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Low-Energy Zeros and Regge Poles from an Extended Amati-Bertocchi-Fubini-Stanghellini-Tonin Multiperipheral Model*

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By inserting the elastic absorptive part from a multiperipheral model into a fixed-momentum-transfer dispersion relation, it is possible to obtain the low-energy amplitude in the crossed channel. We use the ABFST (Amati-Bertocchi-Fubini-Stanghellini-Tonin) version of this model so that the amplitude satisfies a BS (Bethe-Salpeter) equation. The Veneziano model for the $\pi\pi$ amplitude is used to calculate the ABFST input kernel, for which off-shell effects are neglected. We go beyond the usual pion-exchange dominance of the ABFST model by including an "inelasticity" factor in our BS equation. The strength of this factor is fixed by requiring that the Chew-Mandelstam $\pi\pi$ symmetry-point crossing condition is satisfied. It is then found that zeros arise in the S-wave amplitudes, which can presumably be identified with the zeros which are predicted from the Adler self-consistency condition. As a by-product, we also calculate the leading Regge trajectories in our model. These have no difficulty in rising to fairly high angular momenta and lead to relatively narrow reduced widths for the ρ and f^0 resonances, as well as a high-energy total cross section in approximate agreement with experiment.

I. INTRODUCTION

Many successful predictions have recently been made in strong interactions by using the Veneziano model.[1] This model, moreover, is capable of giving zeros in the $\pi\pi$ amplitude[2] consistent with partial conservation of axial-vector current (PCAC).[3,4] Because of the linear nature of the model, however, there are still many undetermined parameters in this approach, particularly if satellite terms are admitted. Also certain features, such as the existence of infinitely-rising Regge trajectories, are not explained dynamically, but have to be assumed from the beginning. For these reasons, it has been emphasized by Chew[5] that a complementary dynamical approach incorporating the nonlinear constraint of unitarity might be desirable. Of course the more the resulting amplitude resembles the Veneziano amplitude, the more likely we are to be on the right track. In particular it would be interesting to see if such a model can give rise to zeros in the amplitude, as predicted by PCAC,[6] as well as Regge trajectories which continue to rise at least for moderately high energies.[7] Up to now unitary models in which the Regge trajectories are not put in by hand at the beginning of the calculation but are generated dynamically have been seen to reproduce either one of these properties, but not both.[7] We shall see that an extended multiperipheral model is able to do so, however.

To see qualitatively how zeros might arise dynamically, consider the usual picture of the Regge trajectories believed to be present in the $\pi\pi$ amplitude. (See, for example, Fig. 1, which gives the trajectories assumed in the Veneziano model.) The absence of an $I=2$ trajectory suggests that the input potential or kernel in a dynamical scheme is weak or repulsive in this state. Thus any zero in the amplitude is expected to arise from a zero in the input potential. On the other hand, the presence of one or more high-lying trajectories implies a strong attractive potential for $I=0$. Unless this potential is very complicated, the most natural way of obtaining a zero is by means of the sort of mechanism used to explain the Ramsauer-Townsend effect in atomic physics.[8] If our zero were above threshold this would correspond to the phase shift passing through 180°, which can certainly be generated by a strong attractive potential. It is just as easy to produce a zero below threshold. We shall see that our dynamical model can generate just this sort of zero.

Let us now turn to the problem of indefinitely rising Regge trajectories. These can arise dynamically if we have some kind of potential which grows parametrically with energy. This was perhaps first noticed by Finkelstein[9] who found that his trajectories showed no tendency to turn around in an equivalent potential calculation. It has since been shown that such trajectories can go to infinity with energy-dependent potentials.[10,11] Our dynamical model will also have this type of input. In this respect it differs from a model such as that of Collins and Johnson[7] which does not have this feature.

The most popular realistic dynamical approach in the last few years has been based on the multi-

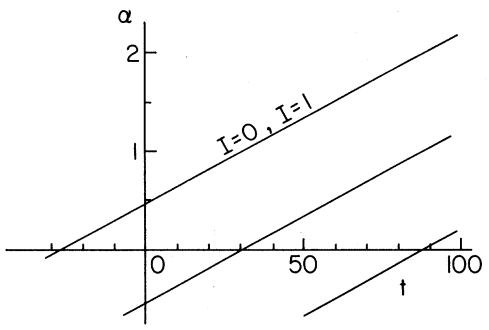

FIG. 1. Regge trajectories assumed in the Veneziano model. There is no $I = 2$ trajectory, and the $I = 0$ and $I = 1$ trajectories are degenerate. (We are using pion-mass units.)

peripheral model.[12-17] This is obtained by inserting a multiperipheral description of production processes into a multiparticle unitary relation. We then have an expression for the absorptive part of the elastic amplitude which is essentially a sum of ladder-type graphs. There is a corresponding integral equation, which enables us to sum these graphs. Most of the applications of this approach have been to high-energy phenomena. On the other hand, our absorptive part contains an essentially correct description of low-energy resonances as well as a Regge behavior at high energies. If we therefore insert it into a fixed-t dispersion relation, we should have a reliable expression for the low-energy amplitude in the crossed channel. This is the region we shall be mainly interested in.

The model we shall actually use is the one originally proposed by Amati, Bertocchi, Fubini, Stanghellini, and Tonin (ABFST).[12] This has recently come to be favored[13-16] over the multi-Regge model,[17] mainly because the important subenergies in the physical production amplitude are of the order of 1 GeV or less,[18,19] whereas the multi-Regge model, unless we rely very heavily on the duality concept, can be expected to be valid only if they are very large ($>>1$ GeV). In Sec. II we shall exploit the equivalence of the ABFST model to a Bethe-Salpeter (BS) equation with a particular form of the input potential.[12,13] This equivalence was first pointed out by Amati *et al.*[12] and has recently been used extensively by Ball and Marchesini.[13] We also go beyond the usual pion-exchange dominance of the ABFST model by introducing an "inelasticity" factor into our BS equation. This is a rough way of including other effects, which seem to be required if we are ever to obtain sufficiently high trajectories in the model.[15]

The fact that the ABFST model furnishes us with an expression for the input potential as well as with a dynamical equation is one of its advantages over other dynamical schemes. It does not, however, tell us anything about the off-shell behavior of the potential, which is needed to solve the equation. In practice, although other extrapolations have also been tried,[13,15] the usual assumption made is that the dependence of the input absorptive-part kernel on the mass of the virtual pions is negligible.[12,15] We shall make the same assumption in Sec. III. It can be justified to some extent by invoking a "criterion of maximal convergence,"[11] according to which we are to pick that off-shell dependence which leads to the most convergent equations[20] without at the same time introducing extra singularities into the problem (see Appendix A). Finally a specific form of the potential is constructed from the Veneziano model.

