uniform convergence of the transverse integrations. It is important to note that the factorization of $b^{(n)}$ into two b's is complete; that is, $b^{(n)}$ factors not only as a function of x's but also as a function of the transverse variables $\vec{k}_i$. This complete factorization is crucial for the asymptotic amplitude of the ladder diagram to behave as a Regge pole, rather than as a more complicated Regge cut. To understand this point, let us return to Eqs. (6.1) and (6.4). We can define the cluster function $c_{\vec{k}}^{(n)}(k_1, k_2, \ldots, k_n)$ according to Eq. (3.6). Note that $c^{(n)}$ is independent of the momentum of other clusters. Thus, the integrated contribution

$$C_n(\vec{k}, s) = \int_{(p+)_{\min}}^{(p+)_{\max}} \prod \left(\frac{dk_{i+}}{k_{i+}} \frac{d^2\vec{k}_i}{(2\pi)^2}\right) c_{\vec{k}}^{(n)}(k_1, \ldots, k_n)$$

$$\equiv \alpha_n(\vec{k}) \ln r + \beta_n(\vec{k}) \tag{6.5}$$

can be computed independently for each cluster. Rotational invariance implies that α_n and β_n are functions of $t = -\vec{k}^2$ only. Therefore, the resultant amplitude is

$$T = \sum T^{(n)} = \beta(t) r^{\alpha(t)}, \tag{6.6}$$

with

$$r=(p_+)_{\max}/(p_+)_{\min}\simeq s\,, \tag{6.7}$$

$$\beta(t)=-ie^{\sum\lambda^n\beta_n(t)}\,,$$

$$\alpha(t)=\sum\lambda^n\alpha_n(t)\,, \tag{6.8}$$

and

$$\lambda=g^2/4\pi\mu^2\,. \tag{6.9}$$

Equations (6.6)–(6.8) have the structure of a Regge pole.

In the case of more complicated diagrams, such as iterated cross diagrams [Fig. 8(a)] in (3+1) dimensions, the factorization is not complete. Instead of obtaining an equation such as (6.4), we find that: (1) the integrand $\rho^{(n)}$ in an iterated cross diagram depends on the transverse momentum of the $(n+1)$st line, so that $\rho^{(n)}=\rho^{(n)}(k_1,k_2,\ldots,k_n;\vec{k}_{n+1})$; and (2) Eq. (6.4) is modified, for $(x_1,x_2,\ldots,x_m)\gg(x_{m+1},\ldots,x_n)$, to the form

$$\rho^{(n)}(k_1,\ldots,k_n,\vec{k}_{n+1})=\rho^{(m)}(k_1,\ldots,k_m,\vec{k}_{m+1})\rho^{(n-m)}(k_{m+1},\ldots,k_n,\vec{k}_{n+1})+O(x_{m+1}/x_m)\,. \tag{6.10}$$

In other words, the integrand $\rho^{(m)}$ depends on the transverse momentum $(\vec{k}_{m+1})$ of the adjacent integrand $\rho^{(n-m)}$. We can still decompose the $\rho^{(n)}$ into clusters. However, the n-particle cluster function, $\tau^{(n)}(k_1,k_2,\ldots,k_n;\vec{k}_{n+1})$ will depend on the first transverse momentum of the cluster right after it. Thus, the clusters remain correlated through their transverse momenta. The contributions of various clusters must in this case be "multiplied" together as matrices, with the matrix elements being of the form $\tau^{(n)}_{\vec{k}_1,\vec{k}_{n+1}}$, where the $\vec{k}_i$ are continuous variables. After summing over contributions due to all clusters, we find that the full amplitude exponentiates in the form $\exp[\alpha_{\vec{k}\vec{k}'}\ln s+\beta_{\vec{k}\vec{k}'}]$ but only as a matrix expression. Diagonalizing the exponent and keeping in mind that the spectra of $\vec{k}_i$ are continuous, we are led to a complicated singularity structure of the type explored earlier by Contogouris, and by Kaschluhn and Zoellner.[24] Thus, we establish a simple criterion for determining whether a set of amplitudes should lead to a Regge pole (series of poles), to a cut, or to an essential singularity; this depends simply on whether the sum over the cluster functions has a discrete or a continuous spectrum. This, in turn, is closely related to whether the integrand $b^{(n)}(\rho^{(n)})$ is completely factorizable or not. This physical criterion and its interpretation are quite general and should be useful for both theoretical and phenomenological analysis.

