$$t_1^i = \max[(t_3^{1/2} - m)^2, (t_4^{1/2} - m)^2], \quad t_1^f = \min[4m^2, (t_3^{1/2} + m)^2, (t_4^{1/2} + m)^2],$$
$$t_2^i = \max[(t_4^{1/2} - m)^2, (t_5^{1/2} - m)^2], \quad t_2^f = \min[4m^2, (t_4^{1/2} + m)^2, (t_5^{1/2} + m)^2],$$
$$t_4^i = \max[(t_1^{1/2} - m)^2, (t_2^{1/2} - m)^2], \quad t_4^f = \min[4m^2, (t_1^{1/2} + m)^2, (t_2^{1/2} + m)^2], \tag{B6}$$
$$s_1^i = \max[(s_3^{1/2} - m)^2, (s_4^{1/2} - m)^2], \quad s_1^f = \min[4m^2, (s_3^{1/2} + m)^2, (s_4^{1/2} + m)^2],$$
$$\overline{s}_1 = \min[4m^2, (s_1^{1/2} + m)^2], \quad \overline{t}_i = \min[4m^2, (t_i^{1/2} + m)^2], \quad i = 1, 2, 4 .$$

*Supported in part by the U. S. Atomic Energy Commission.

[1]A. P. Balachandran and J. Nuyts, Phys. Rev. 172, 1821 (1968).

[2]A. P. Balachandran, W. J. Meggs, J. Nuyts, and P. Ramond, Phys. Rev. 187, 2080 (1969).

[3]A. P. Balachandran, W. Case, and M. Modjtehedzadeh, Phys. Rev. D 1, 1773 (1970).

[4]J. B. Hartle and C. E. Jones, Phys. Rev. 184, 1564 (1969).

[5]In order to be able to express the relations between the Euler angles and the invariants in every channel uniformly, in general we prefer to use different sets of invariants in different channels.

[6]R. Roskies, Nuovo Cimento 65A, 467 (1970); 66, 494 (1970).

[7]B. Bonnier, Nucl. Phys. B17, 525 (1970).

[8]T. W. Kibble, Phys. Rev. 117, 1159 (1960).

[9]Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2, Chap. 10.

[10]Reference 9, p. 158.

[11]A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, 1960), p. 58.

PHYSICAL REVIEW D VOLUME 4, NUMBER 4 15 AUGUST 1971

Self-Consistent Pomeranchukon Singularities. I*

J. B. Bronzan

Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 29 March 1971)

We formulate a functional self-consistency equation for the Pomeranchukon pole trajectory. The only ingredients are unitarity in the t channel (the channel of the Pomeranchukon) and the specification of $\alpha(0)$. The three solutions studied here are: $\alpha(0)=1$, cut dominant; $\alpha(0)=1-\epsilon$, cuts dominant; and $\alpha(0)=1$, pole dominant. We further provide expressions for the t-channel partial-wave amplitudes in terms of a small number of free parameters; these expressions are adequate for the calculation of the high-s behavior of σ_{tot}, $\text{Re}F(s,0)/\text{Im}F(s,0)$, and the diffraction peak in the domain $|t[\ln(s/s_0)]/[\ln\ln(s/s_0)]| \lesssim$ constant. Three prominent conclusions are: (1) multi-Pomeranchukon phase space plays a leading role in determining the relative pole-cut strength; (2) there are fixed cuts in the vacuum partial-wave amplitude that accumulate at $j=1$ even when $\alpha(0)=1-\delta$; (3) Schwarz trajectories $\alpha(t)=1+\gamma t^{1/2}+\cdots$ have a special status.

I. INTRODUCTION

Diffraction scattering has been discussed from so many points of view that we wish to begin with a review of what seem to be the relevant dynamical considerations. Subsequently, we shall state which of these we are able to incorporate in the present paper. Our language is that of S-matrix theory, but a field theorist will readily substitute "t-channel iteration" for "t-channel unitarity", for example.