In Sec. IV our BS equation is projected into partial waves in the t channel and solved by using a Pagels-type approximation.[21] (The detailed formulas are given in Appendix B.) It is then found that the $I=2$ potential, and hence the amplitude, can be made to vanish at the point required by PCAC[6] by a choice of the Veneziano-trajectory parameters consistent with experiment. This potential and amplitude continue to be small in the immediate neighborhood of this point. Now the Chew-Mandelstam symmetry-point[22] condition relates the $I=2$ and $I=0$ amplitudes at a point which lies within this neighborhood. We can therefore choose our "inelasticity" parameter in the $I=0$ state so that the Chew-Mandelstam condition is satisfied exactly, thereby forcing the $I=0$ amplitude to be small there also. Since the $I=0$ potential is fairly large and leads to a relatively rapidly varying amplitude, the latter also has to vanish in the vicinity of this point. The resulting $I=0$ scattering length is bigger than the one obtained on the basis of the usual simple extrapolation from the Adler point, however.[4]

Finally, in Sec. V we calculate the leading $I=0$ and $I=1$ Regge trajectories and their corresponding residue functions in the t channel. We assume that the $I=1$ "inelasticity" factor is the same as the $I=0$ factor calculated in Sec. IV. The $I=0$ trajectory is found to lie higher than the one for $I=1$ and to have an intercept close to unity. It can thus be identified with the Pomeranchuk trajectory and leads to an s-channel high-energy total cross section in approximate agreement with experiment. Both trajectories are somewhat steeper than the experimental ones, but lead to fairly narrow reduced widths for the resonances lying on them. This should be contrasted with the usual situation in a traditional bootstrap calculation, where the reduced widths are invariably too broad.[23]

Although the multiperipheral approach is supposed to be valid only for small t, it is nevertheless interesting to see what happens to the output

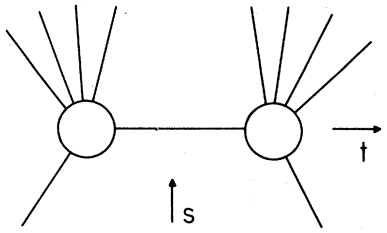

FIG. 2. One-particle-exchange model for a production amplitude.

Regge trajectories for higher t. These are found to have no difficulty in rising at least up to $J=2$. Unfortunately, the approximations we make cannot be used for very high energies, and so we cannot draw any conclusions about whether the trajectories rise indefinitely. If, however, we make the determinantal approximation we are led to an equation of exactly the type considered in Appendix A of Ref. 11. This was shown to lead to an infinitely-rising Regge trajectory.

II. THE EXTENDED ABFST MODEL

The ABFST model[12] assumes that the amplitude M_n for the production of n pions (Fig. 2) can be described by the multiperipheral chain of Fig. 3, where all the particles are pions. If this is then inserted into a multiparticle unitarity relation

$$T_s=\sum_n \int M_n^* M_n, \tag{2.1}$$

we have an expression for the absorptive part of the elastic-scattering amplitude T, which can be calculated from a fixed-t dispersion relation

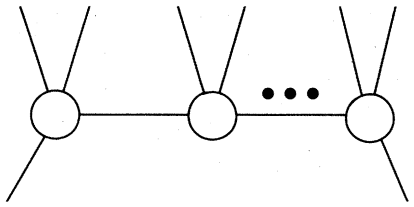

FIG. 3. ABFST multiperipheral model for a production amplitude M_n when all the external lines are pions.

$$T(s,t)=\frac{1}{\pi}\int ds'\frac{T_s(s',t)}{s'-s}, \tag{2.2}$$

where we have neglected the u channel for the time being. This T is thus given by Fig. 4 or Fig. 5, which has the form of a Bethe-Salpeter equation, a fact which was first noted by Amati *et al.*[12] and has recently been developed further by Ball and Marchesini.[13] The input potential V (or kernel) of this equation is the first term on the right-hand side of Fig. 5 and comes from the $n=2$ contribution of Eq. (2.1) to Eq. (2.2). Thus,

$$V(s,t)=\frac{1}{\pi}\int ds'\frac{V_s(s',t)}{s'-s} \tag{2.3}$$

with

$$V_s(s,t)=\frac{q}{2\pi\sqrt{s}}\int d\Omega\, T^*(s,t_1)T(s,t_2), \tag{2.4}$$

where q is the magnitude of the three-momentum and t_1 and t_2 are the squares of the momentum transfers between the intermediate and final and the initial and intermediate states. For a given isospin state I in the c.m. system in the t channel, Fig. 5 gives us the equation

$$T(p',p,t)=V(p',p,t)+\frac{i}{\pi^3}\int d^4p''V(p',p'',t)G(p'',t)T(p'',p,t)R, \tag{2.5}$$

where $P+p$ and $P-p$ are the four-momenta of the initial, and $P+p'$ and $P-p'$ the four-momenta of the final particles in the t channel, while P is the four-vector $(\frac{1}{2}\sqrt{t},0,0,0)$, using a timelike metric. Our amplitude is normalized so that, on the shell, $T\to A=\frac{1}{2}\sqrt{t}\,f$, where f is the physical amplitude which gives a differential cross section $d\sigma/d\Omega=|f|^2$. The Green's function G is given by

$$G^{-1}(p'',t)=[(P+p'')^2-1][(P-p'')^2-1], \tag{2.6}$$

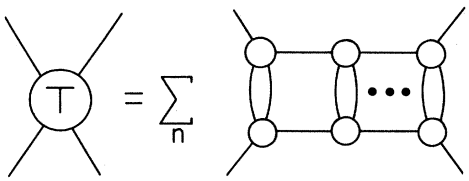

FIG. 4. A graphical illustration of the amplitude calculated from Eqs. (2.1) and (2.2), where M_n is given by Fig. 3.

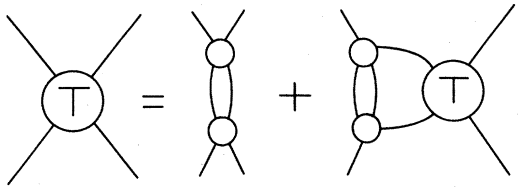

FIG. 5. Bethe-Salpeter equation which gives the amplitude of Fig. 4.

while the potential V can be obtained from Eq. (2.3) with $s=(p-p')^2$. We are taking the pion mass = 1.

The "inelasticity" factor R in Eq. (2.5) does not, of course, represent anything in Fig. 5, which corresponds to $R=1$. We have introduced it to roughly take into account diagrams not included in Fig. 4. These are discussed in more detail in Sec. VI; the most obvious set would arise from the exchange of other particles, such as the ω, in Fig. 3. Our factor R is thus very similar to the coherent inelastic factor λ used in strong-cut models of high-energy scattering.[24]

We will not attempt to calculate R from any particular model, but will simply take a simple parametric form for it. For simplicity we will assume that it depends only on p''^2, so $R=R(p''^2)$. If elastic unitarity is to hold for $t<t_I$, we must then have $R=1$ for $-p''^2<(\frac{1}{4}t_I-1)$. If, moreover, we make a partial-wave projection of Eq. (2.5) and make a Logunov-Tavkhelidze-Blankenbecler-Sugar approximation[25] for it, we find that we can make the identification

$$R(1-\tfrac{1}{4}t)\approx\sigma_l^{\rm tot}(t)/\sigma_l^{\rm el}(t), \tag{2.7}$$

where $\sigma_l^{\rm tot}$ and $\sigma_l^{\rm el}$ are the total and elastic partial-wave cross sections. We are thus assuming that this ratio is approximately independent of l, which is the sort of approximation one makes in a grey-disk model, for example. In such a model one also takes this ratio to be independent of t, which leads to the usual diffraction scattering at high t. This is also approximately true for Pomeranchuk-Regge exchange at sufficiently high t (to within factors of $\ln t$), for which, moreover, the ratio is independent of I.

A simple expression which incorporates all the above properties is[11]

$$R(p''^2)=1+(R_0-1)\theta(-p''^2-\tfrac{1}{4}t_I+1), \tag{2.8}$$

where θ is the usual step function and the constant R_0 is independent of I; it will eventually be determined from the Chew-Mandelstam crossing condition.[22] For t_I we will take the point at which we expect inelastic effects to begin to become important. Now in the s channel, we have seen that the multiperipheral model begins to give inelastic effects at the threshold for the $n=4$ contribution to Eq. (2.1). But the first important contribution to the blobs on the right side of Fig. 4 comes from the ρ resonance. Thus the effective threshold for the $n=4$ contribution is the $\rho\rho$ threshold. This argument does not depend on pion-pole dominance, but continues to be valid if we also exchange, say, the ω in Figs. 3 and 4. By crossing, inelastic effects should therefore also set in at the $\rho\rho$ threshold in the t channel. We will thus take $t_I=4m_\rho{}^2\simeq 120$.