In conclusion, let us discuss two points. First, we should reiterate an important caveat mentioned in Sec. V regarding the use of a "real-gas" analogy to understand certain aspects of strong interactions. We have been able to demonstrate that, within the framework of a multiperipheral model, this analogy has both conceptual and calculational utility. In particular, we have established the validity of a "cluster decomposition" at high energy; this is a direct analogue of the cluster expansions used in treatments of real gases. The variables in which this cluster is simplest, however, are not the actual final-state particle momenta, q_i; rather they are the momenta k_i carried by the sides of the ladder diagrams which comprise our model. In applying the present cluster technique to phenomenological analyses, one must transform from the q_i to the k_i; as discussed in Sec. V, the possible ambiguities in this transformation will impart uncertainties to the final results.

Second, we should discuss the relevance of using a simple multiperipheral model to describe the high-energy s dependence and the multiparticle spectra of an exact theory. At present the exact asymptotic behavior of even a simple φ^3 theory remains unknown. Thus the validity of considering any collection of perturbation-theory diagrams is questionable. One does feel, however, that the larger the class of general principles – for instance, unitarity and/or crossing in given channels – satisfied by given sets of diagrams the more likely that predictions based on them are accurate. On this basis we might argue that some feeling for the correctness (or otherwise) of the multiperipheral model could be obtained by considering a generalization of the model, involving a larger class of diagrams, that satisfies s-u crossing and s-channel unitarity. Those features of the simple multiperipheral model which remain valid in the more general case would then be considered most plausible.

Recently, an attempt was made in this direction by studying the high-energy elastic and inelastic processes in a φ^3 theory calculation based on the s-channel iteration of t-channel ladder diagrams.[25]

This model can be generalized to include the s-channel iteration of an arbitrary multiperipheral process and will be referred to as a unitarized multiperipheral model. One of the important conclusions is that the unitarized theory has the same multiparticle spectra and the same high-energy s dependence, $s^{\alpha_c(t)}$, as the original multiperipheral model, when the Regge intercept $\alpha_c(0)$ obeys $\alpha_c(0) \leq 1$. However, these two models have completely different asymptotic behavior for $\alpha_c(0) > 1$. In the latter case, the original multiperipheral model violates the Froissart bound, whereas the unitarized theory becomes a strong-absorption model and its total cross section saturates the Froissart bound. Interested readers are referred to Ref. 25 for details. The agreement between the unitarized theory and the simple model for $\alpha_c(0) \leq 1$ provides some support for our considerations.

APPENDIX

In Sec. III C we calculated explicitly the first few $c^{(n)}(x_1, \ldots, x_n)$ describing the pionization amplitude in the nearest-neighbor approximation. From these we also calculated C_n, the integrated cluster functions, which determine directly the coefficients in the power series for $\alpha(\lambda)$. In this Appendix we will present a method for determining the C_n and thereby deduce the full power series for $\alpha(\lambda)$.

The first five $c^{(n)}$ are given in Eq. (3.28). As further examples, we list here $c^{(6)}$ and $c^{(7)}$:

$$c^{(6)}(x_1, x_2, x_3, x_4, x_5, x_6) = -\left(16\frac{x_6}{x_1} + 32\frac{x_5x_6}{x_1x_2} + 16\frac{x_5x_6}{x_1x_3} + 16\frac{x_4x_6}{x_1x_2} + 8\frac{x_5x_6}{x_1x_4} + 8\frac{x_4x_6}{x_1x_3} + 8\frac{x_3x_6}{x_1x_2} + 12\frac{x_4x_5x_6}{x_1x_2x_3} + 4\frac{x_3x_5x_6}{x_1x_2x_4}\right) \tag{A1}$$

and

$$\begin{aligned} c^{(7)}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = {} & 32\frac{x_7}{x_1} + 128\frac{x_6x_7}{x_1x_2} + 64\left(\frac{x_5x_7}{x_1x_2} + \frac{x_6x_7}{x_1x_3}\right) + 32\left(\frac{x_4x_7}{x_1x_2} + \frac{x_5x_7}{x_1x_3} + \frac{x_6x_7}{x_1x_4}\right) \\ & + 16\left(\frac{x_6x_7}{x_1x_5} + \frac{x_5x_7}{x_1x_4} + \frac{x_4x_7}{x_1x_3} + \frac{x_3x_7}{x_1x_2}\right) + 72\frac{x_5x_6x_7}{x_1x_2x_3} + 48\left(\frac{x_4x_6x_7}{x_1x_2x_3} + \frac{x_5x_6x_7}{x_1x_2x_4}\right) \\ & + 24\left(\frac{x_4x_5x_7}{x_1x_2x_3} + \frac{x_5x_6x_7}{x_1x_3x_4}\right) + 16\left(\frac{x_4x_6x_7}{x_1x_2x_5} + \frac{x_3x_6x_7}{x_1x_2x_4}\right) + 8\left(\frac{x_3x_5x_7}{x_1x_2x_4} + \frac{x_3x_6x_7}{x_1x_2x_5} + \frac{x_4x_6x_7}{x_1x_3x_5}\right). \end{aligned} \tag{A2}$$