The Froissart bound and its saturation suggest that s-channel unitarity is an important dynamical consideration.[1] This bound implies that total cross sections cannot grow faster than $(\ln s)^2$, and its derivation rests in part on s-channel unitarity. Present experimental evidence is consistent with saturation of the Froissart bound, or at least with total cross sections that do not fall with a detectable power of s.[2] Any theory that agrees with experiment will therefore be on the verge of violating the Froissart bound, and will be saved from doing so only if s-channel unitarity is imposed in a form no weaker than a bound on the magnitude of the s-

channel partial-wave amplitudes, as in Froissart's derivation. Some of the contemporary models of diffraction scattering illustrate this point. For example, Cheng and Wu's study of electrodynamics and the Regge-eikonal model show that s-channel unitarity converts a Regge trajectory $\alpha(t)$ with $\alpha(0)>1$ into a pair of singularities with $\alpha(0)=1$.[3]

"Eikonalization" of the Pomeranchukon singularity is one type of s-channel iteration of the t-channel Pomeranchukon exchange. As such, it should produce Regge cuts, and it does.[4] At just this point t-channel unitarity becomes an important consideration. The discontinuities across the Regge cuts in t-channel partial-wave amplitudes are controlled by t-channel unitarity.[5,6] The discontinuities determine the $(\ln s)$ corrections to the asymptotic forms of amplitudes, so a full calculation must also take t-channel unitarity into account. The cuts are particularly significant for $t<0$, since they have flatter trajectories than the Pomeranchukon pole, and even at $t=0$ they control $\mathrm{Re}F(s,0)/\mathrm{Im}\,F(s,0)$ if $\alpha(0)=1$.[7]

The satisfaction of s-channel unitarity and t-channel unitarity simultaneously is beyond the capacity of contemporary calculations. Current models can be classified as having either s-channel unitarity or t-channel unitarity. All the "eikonalized" models, including those of Cheng and Wu,[3] and Yang and collaborators[8] are among the former. The multi-Regge model studied by Chew and others is an example of the latter.[9] It is true that Cheng and Wu first iterate in the t channel, but their final s-channel iteration of t-channel towers surely upsets the t-channel cut discontinuities. Similarly, in multi-Regge theory one uses s-channel dispersion relations to recover the full amplitude, but this does not impose s-channel unitarity. Hwa's bootstrap recognizes the importance of both types of unitarity, but does not fully exploit t-channel unitarity.[10]

There are at least two further dynamical considerations that should be mentioned. The first is the condensation of an infinite number of Regge cuts on $j=1$ at $t=0$. Suppose there is a pole or cut singularity in the even-signature, vacuum, t-channel partial-wave amplitude with $\alpha(0)=1$. As mentioned, this is allowed by Froissart and suggested by experiment. Then there must be additional Regge cut trajectories $\alpha_n(t)$:

$$\alpha_n(t)=n\alpha(t/n^2)-n+1 \quad (n=2,3,\ldots). \qquad (1.1)$$

These all coincide at $t=0$: $\alpha_n(0)=1$. Since the original singularity must rise through $t=0$, cuts with larger n lead cuts with smaller n in the scattering region. Therefore, a complete understanding of diffraction scattering depends delicately on the summation of all the cut contributions. As we shall see, the situation is not quite as hopeless as this seems to imply, since the threshold behavior of the cut discontinuities weakens as n increases. At any finite s there is an interval of momentum transfer over which only a specified number of cuts are important. This interval increases with the number of cuts included, and decreases with s. Since the interval shrinks as s increases, the Regge description of nonforward scattering becomes unwieldy at ultrahigh energies, although it may be manageable over a limited interval of momentum transfers at presently accessible energies.