III. CONSTRUCTION OF THE INPUT POTENTIAL

To solve Eq. (2.5) we require a knowledge of the potential V. We could, of course, use Eqs. (2.3) and (2.4), but this requires a knowledge of the off-shell amplitude T. We shall therefore make the ABFST assumption[12,15] that

$$V_s(s,t)\simeq\tilde{v}(s,t), \tag{3.1}$$

where $\tilde{v}(s,t)$ is the on-shell value of $V_s(s,t)$ and can thus be determined from the on-shell version of Eq. (2.4),

$$\tilde{v}(s,t)=\frac{q}{2\pi\sqrt{s}}\int d\Omega\tilde{A}^*(s,t_1)\tilde{A}(s,t_2), \tag{3.2}$$

where $A(s,t)$ is the on-shell $\pi\pi$ amplitude, and the tilde means that we are considering a definite isospin state in the s channel. From Eq. (2.3) our input potential is then[26]

$$V(p',p,t)=W((p'-p)^2,t)+(-1)^I W((p'+p)^2,t) \tag{3.3}$$

with

$$W(s,t)=\frac{1}{\pi}\int ds'\frac{v(s',t)}{s'-s}, \tag{3.4}$$

where v is related to $\tilde{v}$ by crossing and where the last term of Eq. (3.3) is the contribution of the u channel, which was left out in Eq. (2.3). This off-shell extrapolation is the one which is favored if we introduce a "criterion of maximal convergence," as discussed in Appendix A.

To obtain an explicit expression for the potential, we shall use the Veneziano model for $\tilde{A}(s,t)$, which is at least approximately consistent with what can be inferred about $\pi\pi$ scattering from production experiments.[27] We do so mainly for convenience, however, since, as we shall see, only its lowest resonances play any important role in our calculation. Our procedure will be very similar to the one followed by Tow,[15] except for the fact that we will take into account the inelasticity of the g meson; it also resembles the one followed by Chew, Rogers, and Snider.[14] Now we cannot simply insert the Veneziano model directly into Eq. (3.2) since this would lead to nonintegrable double poles, at least if we assume exactly linear trajectories. Instead, we expand $\tilde{v}(s,t)$ in partial waves in the s channel. Then Eq. (3.2), when combined with the unitarity relation for A, gives

$$\tilde{v}_l(s)=r_l(s)\,\mathrm{Im}\tilde{A}_l(s), \tag{3.5}$$

where

$$r_l(s)=\sigma_l^{\rm el}(s)/\sigma_l^{\rm tot}(s), \tag{3.6}$$

which can be expected to be approximately equal to the inverse of the factor R introduced in Sec. II.

As in Sec. II, we shall assume that r_l is approximately independent of l so that

$$r_l(s) \approx r(s), \tag{3.7}$$

at least for the important partial waves. This approximation does not have to be a particularly good one if we assume that there is at most only one partial-wave resonance at a given value of s.[27] If we now sum Eq. (3.5) and apply crossing symmetry we have

$$v(s,t) = r(s) A_s(s,t). \tag{3.8}$$

To evaluate $A_s(s,t)$ in Eq. (3.8) or $\mathrm{Im}\tilde{A}_l(s)$ in Eq. (3.5), we use the Lovelace-Veneziano model. Remembering that I refers to an isospin in the t channel, this gives[2]

$$A^{I=0} = \tfrac{3}{2}[F(t,s) + F(t,u)] - \tfrac{1}{2}F(s,u), \tag{3.9}$$

$$A^{I=1} = F(t,s) - F(t,u), \tag{3.10}$$

$$A^{I=2} = F(s,u), \tag{3.11}$$

where

$$F(x,y) = -\bar{\beta}\,\frac{\Gamma(1-\alpha(x))\,\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} \tag{3.12}$$

and $\alpha(x) = \alpha_0 + \alpha' x$. Equation (3.13) depends on three parameters. We can determine two of them by requiring that the ρ mass and width be the same as that given by experiment, which gives $\bar{\beta} = 0.6$ and $\alpha(30) = 1$. The third parameter will be considered later. If we now take the absorptive part in the s variable, we always obtain an expression of the form

$$A_s(s,t) = \sum_{n=1}^{\infty} \gamma_n(t)\,\delta(s - s_n), \tag{3.13}$$

where the $\gamma_n(t)$ are polynomials in t and the s_n are the positions of the resonances, given by $\alpha(s_n) = n$. If we then insert Eq. (3.13) into Eq. (3.8), we have

$$v(s,t) = \sum_{n=1}^{\infty} r(s_n)\gamma_n(t)\,\delta(s - s_n). \tag{3.14}$$

To determine the $r(s_n)$ we turn to experiment. Thus, it is known that the ρ and f^0 are purely elastic resonances, and so Eqs. (3.6) and (3.7) give $r(s_1) = r(s_2) = 1$. At $s = s_3$, i.e., at the position of the g meson, we can obtain the total g width Γ_g^{tot} from experiment, and the partial width $\Gamma_g^{\pi\pi}$ for decay into $\pi\pi$ from the Veneziano model. This gives us

$$r(s_3) = \Gamma_g^{\pi\pi}/\Gamma_g^{\mathrm{tot}} \simeq \tfrac{1}{3}.$$

For higher resonances we expect $r(s)$ to be even smaller. In the high-energy region we can get some idea of what r should be from σ_{tot} and σ_{el}. These can be obtained from the corresponding pp and πp cross sections by factorization, which gives $\sigma_{\mathrm{tot}} = 16$ mb and $\sigma_{\mathrm{el}} = 2.5$–$3.0$ mb at least up to about 50 GeV^2. If we again make the assumption (3.7) that all important partial waves have approximately the same $r_l(s)$, we obtain from Eqs. (3.6) and (3.7) the result $r(s) = \sigma_{\mathrm{el}}/\sigma_{\mathrm{tot}} \approx \frac{1}{5} - \frac{1}{6}$. [Actually, Eq. (3.8) is exact at $t = 0$ with this expression for $r(s)$.] Since this is a relatively small value, we shall simply take $r = 0$ for $s > s_3$. Our input thus contains only a finite number of resonances. Previous calculations with the ABFST model suggest that this is a reasonable first approximation.[14] (See also Appendix C.)

If we insert Eq. (3.14) into (3.4), we have

$$W(s,t) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{r(s_n)\gamma_n(t)}{s_n - s}. \tag{3.15}$$

Now $W(s,t)$ will be needed only for $s < 0$ when we solve the BS equation, and in the $I = 0$ and $I = 1$ states all the terms in the summation of (3.15) are positive there, at least for the values of t we will be considering. For these states we shall thus make the further approximation of replacing W by a single effective pole

$$W(s,t) \simeq \frac{c(t)}{s_R - s}, \tag{3.16}$$

where c is the sum of the residues in Eq. (3.15),

$$c(t) = \frac{1}{\pi}\sum_{n=1}^{\infty} r(s_n)\gamma_n(t), \tag{3.17}$$

and s_R is adjusted so that, at each value of t, the approximation is exact at $s = 0$. Equation (3.17) guarantees that Eq. (3.16) is also exact at $s = -\infty$, and it can be checked explicitly that it is a good approximation for other negative values of s at various typical values of t. The expression (3.16) is, of course, equivalent to taking

$$v(s,t) \simeq \pi c(t)\,\delta(s - s_R). \tag{3.18}$$

IV. CHEW-MANDELSTAM CROSSING CONDITION AND PCAC PREDICTIONS

As discussed already, PCAC predicts zeros in the S-wave amplitudes.[6] These occur at $t = 2$ in the $I = 2$ state and at $t = \frac{1}{2}$ in the $I = 0$ state, if we make the usual linear extrapolation from the off-shell Adler point. Now a zero can be easily produced in our $I = 2$ potential at $t = 2$ by simply choosing our remaining Veneziano parameter so that $\alpha(1) = \frac{1}{2}$, a value which agrees approximately with experiment.[28] This leads to a vanishing at $t = 2$ of all the $\gamma_n(t)$ which come into Eqs. (3.13) and (3.14), and hence of the potential V, as given by Eqs. (3.3) and (3.4). This, in turn, means that the amplitude calculated from the BS equation (2.5) also has a zero at this value of t. This zero occurs in all partial waves.