These examples illustrate several properties common to all $c^{(n)}$.

(1) The general $c^{(n)}$ contains only terms consisting of products of simple ratios of its arguments, $x_i/x_j \cdots x_k/x_l$. The maximum number of factors in a product is $\mathrm{Int}(n/2)$, where $\mathrm{Int}(n/2)$ is the largest integer less than or equal to $n/2$; if we call terms involving a product of m ratios (x_i/x_j) "m-factor terms," then $c^{(n)}$ contains m-factor terms of all orders from $m = 1$ to $\mathrm{Int}(n/2)$.

(2) Of all possible products of ratios only those which satisfy the following conditions appear:

(a) a factor of x_n/x_1 must always appear;

(b) it must be possible to put the other factors into ratios x_i/x_j, where $i > j$;

(c) no index can ever appear twice in a given product of ratios.

(3) There is an over-all factor of $(-1)^{n-1}$.

(4) We consider the ratios (x_ix_j/x_kx_1) and (x_ix_j/x_1x_k) to be equivalent.

In terms of the $c^{(n)}(x_1, \ldots, x_n)$, C_n is given by[26]

$$\begin{aligned} C_n(s) &= \int_{1/s}^{1} \frac{dx_1}{x_1} \int_{1/s}^{x_1} \frac{dx_2}{x_2} \cdots \int_{1/s}^{x_{n-1}} \frac{dx_n}{x_n} c^{(n)}(x_1, \ldots, x_n) \\ &\equiv \sum_{m=1}^{\mathrm{Int}(n/2)} C_n^m(s) = \sum_{n=1}^{\mathrm{Int}(n/2)} (\alpha_n^m \ln s + \beta_n^m), \end{aligned} \tag{A3}$$

where C_n^m indicates the total contribution to C_n from those terms in $c^{(n)}(x_1, \ldots, x_n)$ containing exactly m factors (x_k/x_l). We now make the crucial observation that, for a given n, each allowed m-factor term contributes the same amount to the coefficients of $\ln s$ in C_n; for example, if we examine C_5 we see that there are three two-factor terms:

$$8\frac{x_4x_5}{x_1x_2}, \quad 4\frac{x_3x_5}{x_1x_2}, \quad \text{and} \quad 4\frac{x_4x_5}{x_1x_3}.$$

By direct application of (3.9), we find that each of these terms contributes $2 \ln s$ to $C_5(s)$. In general, we can observe that the contribution to α_n, the coefficient of $\ln s$ in $C_n(s)$, of each term in $c^{(n)}(x_1, \ldots, x_n)$ with

m factors is exactly $2^{(n-2m)}$. Thus the problem reduces to determining how many terms with m factors satisfy the restrictions (1)–(4) listed above.

To solve this problem we begin by considering the simple combinatoric effects. In the case of C_n there are n x_i's from which to choose; selecting $2m$ of them to form an m-factor expression, subject to the proviso that both x_n and x_1 be selected, can be done in

$$\binom{n-2}{2(m-1)} = \frac{(n-2)!}{(n-2m)![2(m-1)]!}$$

ways.

Having thus selected the $2m$ x_i's, we eliminate x_1 and x_n, order the remaining $2m-2$ x_i's, and then relabel these sequentially from 1 to $2m-2$; thus

$$x_1 < x_2 < \cdots < x_{2m-3} < x_{2m-2} . \tag{A4}$$

The problem is now reduced to finding the number of ways these $2m-2$ numbers can be arranged in $m-1$ ratios in accordance with the above requirements; this number is independent of n, the order of C_n. Let us call this number A_{m-1}; then the total number of m-factor terms is given by

$$\binom{n-2}{2(m-1)} A_{m-1} .$$

We can calculate A_{m-1} by observing the equivalence of our problem to a particular one-dimensional random walk. In the general case, we find