In view of this difficulty, it is worth mentioning that Schwarz has put forward a unique method for reducing the number of Regge cuts to only two.[11] Schwarz cuts have been used by Hwa,[10] and they arise naturally from the eikonalization of a trajectory with $\alpha(0)>1$. This suggests that they may be forced on us by s-channel unitarity. Finally, they appear in the discussion of amplitudes that violate the Pomeranchuk theorem.[12] Schwarz imposes the condition $\alpha_n(t)=\alpha(t)$ for all n. If we write $\alpha(t)$ in the form

$$\alpha(t)=1+t^{1/2}\psi(\ln t), \qquad (1.2)$$

Schwarz's functional equation becomes

$$\psi(x)=\psi(x-2\ln n), \qquad (1.3)$$

and one finds that $\psi(x)$ must be a periodic function of x with the logarithm of the square of every rational number as a period. The only smooth solution is $\psi(x)=\gamma$, a constant. In order that the t-channel partial-wave amplitude have no spurious fixed singularity at $t=0$, the two branches $1\pm\gamma t^{1/2}$ must be Regge cut trajectories.

This brings us to the last item on our list of dynamical considerations. It is clear that any deep discussion of diffraction scattering in the Regge framework cannot ignore cuts. At present there is no complete dynamical scheme for taking account of them. This difficulty is most concisely expressed in Gribov's Reggeon calculus, where the relevant dynamical object is the amplitude for particle + particle → Pomeranchukon + Pomeranchukon.[13] There is no reliable way of calculating this amplitude in terms of familiar amplitudes or parameters.

The foregoing observations have colored the approach we take in the present paper. Since diffraction scattering is complicated, we wish to make only modest and clearly identified assumptions, and to derive results that are reliable, if incomplete. Our assumptions are the following:

(1) We take s-channel unitarity and experiment into account only to the extent that the leading Pomeranchukon singularity for $t>0$ is assumed to have $\alpha(0)=1$. In Sec. III, we examine the case

$\alpha(0)=1-\delta$.

(2) We assume that this singularity is a pole, and that the full complement of Regge cuts of Eq. (1.1) is present. In a future paper we shall examine the interesting possibility raised by Schwarz's self-reproducing cuts. Schwarz cuts turn out to have unique properties associated with their self-reproducing character, so their discussion naturally constitutes a separate topic. Here we merely report that self-consistency akin to what we find in this paper is also possible for Schwarz cuts.

(3) We exploit t-channel unitarity completely. For this purpose we use the unitarity study of Gribov, Pomeranchuk, and Ter-Martirosyan.[5] This study includes intermediate states of arbitrary number of particles, and extracts from the unitarity integrals the contributions of intermediate states of any number of Reggeons. The Reggeon unitarity relations are solved to obtain the behavior of t-channel partial-wave amplitudes in the presence of Regge cuts generated by n Reggeons. Our work starts with this representation of the t-channel partial-wave amplitude. The unitarity work of Gribov[5] should be distinguished from his later papers on the Reggeon calculus.[13,14] The latter papers start with sums of Feynman diagrams rather than unitarity.

(4) There are a few technical assumptions whose significance will be clarified in later sections of the paper. One is that there are only "weak" fixed cuts at $j=1$. The self-consistency of this assumption is subject to a partial check. A second assumption is that the Pomeranchukon pole and its attendant cuts "interact." This question has an analog in the Regge theory of potential scattering. In general, when two Regge trajectories cross, the trajectory functions are singular at the energy of the crossing.[15] If singularities do develop, we say that an interaction occurs, otherwise not. We shall give arguments that strongly indicate that the Pomeranchukon pole must interact with the multi-Pomeranchukon cuts.

The spirit and results of our work are comparable to at least two other lines of research. The results obtained by analyticity methods in Sec. IV are similar to results found by Gribov and Migdal by analyzing the vertices and Green's functions of the Reggeon field theory.[14] The Pomeranchukon pole-cut relationship we find has been studied by others, but less conclusively.[16,17]

In Sec. II we use Gribov's representation to set up a functional equation for the Pomeranchukon pole trajectory. This equation is a self-consistency or bootstrap condition on the pole trajectory in the presence of its cuts. We use the solution to obtain an explicit representation of the partial-wave amplitude near $t=0$ and $j=1$, and discuss the implications for diffraction scattering. In Sec. III we discuss the modifications that occur if $\alpha(0)$ is slightly less than one. In Sec. IV we observe that the solution of Sec. II does not have Mandelstam's sign for the two-Pomeranchukon cut,[4] and we determine what further modifications are required to obtain agreement with Mandelstam. In Sec. V we review the insights that have emerged, and show why the Schwarz trajectories have a special status.