The vanishing of the potential in the $I = 2$ state

arises from a cancellation of the contributions of isospin 0 and 1 states in the s channel, since the crossing matrix elements connecting these to the $I=2$ state in the t channel have opposite signs. This cancellation tends to occur even if we did not have the Veneziano model and is closely connected with the absence of resonances in the $I=2$ state. On the other hand, no such cancellations occur in the $I=0$ state, for which the potential is quite large at $t=\frac{1}{2}$. It is thus necessary to solve the BS equation (2.5) explicitly to see whether it generates a Ramsauer-Townsend zero[8] near this point.

In practice, Eq. (2.5) was solved by making a partial-wave projection and solving the resulting equation by using a Pagels-type approximation.[21] Unlike some approximations to the BS equation, such as the determinantal,[29] this does not destroy the Ramsauer-Townsend zero. It is also extremely simple, since it reduces the BS equation to algebraic form. The detailed equations are given in Appendix B.

The only parameter which is still free in our equations is the "inelasticity" constant R_0. To determine this we impose the Chew-Mandelstam crossing condition[22]

$$A^{I=0}=\tfrac{5}{2}A^{I=2} \tag{4.1}$$

at the symmetry point $s=t=u=\frac{4}{3}$. This condition is a particularly convenient one for our purposes, since it does not involve the $I=1$ amplitude and is applied at a point in the region we are interested in. Now $V^{I=2}$, which vanished at $t=2$, is still very small at the symmetry point. We can therefore set

$$A^{I=2}\simeq V^{I=2}. \tag{4.2}$$

If we also make the usual assumption that $A^{I=0}$ is dominated by its S wave, Eq. (4.1) becomes

$$A_{l=0}^{I=0}(\tfrac{4}{3})=\tfrac{5}{2}V_{s=t=u=4/3}^{I=2}. \tag{4.3}$$

We can now vary R_0 until Eq. (4.3) is satisfied. This leads to $R_0=4.82$. If we make the identification (2.7), this should be compared with the experimental value $\sigma_{\text{tot}}/\sigma_{\text{el}}\approx 5-6$. [See the discussion following Eq. (3.14).]

Now since $V^{I=0}$ is small at the symmetry point, Eq. (4.3) forces $A_{l=0}^{I=0}$ to be small there also. But since the $I=0$ potential is large and attractive, $A_{l=0}^{I=0}$ varies fairly rapidly. We can thus expect a zero in the immediate vicinity of $t=\frac{4}{3}$. With our approximations this zero appears at $t\simeq 0.8$, which is fairly close to the PCAC zero at $t=0.5$.[6]

The above argument for the existence of an $I=0$ zero seems to be fairly insensitive to the particular dynamical approximations we have made, since it is almost forced on us by the relation (4.2). As we go away from the zero, we will, of course, be depending much more on the approximations. This would be true, for example, of the $I=0$ scattering length, which turns out to be $a_0=0.58$. This is much larger than the Weinberg value[4] of $a_0=0.15$ or the Lovelace-Veneziano value[2] of $a_0=0.24$. A large part of the difference comes from the fact that nonlinear terms in t are important in our amplitude. If we drop such terms, we get $a_0=0.34$, which is still somewhat large but much closer to the other values. The fact that we were able to obtain such important nonlinear terms suggests that the usual linear extrapolations made in current algebra may not always be reliable. Of course, it could also be that a more accurate method than the one we used would lead to less important nonlinear terms.

The $I=2$ scattering length a_2 depends much less on specific dynamical assumptions since it can be calculated from Eq. (4.2). This gives $a_2=-0.06$, which agrees quite well with the Weinberg value.[4]

V. REGGE TRAJECTORIES

The partial-wave projection of Eq. (2.5) can be readily extended to unphysical l,[29] and solved by the same Pagels approximation as was used in obtaining the results of the preceding section. (See Appendix B.) A resonance or bound state in A_l will now occur at $t=t_l$ when

$$A_l^{-1}(t_l)=0. \tag{5.1}$$

The corresponding residue is $(\frac{1}{4}t_l-1)^l\Gamma_l$, where

$$\Gamma_l=-\left\{\left[\frac{\partial}{\partial t}\frac{(\frac{1}{4}t-1)^l}{A_l(t)}\right]_{t=t_l}\right\}^{-1}. \tag{5.2}$$

The equation for a Regge trajectory $\alpha(t)$ is then given by

$$l=\alpha(t_l), \tag{5.3}$$

while the residue of the corresponding pole of A_l in the l plane is

$$\beta(t)=(\tfrac{1}{4}t-1)^{\alpha(t)}\Gamma_{\alpha(t)}\alpha'(t). \tag{5.4}$$

In practice our BS equation was solved only for $l=0.6$, 1.0, and 2.0, and only for the $I=0$ and $I=1$ states, using the same parameters as in the preceding sections. A polynomial in t was then interpolated through the resulting $\alpha(t_l)$ to obtain an explicit expression for $\alpha(t)$ at other values of t. Some of the results for the leading trajectory are given in Table I and Fig. 6. The Table I parameters for the resonances in the $I=1$, $l=1$ and $I=0$, $l=2$ states should be compared with experimental values of $s_1=30$, $\Gamma_1=0.2$ for the ρ and $s_2=80$, $\Gamma_2=0.064$ for the f. They should also be compared with the Veneziano parameters in Table II. The masses are thus too low but this may, to some extent, be due to the crude method used in solving

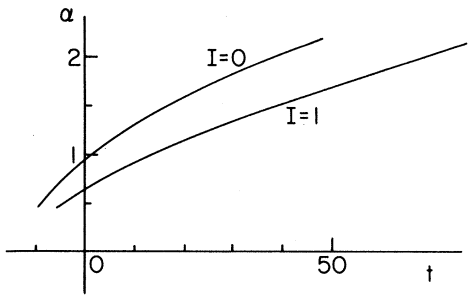

FIG. 6. Output Regge trajectories calculated from Eqs. (5.1) and (5.2).

our BS equation. The reduced widths Γ_l, on the other hand, seem to agree quite well with experiment, in sharp contrast with the reduced widths obtained in most traditional bootstrap calculations, which are often several times larger than the experimental values.[23] Our Regge intercepts of $\alpha^{I=1}(0)=0.63$ and $\alpha^{I=0}(0)=0.93$ are also in approximate agreement with the experimental intercepts for the ρ and P (Pomeranchuk) trajectories, respectively. Presumably the inclusion of a high-s "tail" in $v(s,t)$ would lead to a splitting of the latter trajectory into P and P' trajectories,[13,30] and hopefully give an even better intercept for the P.

Our trajectories are somewhat steeper than those expected from experimental considerations or those assumed in the Veneziano model, as can be seen by comparing Figs. 1 and 6, for example. They clearly have no difficulty in rising at least up to $l=2$. We cannot really go to much higher t since our Pagels approximation is really applicable only to small t. If, however, we make a determinantal approximation[29] to Eq. (2.5) with a potential given by Eqs. (3.3), (3.4), and (3.14), we are led to an equation of the type considered in Appendix A of Ref. 11. This was shown to give an infinitely rising Regge trajectory.

We have not calculated our trajectories explicitly very far below $t=0$. Here one difficulty which might arise is the possibility of a finite $\beta^{I=0}(t)$ at the point where $\alpha=0$; this would lead to an unphysical t pole in the physical region of the s channel. However, we find that the potential has a tendency to vanish at about the point where we expect this to happen. Moreover it is shown in Appendix B that, at least in the simple Pagels approximation we are using, α will in fact pass through zero at exactly the point where the on-shell potential vanishes. This guarantees that $\beta=0$ at that point.