$$A_m = \frac{1\times 3\times 5\times\cdots\times(2m-1)2^m}{(m+1)!} . \tag{A5}$$

The result can be expressed more compactly in terms of a generating function

$$A(x) = \sum A_m x^m = \frac{1}{2x}[1-(1-2x)^{1/2}] . \tag{A6}$$

From this result, we find that

$$\alpha_n = (-1)^{n-1} \sum_{m=1}^{\mathrm{Int}(n/2)} \binom{n-2}{2(m-1)} A_{m-1} 2^{n-2m} , \quad n \geq 2 \tag{A7}$$

and

$$\alpha_1 = 1 .$$

The trajectory function is

$$\begin{aligned} \alpha(\lambda) &= \sum_{n=1}^{\infty} \alpha_n \lambda^n , \\ &= \lambda + \sum_{n=2}^{\infty} \sum_{n=1}^{\mathrm{Int}(n/2)} (-1)^{n-1} \binom{n-2}{2(m-1)} A_{m-1} 2^{n-2m} \lambda^n . \end{aligned} \tag{A8}$$

Interchanging the order of summation and using (A6), we have

$$\alpha(\lambda) = [-1+(1+4\lambda)^{1/2}]/2 \tag{A9}$$

which agrees with the exact result given by (3.30) and (3.40).

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[8]The diagrams with twists on the left ends can be shown to be equivalent to those in Fig. 1(b).

[9]S.-J. Chang and S. Ma, Phys. Rev. 180, 1506 (1969); 188, 2385 (1969).

[10]S. Weinberg, Phys. Rev. 150, 1313 (1966).

[11]It develops that the amplitude corresponding to Fig. 2(b) is just $A_3(-s)$.

[12]$A(s)$ is the invariant T-matrix element in the convention of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[13]See Ref. 5, Chap. 14; J. D. Mayer and M. G. Mayer, *Statistical Mechanics* (Wiley, New York, 1940), Chap. 13 and Appendix.

[14]In fact we shall see that the appropriate variable corresponding to the separation of the molecules in our gas analogy is the rapidity variable $z_i = \ln x_i$; this is a detail which will become clear later.

[15]See also the discussion in S.-J. Chang and P. M. Fishbane, Phys. Rev. D 2, 1104 (1970), Secs. IV C and VII.

[16]Hence the restriction to finite Lorentz transforms, that is, to $\lambda \leq O(\ln s)$.

[17]We ignore constant terms in the higher $C^{(n)}$ as we are primarily interested in establishing in detail the validity of the cluster decomposition for calculating the s dependence of the amplitude. Further, all terms of order $1/s$ or smaller are ignored.

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Internal Symmetry Propagation in the Strong-Interaction S Matrix*

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We consider a strong-interaction S matrix in which the $\pi\pi$ system has the lowest threshold, and the $K\bar{K}$ system the next threshold (i.e., only two-body states are allowed up to and slightly beyond the $K\bar{K}$ threshold). We show that if the $\pi\pi$ scattering amplitudes below the $K\bar{K}$ threshold are isospin-invariant, then unitarity, crossing symmetry, and sufficient analyticity demand isotopic spin as the least trivial symmetry of the $K\bar{K} \to K\bar{K}$, $KK \to KK$, and $K\pi \to K\pi$ amplitudes. In every case, *all* of the isospin relations are obtained. It is also pointed out that an assumption that various partial waves are correlated can be used to replace the assumption that the S matrix can be diagonalized by a constant matrix.

I. INTRODUCTION

A few years ago Blankenbecler *et al.*[1] formulated an approach to internal symmetries based on the unitarity, crossing symmetry, and analyticity of the strong-interaction S matrix. Their basic assumption (or definition) was that there exists a *constant* unitary matrix which will diagonalize the S matrix. They first studied the $\pi\pi$ system for isospin symmetry. This was extended to $SU(3)$ for mesons,[1] to broken-symmetry mass sum rules,[2] and to the octet of baryons.[3] For the $\pi\pi$ system BCR proved that isospin symmetry was the least trivial symmetry (i.e., that symmetry consistently requiring the least number of relations among scattering amplitudes). They then considered the coupled $K\pi$ system, assuming that the $K\bar{K}$ channel was the first inelastic one to open up in $\pi\pi$ scattering and that Bose statistics, charge-conjugation invariance, and time-reversal invariance were valid. They also assumed that the S matrix could be diagonalized by constant orthogonal matrices in all channels. They obtained all but two of the usual isospin relations for this coupled system.