II. POMERANCHUKON SINGULARITIES WITH $\alpha(0)=1$

Gribov *et al.* have shown that an elastic partial-wave amplitude has the representation

$$f(t,j)=\frac{A(t,j)}{B(t,j)+[1/\alpha'(\tfrac{1}{4}t)]\ln[j-2\alpha(\tfrac{1}{4}t)+1]}, \quad (2.1)$$

where A and B are real analytic functions of t and j at the branch point of the Regge cut due to intermediate states of two Pomeranchukon poles.[5,18] Actually, the discontinuity across the j-plane cut of the denominator of Eq. (2.1) is not exactly constant. What Gribov *et al.* have shown is that the threshold behavior of the discontinuity is a constant. This means that B is not strictly analytic at $j=2\alpha(\frac{1}{4}t)-1$, but contains weakly singular terms like

$$[j-2\alpha(\tfrac{1}{4}t)+1]^n\ln[j-2\alpha(\tfrac{1}{4}t)+1] \quad (n=1,2,\ldots). \quad (2.2)$$

We shall later verify that such terms do not affect our study, and can therefore be ignored. B will further contain singular terms due to intermediate states of n Pomeranchukon poles. Gribov *et al.* have shown that such terms have the threshold behavior

$$[j-n\alpha(t/n^2)+n-1]^{n-2}\ln[j-n\alpha(t/n^2)+n-1] \quad (n=3,4,\ldots). \quad (2.3)$$

These terms are potentially significant near $j=1$ and $t=0$, but they too will turn out to be negligible. The coefficient of the logarithm in Eq. (2.3) arises from the threshold behavior of n-Pomeranchukon phase space, and it vanishes strongly for the higher cuts. It has the consequence that the behavior of $\alpha(t)$ near $t=0$ is controlled by the collision of the pole with the two-Pomeranchukon cut only. Also, it provides an extra factor of $(\ln s)^{-n}$ in the contribution of the n-Pomeranchukon cut to the asymptotic amplitude in the s-channel. These two simplifications are of central importance in the Regge theory of diffraction scattering. Without them it would be impossible to construct an expression for $f(t,j)$ in terms of a finite number of parameters.

An improved statement of the results of Gribov *et al.* is provided by the representation

$$f(t,j)=\frac{A(t,j)}{B(t,j)+\frac{1}{2}t'(j)\ln\{[t(j)-t]/t'(j)\}}, \qquad (2.4)$$

where $t(j)$ is the moving two-Pomeranchukon cut in the t plane.[19] $t(j)$ is given implicitly by the equations

$$2\alpha(\tfrac{1}{4}t(j))-1=j,$$
$$\tfrac{1}{4}t(2\alpha(t)-1)=t. \qquad (2.5)$$

Equation (2.1) should be regarded as a special case of Eq. (2.4), valid where $\alpha(t)$ is analytic. It is inadequate here because we anticipate that $\alpha(t)$ has a branch point at $t=0$, where the pole and cuts collide. Equation (2.1) would have $f(t,j)$ manifest that fixed t-cut, in violation of the analyticity implied by the Mandelstam representation and the Froissart-Gribov definition of the partial-wave amplitude. Equation (2.4) results from Eq. (2.1) when one removes the t-cut by attributing an appropriate fixed t-cut to B. In general, $t(j)$ and $f(t,j)$ in Eq. (2.4) will have fixed j-cuts if $\alpha(t)$ has a fixed t-cut, but fixed j-cuts violate no general principles. In fact, the fixed j-cut of $f(t,j)$ is unavoidable because if we try to remove it by attributing an appropriate fixed j-cut to B, we are led back to Eq. (2.1) with its illegal fixed t-cut. In conclusion, Eq. (2.4) is the preferred statement of the results of Gribov, *et al.* Equation (2.1), which is the representation given by Gribov *et al.*, is valid only where $\alpha(t)$ is analytic.