TABLE I. Some of the values of t_l and Γ_l calculated from Eqs. (5.1) and (5.2).

	$I=1$		$I=0$	
l	t_l	Γ_l	t_l	Γ_l
0.6	−1	0.50	−6.7	
1.0	14	0.22	1.7	0.17
2.0	68	0.0051	39	0.0069

TABLE II. Some of the values of t_l and Γ_l for the leading trajectory of the Veneziano model of Eqs. (3.9)–(3.12) with $\bar{\beta}=0.6$, $\alpha(30)=1$, and $\alpha(1)=0.5$.

	$I=1$		$I=0$	
l	t_l	Γ_l	t_l	Γ_l
1.0	30	0.20	30	0.30
2.0	88	0.0032	88	0.0048

We have seen that our leading $I=0$ trajectory does not pass exactly through $\alpha=1$ at $t=0$. If, however, we imagine adding some perturbation which causes it to do so, we can immediately calculate the asymptotic cross section from the formula $\sigma_{\text{tot}}=-8\pi^2\beta(0)$, with $\beta(0)$ given by Eq. (5.4); in practice we also assumed that $\beta(0)\simeq\beta(-1)$. This gives $\sigma_{\text{tot}}=12.50$ mb, which should be compared with the experimental value $\sigma_{\text{tot}}=16$ mb. [See the discussion following Eq. (3.14).]

Another quantity which is of some interest is the average multiplicity $\langle n\rangle$ for the production of pions. Here the situation is less clear-cut. With pion-exchange dominance in Fig. 3, i.e., with the inelasticity factor $R=1$, this would be given by

$$\langle n\rangle = 2\left(g\frac{\partial\alpha}{\partial g}\right)_{g=1}\ln s, \tag{5.5}$$

where g is the over-all strength of the first term on the right-hand side of Fig. 5. However, we have seen that the symmetry-point crossing condition requires a fairly large value of R for $t>t_I$. In the absence of details about the actual processes which go into this R, it is not clear how Eq. (5.5) is to be modified in this case. Presumably it continues to be valid if the dominant exchanges in Fig. 3 are the Regge recurrences of the π, since Eq. (5.5) does apply to a multi-Regge as well as to an ABFST model.[31] On the other hand, there may be other production diagrams which would lead to a different expression for $\langle n\rangle$. If we simply assume these to be unimportant, however, we would obtain $\langle n\rangle=0.48\ln s$. This is somewhat smaller than the value $\langle n\rangle\simeq 1\ln s$ inferred from experiment.[32]

Finally we can check to see how well our equations satisfy average duality,[33] by considering the lowest-moment finite-energy sum rule[33,34]

$$\tau\equiv\frac{\alpha+1}{N^{\alpha+1}}\int_0^N d\bar{\nu}A_s=\frac{b}{\pi}, \tag{5.6}$$

where $\bar{\nu}=\frac{1}{2}(s-u)$ and

$$\frac{b}{\pi}=\frac{4}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+1)}\,\frac{\beta}{(\frac{1}{4}t-1)^{\alpha}}\,. \tag{5.7}$$

Of course, any amplitude with Regge asymptotic behavior will automatically satisfy Eq. (5.6) for sufficiently large N. Duality requires that it should continue to hold even for fairly low values of N, provided this is taken midway between two resonances. We took N to be midway between the f and g resonances; this is below the $\rho\rho$ threshold so that the A_s in Eq. (5.6) is given exactly by the resonance expression (3.13). We also considered only small values of t, where the approximations we have made are presumably most reliable. Thus in the $I=1$ state at $t=-1$, we have $\tau=0.033$ and $b/\pi=0.036$, which agree quite well. On the other hand, in the $I=0$ state at $t=-1.68$, we find $\tau=0.010$ and $b/\pi=0.023$. This disagreement should not be surprising, however, if we remember that our leading trajectory is more like a Pomeranchuk than a P' trajectory. Such a trajectory is not expected to be dual with a low-energy A_s which is dominated purely by resonances.[35]

VI. CONCLUSION

We have seen that S-wave zeros arise fairly naturally in an ABFST multiperipheral model which is modified by an "inelasticity" factor to take some account of other exchanges besides the pion. The strength of this factor is approximately related to the high-energy value of $\sigma_{\rm tot}/\sigma_{\rm el}$, and the value which leads to zeros resembling most closely the ones predicted by PCAC agrees quite well with the experimental value of this ratio, as we have seen. Its precise value was not fixed in this way in practice, however, but was instead adjusted so as to guarantee crossing at the Chew-Mandelstam symmetry point. As argued in Sec. IV, this ensures that the $I=0$ zero will remain close to the $I=2$ zero, essentially independently of the more specific dynamical approximations that may be made. The location of the $I=2$ zero is also largely independent of such approximations since the weakness of the potential V guarantees that $A\approx V$ in almost any dynamical scheme. But this also means that it does depend on the details of the potential. We have seen, however, that such a potential has a natural tendency to vanish at the correct place because of cancellations between $I=0$ and $I=1$ exchanges in this state, and that such a vanishing does occur if we take for it the exchange of the lowest resonances of a Veneziano model with the experimentally reasonable intercept of $\alpha(1)=\frac{1}{2}$.

In addition to low-energy zeros we were also able, at the same time, to obtain high-rising Regge trajectories. This feature, as we have seen already, does not depend on the details of our model but is a result of the parametric growth of the potential V in Eq. (2.5) with the energy variable t in the t channel. Our trajectories are also roughly consistent with the corresponding trajectories in the Veneziano model, at least for moderate values of t (compare Figs. 1 and 6), and lead to reduced widths for ρ and f^0 in approximate agreement with experiment. We thus have a unitary amplitude which resembles the Veneziano model, at least in some of its dominant features; by construction the elastic resonances in the s and u channels are, of course, the same as in the Veneziano model.

Our calculation required a fairly large inelasticity factor R. This has the effect of increasing the strength of the kernel in our BS equation and of raising the output trajectories to higher values than they would otherwise have had.[15] Although the strength of R was fixed by a crossing condition in our calculation, it would naturally be desirable to calculate at least some of the effects which go into it explicitly. One class of diagrams, which still falls within the pion-exchange approximation, is obtained by rearranging the final particles along the multiperipheral chain of Fig. 3. This leads to interference terms, which correspond to nonplanar graphs, such as the one shown in Fig. 7. Such graphs would lead to extra inelastic effects beginning at the 4π threshold in the t channel. The fact that the ρ and f^0 resonances are almost purely elastic, however, would seem to indicate that such graphs are probably not too important, at least for low values of t.

Another class of diagrams which contribute to R would come from the exchange of other particles besides pions in the chain of Fig. 3. Now $\pi\pi$ particles, such as ρ and f^0, are presumably excluded since their exchange already contributes to the blobs in Fig. 3, and we would be double-counting if we explicitly added their exchange in Fig. 3. The exchange of strange particles and baryons is allowed. If such exchanges are important, however, blobs involving these particles in Fig. 3 become significant and it becomes difficult to explain the relative absence of strange-particle and baryon production at high energies, since we would expect

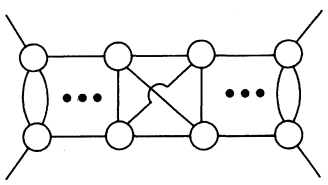

FIG. 7. A nonplanar graph arising from interference terms.