We note in passing that the Schwarz trajectories belong to the small class of trajectories that are singular at $t=0$, but which give an analytic $t(j)$. For the Schwarz trajectories, $t(j)=(j-1)^2/\gamma^2$.

The next step in our argument is to discuss the analyticity of B. In the t plane, B can have no cuts near $t=0$ for $j\approx 1$ once we have dropped the weak cuts of Eqs. (2.2) and (2.3). This follows from the standard energy analyticity of partial-wave amplitudes. $B(t,j)$ can have fixed or moving poles of arbitrary order that are near $t=0$ for $j\approx 1$, and in Sec. IV we shall makes use of this possibility. Here we assume that no such poles are present. In the j plane, aside from the weak moving cuts, B can have a fixed j-cut at or near $j=1$. We assume that any fixed j-cuts in B are so weak that they can be ignored, like those envisioned in Ref. 19. Later we shall report on estimates that partially justify this assumption. Subject to these assumptions, we can expand $B(t,j)$ in a power series about $t=0$ and $j=1$.

$$B(t,j)=b_0+b_1t+b_2(j-1)+\cdots. \qquad (2.6)$$

It is traditional to write $f(t,j)$ in the form

$$f(t,j)=A(t,j)/D(t,j),$$
$$D(t,j)=\tfrac{1}{2}t'(j)\ln[(t(j)-t)/t'(j)] \qquad (2.7)$$
$$+b_0+b_1t+b_2(j-1)+\cdots.$$

Since $f(t,j)$ is the even-signature vacuum amplitude, it has a Regge pole at $j=\alpha(t)$. This pole can appear either as a pole of A or a zero of D in Eq. (2.7). In the former case the Pomeranchukon pole does not interact with its cuts, and in the latter case it does. There is good reason to believe that the noninteracting case is unphysical. Gribov *et al.* show that the amplitude for particle + particle → Pomeranchukon + Pomeranchukon has the form $\sqrt{A}/D$, and the amplitude for Pomeranchukon + Pomeranchukon → Pomeranchukon + Pomeranchukon has the form $1/D$. Therefore, if the Pomeranchukon pole is in A rather than D, the Pomeranchukon production amplitude has a square-root Regge cut that is not generated by the Mandelstam mechanism.[4] In addition, the three-Pomeranchukon vertex would vanish *identically*. Both of these consequences of a pole in A are implausible, so we assume that the Pomeranchukon pole is a zero of D. $\alpha(t)$ satisfies the self-consistency or bootstrap equation

$$D(t,\alpha(t))=0. \qquad (2.8)$$

This equation is most easily solved by making the substitution $t\to\frac{1}{4}t(j)$. Using Eqs. (2.5) and (2.7), it becomes

$$0=D(\tfrac{1}{4}t(j),\tfrac{1}{2}(j+1))$$
$$=\tfrac{1}{2}t'(\tfrac{1}{2}(j+1))\ln\frac{t(\frac{1}{2}(j+1))-\frac{1}{4}t(j)}{t'(\frac{1}{2}(j+1))}$$
$$+b_0+\tfrac{1}{4}b_1t(j)+\tfrac{1}{2}b_2(j-1)+\cdots. \qquad (2.9)$$

In this section we are interested in solutions of Eq. (2.9) with $t(1)=0$, which corresponds to $\alpha(0)=1$. At first glance it seems impossible for Eq. (2.9) to have solutions with $t(1)=0$, because then the logarithm diverges at $j=1$ while all other terms seem to remain finite. However, if $t'(j)$ vanishes like $[\ln(j-1)]^{-1}$ at $j=1$, the cut term in Eq. (2.9) can take on any value at $j=1$, including $-b_0$. This shows that we can expect $t(j)$ to behave like $(j-1)/\ln(j-1)$ at $j=1$. Further inspection suggests that the solution of Eq. (2.9) with $t(1)=0$ has the expansion

$$t(j)=\frac{j-1}{\ln(j-1)}\left(\lambda_0+\lambda_1\frac{\ln[-\ln(j-1)]}{\ln(j-1)}+\frac{\lambda_2}{\ln(j-1)}+\cdots\right). \qquad (2.10)$$