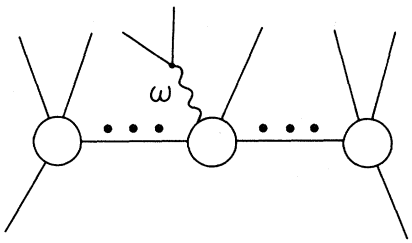

FIG. 8. An ABFST-type diagram with ω production at one or more of the blobs. All external particles are pions. The class of such diagrams is dual with the class represented by Fig. 3.

blobs leading to the production of these particles in Fig. 3 to be important also. Of course, kinematic or off-shell effects may force us to modify this argument.

If the above arguments are correct, we are left only with the exchange of particles which decay into more than two pions. This would include particles like the ω, A_1, and A_2. Of course, in this case we would also expect diagrams such as Fig. 8 to be important. These are consistent with pion-production predominance at high energies. However, the class of such diagrams is dual with the class represented by Fig. 3. We would thus be double-counting if we included them explicitly.

If we thus insert our generalized Fig. 3 into Eq. (2.4), we obtain generalized Figs. 4 and 5 which correspond to a multichannel BS equation, but with only two-pion exchange in each channel. In the $\pi\omega \to \pi\omega$ channel, for example, this means that B-meson exchange is to be suppressed. Of course, by including also the exchange of the Regge recurrences of the π, ω, A_1, A_2, etc., we would have an infinite number of channels, or, equivalently, a generalized multi-Regge model in which we have several Regge exchanges, each of which couples to two pions rather than one.

APPENDIX A: A CRITERION OF MAXIMAL CONVERGENCE

As we have seen, there is no unique or *a priori* way of continuing the ABFST input potential off shell. We shall argue, however, that if we make certain assumptions about $v(s,t)$, a "criterion of maximal convergence" would tend to favor the extrapolation given by Eqs. (3.3) and (3.4).[11,20] According to this criterion we must pick that off-shell behavior which leads to the most convergent BS equation without at the same time generating any additional singularities in the amplitude (and thereby violating maximal analyticity). This criterion is actually closely tied in with some of the basic assumptions of the multiperipheral model itself, since an equation with less-convergent integrals can be expected to have increased average momentum transfers.

We will begin by assuming that $v(s,t)$ grows with t for large t. This would be true if v were dominated by single-particle exchange or Regge-pole or -cut exchange with sufficiently high $\alpha(s)$; in particular, it is true of Eq. (3.14). Let us now consider various possible modifications of the off-shell extrapolation given by Eqs. (3.3) and (3.4). One possibility would be to make the replacement $t \to f(p', p, t)$ in $v(s', t)$, where f reduces to t on shell. This is the sort of continuation made in perturbation theory, for example. Now if we are not to introduce additional singularities into the amplitude, f must be a polynomial in t, p', and p. Since v grows with f, we therefore have a v which grows with p and p', unless $f = t$ everywhere. Such a growth would lead to less-convergent integrals in Eq. (2.5), with Eq. (3.3) for V, than would be obtained with the unmodified Eq. (3.4). This can be seen, for example, if we make an iterative expansion of Eq. (2.5). In fact we might even obtain a downright divergence; we then have to introduce a cutoff which necessarily brings an extra singularity into the amplitude.

Another type of modification which might be tried at the same time, would be to multiply v by a factor depending on p and p', which reduces to unity on the shell. Or, we could add a function of these variables, which reduces to zero on the shell. Both of these must again be polynomials if we are not to introduce any additional singularities into the amplitude, and would thus lead to a more divergent behavior than is obtained by simply taking $v(s', t)$ itself in Eq. (3.4). Other extrapolations can also be tried, but always seem to give more-divergent behavior. We are thus left with the one given by Eqs. (3.3) and (3.4).

APPENDIX B: EQUATIONS FOR A PAGELS-TYPE APPROXIMATION TO THE BS EQUATION

To solve Eq. (2.5) we used an extension of the Pagels approximation proposed by Balázs and Patil.[21] We shall not discuss here the justification for this type of approximation, for which the reader is referred to Ref. 21, but merely write down some of the formulas which lead up to it. The detailed way in which we actually make the approximation is somewhat different from that used in Ref. 21 and resembles more the one in Ref. 20.

We begin by making a partial-wave projection of Eqs. (2.5), (3.3), and (3.4). If we write $p=(\omega,\vec{p})$, $p'=(\omega',\vec{p}')$, and $p''=(\omega'',\vec{p}'')$, this gives

$$T_l(q',\omega';q,\omega;t)=V_l(q',\omega';q,\omega;t)+i\frac{4}{\pi^2}\int_0^\infty q''^2dq''\int_{-\infty}^{\infty}d\omega''V_l(q',\omega';q'',\omega'';t)\times R(\omega''^2-q''^2)G(q'',\omega'',t)T_l(q'',\omega'';q,\omega;t) \quad \text{(B1)}$$

with

$$V_l(q',\omega';q'',\omega'';t)=\frac{1}{\pi}\int ds'\frac{v(s',t)}{2q'q''}\left[Q_l\left(\frac{s'-(\omega'-\omega'')^2+q'^2+q''^2}{2q'q''}\right)+Q_l\left(\frac{s'-(\omega'+\omega'')^2+q'^2+q''^2}{2q'q''}\right)\right] \quad \text{(B2)}$$

and

$$G^{-1}(q'',\omega'',t)=[(\omega''+\tfrac{1}{2}\sqrt{t}\,)^2-q''^2-1][(\omega''-\tfrac{1}{2}\sqrt{t}\,)^2-q''^2-1], \quad \text{(B3)}$$

where $q^2=\vec{p}^2$, $q'^2=\vec{p}'^2$, and $q''^2=\vec{p}''^2$; we shall take $\omega=0$ and $q^2=\frac{1}{4}t-1$ from now on. Equations (B1)–(B3) apply both to physical and unphysical values of l.[29]

Following Levine, Wright, and Tjon,[36] we now make a Noyes-Kowalski reduction[37] by writing

$$(q'q)^{-l}T_l(q',\omega';q,0;t)=q^{-2l}g(q',\omega',t)A_l(t), \quad \text{(B4)}$$

where $A_l(t)$ is the on-shell value of T_l and can be written in terms of the phase shift δ as $\frac{1}{2}\sqrt{t}\,e^{i\delta}\sin\delta/q$ in the elastic region. Equation (B1) then becomes

$$g(q',\omega',t)=\frac{U(q',\omega';q,0;t)}{U(t)}-i\frac{4}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_{-\infty}^{\infty}d\omega''\left[U(q',\omega';q'',\omega'';t)-\frac{U(q',\omega';q,0;t)}{U(t)}U(q,0;q'',\omega'';t)\right]\times R(\omega''^2-q''^2)G(q'',\omega'',t)g(q'',\omega'',t), \quad \text{(B5)}$$

where

$$U(t)=U(q,0;q;0;t)$$

and

$$U(q',\omega';q'',\omega'';t)=(q'q'')^{-l}V_l(q',\omega';q'',\omega'';t).$$

Once we have solved the above equation for g, we can obtain $A_l(t)$ from

$$q^{-2l}A_l(t)=U(t)/\Delta(t) \quad \text{(B6)}$$

with

$$\Delta(t)=1+i\frac{4}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_{-\infty}^{\infty}d\omega''U(q,0;q'',\omega'';t)R(\omega''^2-q''^2)G(q'',\omega'',t)g(q'',\omega'',t). \quad \text{(B7)}$$

It will prove convenient for us to rewrite Eq. (B6) as

$$\Delta(t)=1+U(t)H(t)+i\frac{4}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_{-\infty}^{\infty}d\omega''G(q'',\omega'',t)\times\left[U(q,0;q'',\omega'';t)R(\omega''^2-q''^2)g(q'',\omega'',t)-R(-q''^2)\theta(\Lambda-q''^2)U(t)\right] \quad \text{(B8)}$$

where we have added and subtracted UH with

$$H(t)=i\frac{4}{\pi^2}\int_0^\infty q''^{2l+2}dq''\int_{-\infty}^{\infty}G(q'',\omega'',t)R(-q''^2)\theta(\Lambda-q''^2), \quad \text{(B9)}$$

an expression which can be evaluated analytically. So far, Λ can take on any value we please.