Substituting this expansion into Eq. (2.9), we find

$$0=(b_0+\tfrac{1}{2}\lambda_0)+\tfrac{1}{2}\lambda_1\frac{\ln[-\ln(j-1)]}{\ln(j-1)}+\tfrac{1}{2}(\lambda_2-\lambda_0-\lambda_0\ln 2)\frac{1}{\ln(j-1)}+R_1+R_2\,. \qquad (2.11)$$

The remainder term R_1 receives contributions from the higher terms in Eq. (2.10), and it vanishes at $j=1$ "logarithmically." R_2 receives contributions from b_i $(i>0)$ and the weak singularities of Eqs. (2.2) and (2.3). Up to logarithmic factors, R_2 vanishes linearly at $j=1$, so R_2 is negligible compared to R_1. This means that all the terms in $t(j)$ that vanish linearly at $j=1$ (up to logarithms) are determined by b_0 and the two-Pomeranchukon cut. The weak moving singularities and b_i $(i>0)$ determine terms in $t(j)$ that vanish at least quadratically at $j=1$, and which are not even shown in Eq. (2.10). This justifies dropping the weak moving singularities in Eq. (2.9), and we see that we could have dropped b_1 and b_2 as well. The first three expansion constants in Eq. (2.10) are

$$\lambda_0=-2b_0,\quad \lambda_1=0,\qquad \lambda_2=-2b_0(1+\ln 2)\,. \qquad (2.12)$$

We now have $D(t,j)$ near $t=0$ and $j=1$:

$$D(t,j)=b_0-\frac{b_0}{\ln(j-1)}\left(1+\frac{\ln 2}{\ln(j-1)}\right)\times\ln\left((j-1)+\frac{j-1}{\ln(j-1)}+\frac{t}{2b_0}\ln\tfrac{1}{2}(j-1)\right). \qquad (2.13)$$

The Pomeranchukon pole trajectory can be calculated from Eqs. (2.5), (2.10), and (2.12). The result is

$$\alpha(t)=1-\frac{t}{b_0}\left(\ln\frac{t}{b_0}\right)\left(1+\frac{\ln[-\ln(t/b_0)]}{\ln(t/b_0)}-\frac{1}{\ln(t/b_0)}+\cdots\right). \qquad (2.14)$$

This trajectory is acceptable only if $b_0=B(0,1)$ is positive, since the real part of the trajectory must rise through 1. However, this is the only dynamical condition that must be met for a self-consistent trajectory. By examining D as a function of t for $j>1$, we find there is just one Pomeranchukon pole on the physical sheet of the t plane, with $t>0$. In the j plane, this pole and the two-Pomeranchukon branch point move as a function of t and become complex as t is decreased through $t=0$. At $t=0$ the pole residue vanishes. A second pole and two-Pomeranchukon cut emerge from behind the fixed cut at $j=1$ and take up complex-conjugate positions so that $f(t,j)$ remains a real analytic function of j for $t<0$. Real analyticity is maintained only because we have used Eq. (2.4) rather than Eq. (2.1) as our starting point. There are infinite numbers of poles and cuts on the Riemann sheets entered by winding around the fixed and moving j-cuts, but they all remain off the physical sheet for real t.[20] Therefore, they do not contribute to asymptotic amplitudes in the s channel. The pole and cuts in the j plane that we have described are depicted in Figs. 1 and 2.

At $t=0$ the poles and cuts of Figs. 1 and 2 condense into a single cut with branch point at $j=1$. It is easy to evaluate the Sommerfeld-Watson integral to find the leading asymptotic expressions

$$\sigma_{\text{tot}}\xrightarrow[s\to\infty]{}(\text{const})^2\frac{A(0,1)}{b_0\ln(s/s_0)}\,,\qquad \frac{\operatorname{Re}F(s,0)}{\operatorname{Im}F(s,0)}\xrightarrow[s\to\infty]{}\frac{-\pi}{2\ln(s/s_0)}\,. \qquad (2.15)$$

Note that we must require $A(0,1)>0$, and that the leading behavior of the cross section is controlled by the two-Pomeranchukon cut since the pole residue vanishes at $t=0$. The corrections provided by b_2 and the weak cuts of Eqs. (2.2) and (2.3) are down by factors of $[\ln(s/s_0)]^{-1}$.