If we now make a Wick rotation to the imaginary axis for the ω variable, we find that Eqs. (B5) and (B8) are nonsingular – actually Eq. (B8) is nonsingular only for $t<4(\Lambda+1)$, but that is sufficient for our purposes. Now $G(q'',i\omega'',t)$ is peaked around $\omega''=0$ at $t=0$. For $t\neq 0$ this is not quite true, but continues to be a

reasonable approximation. We shall therefore simply approximate it by a δ function

$$G(q'', i\omega'', t) \simeq c'(q'', t)\delta(\omega'') \tag{B10}$$

with c' such that the integral of both sides of Eq. (B10) over ω'' is the same, which guarantees that both sides are equal on the average. Explicitly this gives

$$c'(q'', t) = \frac{\pi}{4} \frac{1}{(q''^2+1)^{1/2}(q''^2 - \frac{1}{4}t + 1)}. \tag{B11}$$

By the arguments of Ref. 21 we can make the further approximation

$$q''^{2l+2}/(q''^2+1)^{1/2} \simeq c\,\delta(q''^2 - a^2), \tag{B12}$$

with c and a such that

$$\int_0^\infty dq''^2 \frac{q''^{2l+1}}{(q''^2+1)^{1/2}} \frac{R(-q''^2)\theta(\Lambda - q''^2)}{q''^2 - q^2} \simeq \frac{c}{(a^2 - q^2)} \tag{B13}$$

is a good approximation when the constant Λ is chosen so as to simulate the large q''^2 behavior in U and g.[20] If we make a Born approximation for g, this means that we would have $\sqrt{\Lambda} \approx$ highest exchanged mass; we therefore took $\Lambda = 130$, to correspond to the g-meson mass. In practice Eq. (B13) was required to be exact at $q^2 = -\infty$ and $q^2 = -\Lambda$. It is then a good approximation in the region $-m_l^2 - \frac{1}{4}t + 1 \gtrsim q^2 \gtrsim -\infty$, as it has to be for the approximation (B12) to be meaningful[21]; here m_l is the lowest exchanged mass. The approximations (B10)–(B13) turn out[20] to be approximately equivalent to inserting

$$R(\omega''^2 - q''^2)G(q'', \omega'', t) \simeq i\frac{\pi}{4} c \frac{\delta(\omega'')\delta(q'' - a)}{a^2 - q^2} \tag{B14}$$

into the integrals of Eqs. (B5) and (B8), thereby reducing these equations to purely algebraic form. Actually for t below a point about midway between $t = 4$ and the first singularities of $U(t)$, it is more appropriate to insert (B14) into Eq. (B7) than into (B8) since the latter has singularities in the same places as $U(t)$. In particular, if we now evaluate Eq. (B5) at $q' = a$ and $\omega' = 0$, we have a linear equation for $g(a, 0, t)$. When we solve this it is possible, if U is strong enough, to obtain $g(a, 0, t) = \infty$. If we now insert this into Eq. (B8), after having made the approximation (B14), we obtain $\Delta = \infty$ at this value of t. From Eq. (B6) this, in turn, leads to $A_l = 0$, which, in fact, is the Ramsauer-Townsend-type zero which we obtain from the $I = 0$ state.

Finally, let us consider a situation where the on-shell potential $U(t) \to 0$, but $U(a, 0; q, 0; t)$ stays finite. As we saw in Sec. V, there is a tendency for this to happen at about the point where we expect $\alpha(t) = 0$ in the $I = 0$ S state. In that case a solution of (B5) with the approximation (B14) gives

$$g^{-1}(a, 0, t) = \frac{1}{\pi} \frac{c}{a^2 - q^2} U(a, 0; q, 0; t). \tag{B15}$$

Now in this region, as mentioned in the preceding paragraph, it is more appropriate to use Eq. (B7) than Eq. (B8). If we insert Eq. (B15) into the latter equation and use the approximation (B14), we obtain $\Delta(t) = 0$ at this point. This implies a bound state, and hence $\alpha(t) = 0$, but with a residue which vanishes at exactly the same point. From Eq. (5.4) this in turn means that $\beta = 0$ at this point.

APPENDIX C: HIGH-s "TAIL" OF $\nu(s,t)$

In the paragraph following Eq. (3.14) we took $r = 0$ for $s > s_3$. But the estimates made in the same paragraph show that actually we should take something like $r = r_\infty \approx \frac{1}{5}$ to $\frac{1}{6}$ in this region. We will now argue that, at least if the approximations (3.16) and (3.17) continue to be valid, the contribution of the extra "tail" for $s > s_3$ is small in this case also.

Let us begin by considering its effect on $c(t)$. If we use Eq. (3.13), the extra contribution is

$$\Delta c(t) \simeq r_\infty \frac{1}{\pi} \sum_{r=4}^{\infty} \gamma_n(t) = r_\infty \frac{1}{\pi} \int_\eta^\infty ds' A_s(s', t), \tag{C1}$$

where η is, say, midway between s_3 and s_4. Now, strictly speaking, the last integral usually diverges. It can, however, be defined by continuing the superconvergence relation

$$\int_\eta^\infty ds' A_s(s', t) = -\int_0^\eta ds' A_s(s', t) = -\sum_{n=1}^{3} \gamma_n(t), \tag{C2}$$

where we have again used Eq. (3.13). If we then combine Eqs. (C1), (C2), and (3.17), we have

$$c(t) \simeq \frac{1}{\pi} \sum_{n=1}^{3} [r(s_n) - r_\infty]\gamma_n(t), \tag{C3}$$

since $r(s_1) = r(s_2) = 1$ and $r(s_3) \simeq \frac{1}{3}$, we see that the effect of r_∞ is not very large on $c(t)$.

Now s_R was adjusted so that Eq. (3.16) is exact at $s = 0$. From Eq. (3.15) this means that

$$\frac{c(t)}{s_R} = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{r(s_n)\gamma_n(t)}{s_n}. \tag{C4}$$

If we can show that the contributions of the terms $n > 3$ in this sum are small, then the result of the previous paragraph shows that s_R is not shifted much by including a high-s tail. But the terms $n > 3$ can be written as

$$r_\infty \frac{1}{\pi}\sum_{n=4}^{\infty}\frac{\gamma_n(t)}{s_n} = r_\infty \frac{1}{\pi}\int_\eta^\infty ds' \frac{A_s(s',t)}{s'}, \tag{C5}$$

where we have again used Eq. (3.13). Suppose we now write the Veneziano amplitude as

$$A^I = B(t,s) + (-1)^I B(t,u), \tag{C6}$$

where the functions B can be readily calculated from Eqs. (3.9)–(3.12) and satisfy the dispersion relation

$$B(t,s) = \frac{1}{\pi}\int_0^\infty ds' \frac{A_s(s',t)}{s'-s}. \tag{C7}$$

When we combine Eq. (C7) with Eqs. (C4) and (C5), we obtain

$$\frac{c(t)}{s_R} = r_\infty B(t,0) + \frac{1}{\pi}\sum_{n=0}^{\infty}[r(s_n)-r_\infty]\frac{\gamma_n(t)}{s_n}. \tag{C8}$$

The effect of r_∞ on the last term on the right-hand side of Eq. (C8) is small for the same reason that it was small for Eq. (C3). To see how large $r_\infty B(t,0)$ is, let us consider the $I=1$ state, for example. In this case we simply have $B(t,0)=F(t,0)$ which is certainly small for $t\approx 0$. In fact it vanishes at a point in this region. In the $I=0$ and $I=2$ states we also have to consider the effect of $F(s,u)$, but its contribution remains quite small in the entire region of interest. As t approaches the ρ resonance, however, an unmodified $F(t,0)$ would blow up because of one of the Γ functions in the numerator of Eq. (3.12). We must therefore modify Eq. (3.12) so that the amplitude at least satisfies the unitarity bound. We can do this by projecting $B(t,s)$ into partial waves in the s channel. Then from Eq. (C6)

$$B(t,0) = \tfrac{1}{2}\sum_{l=0}^{\infty}(2l+1)A_l(t),$$

where $A_l(t)$ are the usual partial-wave amplitudes, which are bounded by $\sqrt{t}/2q \approx 1$ near the ρ meson. Since at this position the only partial waves which are important in the Veneziano model are $l=0$ and $l=1$, we thus have

$$B(t,0) \lesssim 2.$$

With $r_\infty = \frac{1}{6}$ we then have $r_\infty B(t,0) \approx \frac{1}{3}$ whereas, for the $I=1$ state, for instance, the last term in Eq. (C8) has the value ≈ 2.2 without the r_∞. The inclusion of r_∞ in this term adds an extra contribution which tends to cancel the effect of $r_\infty B(t,0)$.

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[1]G. Veneziano, Nuovo Cimento 57, 190 (1968).

[2]C. Lovelace, Phys. Letters 28B, 264 (1968); see also E. Cremmer, J. Nuyts, and H. Sugawara, Phys. Rev. D 3, 478 (1971). For an earlier derivation of the Adler self-consistency condition from analyticity, crossing, and Regge behavior, see L. A. P. Balázs and J. M. Cornwall, Phys. Rev. 160, 1313 (1967).

[3]S. Adler, Phys. Rev. 137, B1022 (1965).

[4]S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[5]See, for example, G. F. Chew, Phys. Rev. Letters 22, 364 (1969).

[6]Since we shall be calculating on-shell partial-wave amplitudes, the zeros we are referring to are the ones which arise in the S-wave amplitudes when we make a smooth extrapolation from the off-shell Adler point. See, for example, L. S. Brown and R. L. Goble, Phys. Rev. Letters 20, 346 (1968). Of course the precise locations of such zeros depend to some extent on the details of the particular extrapolation which is assumed.

[7]The $\pi\pi$ strip-approximation calculation of R. C. Johnson and P. D. B. Collins, Phys. Rev. 185, 2020 (1969), gave small S-wave $\pi\pi$ scattering lengths, which strongly suggests the presence of zeros below threshold. Their trajectories, however, were unable to rise very much above $J=1$; see P. D. B. Collins and R. C. Johnson, Phys. Rev. 182, 1755 (1969). It has also been argued that certain features of their results suggest that the physical ρ should appear as a Castillejo-Dalitz-Dyson pole in their N/D equations; see D. H. Lyth, Phys. Rev. D 3, 1991 (1971). Another type of calculation is that exemplified by J. B. Carotte and R. C. Johnson, Phys. Rev. D 2, 1945 (1970). In such a calculation one does not calculate Regge trajectories, however.

[8]See, for example, the discussion in L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), Third Edition, p. 123.

[9]J. Finkelstein, Phys. Rev. 145, 1185 (1966).

[10]S. H. Patil, Phys. Rev. 179, 1405 (1969); G. Tiktopoulos, Phys. Letters 29B, 185 (1969); U. Trivedi, Phys. Rev. 188, 2241 (1969); see also A. R. Swift and R. W. Tucker, Phys. Rev. D 2, 2486 (1970).

[11]L. A. P. Balázs, Phys. Rev. D 2, 999 (1970).

[12]L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento 25, 626 (1962); D. Amati, A. Stanghellini, and S. Fubini, *ibid.* 26, 896 (1962).

[13]J. S. Ball and G. Marchesini, Phys. Rev. 188, 2508 (1969); Phys. Rev. D 2, 2665 (1970).

[14]G. F. Chew, T. W. Rogers, and D. R. Snider, Phys. Rev. D 2, 765 (1970).

[15]D. M. Tow, Phys. Rev. D 2, 154 (1970).

[16]V. N. Akimov, S. S. Chernavskiĭ, I. M. Dremin, and I. I. Royzen, Nucl. Phys. B14 (2), 285 (1969). These authors actually consider a generalization of the ABFST model.

[17]G. F. Chew and A. Pignotti, Phys. Rev. 176, 2112 (1968); G. F. Chew, M. L. Goldberger, and F. E. Low, Phys. Rev. Letters 22, 208 (1969); I. G. Halliday, Nuovo Cimento 60A, 177 (1969); I. G. Halliday and L. M. Saunders, *ibid.* 60A, 494 (1969); L. Caneschi and A. Pignotti, Phys. Rev. 180, 1525 (1969); 184, 1915 (1969); G. F. Chew

and W. R. Frazer, *ibid.* 181, 1914 (1969); J. S. Ball and G. Marchesini, *ibid.* 188, 2209 (1969); S. Pinsky and W. I. Weisberger, Phys. Rev. D 2, 1640 (1970).

[18]See, for example, S.-J. Chang and R. Rajaraman, Phys. Rev. 183, 1517 (1969).

[19]Aachen-Berlin-Bonn-CERN-Warsaw Collaboration, Nucl. Phys. B8, 471 (1968).

[20]L. A. P. Balázs, Phys. Rev. 176, 1769 (1968).

[21]L. A. P. Balázs and S. H. Patil, Phys. Rev. 171, 1633 (1968).

[22]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[23]See, for example, J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. 137, B1242 (1965). On the other hand, certain other calculations were able to obtain at least comparatively narrow widths. See, for instance, Ref. 7 or L. A. P. Balázs, Phys. Rev. 132, 867 (1963); 134, AB1 (1964).

[24]See, for example, F. Henyey, G. L. Kane, J. Pumplin, and M. H. Ross, Phys. Rev. 182, 1579 (1969).

[25]A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento 29, 380 (1963); R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).

[26]This is the same type of potential as was used in Ref. 19.

[27]The lower-lying trajectories, which are least well known experimentally, were shown not to contribute much to the potential in Ref. 20, where the construction of the potential was in practice very similar to the one used here.

[28]A small change in $\alpha(1)$ will merely lead to a small shift in the position of our zero and will not affect our subsequent arguments in any substantial way. Note that $\alpha(1) = 1$ also happens to be the value which guarantees that the Veneziano amplitude vanishes at the Adler point. See Ref. 2.

[29]B. W. Lee and R. F. Sawyer, Phys. Rev. 127, 2266 (1962).

[30]G. F. Chew and D. R. Snider, Phys. Rev. D 1, 3453 (1970); 3, 420 (1971).

[31]See, for example, P. D. Ting, Phys. Rev. D 2, 2982 (1970).

[32]L. W. Jones *et al.*, Phys. Rev. Letters 25, 1679 (1970), obtain (0.7 ± 0.1) ln$s$ for the average multiplicity of charged particles produced. Assuming pion-production dominance, this must be multiplied by $\frac{3}{2}$ to obtain $\langle n \rangle$.

[33]R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters 19, 402 (1967).

[34]A. A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze, Phys. Letters 24B, 181 (1967); K. Igi and S. Matsuda, Phys. Rev. Letters 18, 625 (1967); R. Gatto, *ibid.* 18, 803 (1967); L. A. P. Balázs and J. M. Cornwall, Ref. 2.

[35]H. Harari, Phys. Rev. Letters 20, 1395 (1968); P. Freund, *ibid.* 20, 235 (1968).

[36]M. Levine, J. Wright, and J. Tjon, Phys. Rev. 154, 1433 (1967).

[37]H. P. Noyes, Phys. Rev. Letters 15, 538 (1965); K. L. Kowalski, Phys. Rev. Letters 15, 798 (1965); 15, 908 (E) (1965).