According to Gribov's Reggeon calculus,[13] it is impossible to construct a field theory in which the two-Pomeranchukon cut contributes positively to the total cross section, as in Eq. (2.15). Gribov's observation is based on a very general principle, namely, the real analyticity of Feynman amplitudes. Gribov and Migdal use this observation to rule out a "strong coupling" solution of the Reggeon field theory, and what we have calculated here is the S matrix version of a strong coupling solution.[14] Gribov's stipulation implies that it is inconsistent with the general principles of quantum mechanics to have $\alpha(0)=1$ and a vanishing Pomeranchukon pole residue at $t=0$. We believe this argument is correct, and in Sec. IV we shall investigate what modifications are required to make the pole residue nonvanishing at $t=0$. Once this is done, the two-Pomeranchukon cut can contribute negatively to σ_{tot}.

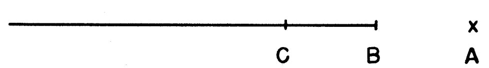

FIG. 1. Poles and cuts in the j plane for $t>0$. A: $\alpha(t)$; B: $2\alpha(t/4)-1$; C: 1.

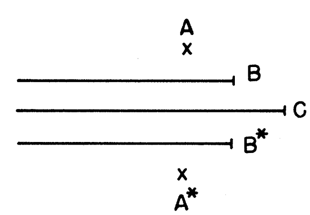

FIG. 2. Poles and cuts in the j plane for $t<0$. A: $\alpha(t)$; B: $2\alpha(t/4)-1$; C: 1.

The issue of the sign of the two-Pomeranchukon cut is related to a more familiar dispute. In the first calculations of Regge cuts, Mandelstam's calculation[4] gave a sign for the two-Reggeon cut that disagreed with the sign found by Amati, Fubini, and Stanghellini.[21] The disagreement has been handed down to the successors of these calculations. Models that are based on s-channel unitarity, like the multi-Regge model, tend to give the Amati-Fubini-Stanghellini sign,[22] while models based on the full amplitude, like the absorption model, tend to agree with Mandelstam. Most commentators favor the Mandelstam sign.[23] The sign we have found in this section is the Amati-Fubini-Stanghellini sign.

The most intriguing prediction of Eq. (2.15) is the asymptotic ratio of the real and imaginary forward amplitudes. It contains no arbitrary parameters, and it is in order-of-magnitude agreement with experiment. Foley *et al.* have measured the ratio for the pion-nucleon amplitudes.[24] They find a ratio of about -0.1 for $\pi^- -p$ and about -0.2 for $\pi^+ -p$ between 8 and 20 GeV/c. Using the conventional s_0 of 1 GeV^2, we predict about -0.4. However, since $\ln(s/s_0)$ is only 3.7 at 20 GeV/c, there will be 30% corrections from the multi-Pomeranchukon cuts, and the charge dependence of the experimental results shows that secondary trajectories play a role. In any case, the modifications of Sec. IV introduce a parameter into the ratio and change the sign of the prediction.

The diffraction scattering predicted by the distribution of singularities shown in Fig. 2 is difficult to assess analytically. One might guess that the combination of a fixed cut and singularities moving rapidly to the left will give an elastic cross section with both shrinking and nonshrinking components. It is easier to estimate the interval of $-t$ over which the multi-Pomeranchukon cuts are negligible. The threshold behavior of n-Pomeranchukon phase space introduces a factor $(\ln s)^{2-n}$ in the contribution of the higher cuts. This factor allows the pole and two-Pomeranchukon cuts to dominate near the forward direction, even though the higher cuts dominate for large enough $-t$. We estimate that the three-Pomeranchukon cut is small compared to the two-Pomeranchukon cut for momentum transfers such that

$$\left(\frac{s}{s_0}\right)^{1+t/2b_0} \gtrsim \frac{1}{\ln(s/s_0)}\left(\frac{s}{s_0}\right)^{1+t/3b_0}. \qquad (2.16)$$

The interval of momentum transfer is

$$0 \leq -\frac{t}{b_0} \lesssim \frac{\ln[\ln(s/s_0)]}{\ln(s/s_0)}. \qquad (2.17)$$

When $-t$ is at the upper limit of this range, the elastic cross section has dropped by a factor of $[\ln(s/s_0)]^{-2}$. If we continue to take $s_0 = 1$ GeV^2, at 20 GeV/c the three-Pomeranchukon cut comes in when the elastic cross section has dropped by an order of magnitude. Experimentally, this range is $-t \lesssim 0.5$ GeV^2, and b_0 is about 1 GeV^2.

Equation (2.4) incorporates the effects of intermediate states of two Pomeranchukon poles. As we have seen, the cuts generated by more than two Pomeranchukon poles are unimportant near $t=0$, both in the sense that they do not affect $\alpha(t)$, and in the sense that they contribute subordinate terms to the asymptotic s-channel amplitudes near $t=0$. In principle, Eq. (2.4) should also include terms coming from intermediate states of several poles and cuts, or of several cuts alone. Here again, phase-space arguments show that such terms are ordinarily negligible near $t=0$ because, for example, two-cut intermediate states give a branch point whose position and threshold discontinuity are like that of intermediate states of four poles. However, in the present section, where the pole residue vanishes at $t=0$, such usual suppression of intermediate states with cuts no longer holds. Given the linear vanishing (up to logarithms) of the pole residue at $t=0$, the two-cut intermediate state gives a contribution at $t=0$ that is comparable to the two-pole intermediate state. Fortunately, in Sec. IV, where the pole residue is finite, no such problems arise, and all the approximations made in setting up the equation for $\alpha(t)$ are controlled, except the assumption that any fixed cut at $j=1$ in $B(t,j)$ is weak. To discuss this assumption, recall that in Gribov's original unitarity paper,[5] one obtains expressions for $f(t,j) - f^{(2n)}(t,j)$, where $(2n)$ indicates continuation around the $2n$-particle unitarity threshold. For the n-Pomeranchukon cut, where $\alpha(t) \neq \alpha^{(2n)}(t)$, one can separately determine the Regge-cut discontinuities of f and $f^{(2n)}$, arriving at the representation we have quoted. This separation cannot be made for a fixed cut, so it cannot be verified that the fixed cut of Eq. (2.13) contributes negligibly to $f(t,j)$ when included in the list of intermediate-state j singularities. What can be shown is that $f - f^{(4)}$ has a fixed cut of negligible strength. This is a partial check on consistency, since it is unlikely that f would have a weak cut if $f - f^{(4)}$ had a strong cut. One can further conjecture that f and $f - f^{(2n)}$ should have similar discontinuities across fixed cuts because they have similar discontinuities when the cut moves, however slowly.

We should mention one exceptional situation that allows the Pomeranchukon pole to appear as a zero of D and still not interact with its two-Pomeranchukon cut. In Ref. 19 we point out that the factor $t'(\frac{1}{2}(j+1))$ in the argument of the logarithm of

Eq. (2.9) can be replaced by a constant or various other expressions, with only trivial changes in Eq. (2.12). However, if $t'(\frac{1}{2}(j+1))$ is replaced by some function that behaves like $c(j-1)$ at $j=1$, $\alpha(t)$ becomes analytic at $t=0$. We see no reason why this particular choice should be the physical one.

III. POMERANCHUKON SINGULARITIES WITH $\alpha(0)=1-\delta$

When the intercept $\alpha(0)$ of the Pomeranchukon pole is at $1-\delta$ (δ small), it is still possible to obtain a self-consistent trajectory. The possibility of self-consistency for a continuous range of $\alpha(0)$ is at odds with our expectation for trajectories determined by a functional equation like (2.8). We might expect Eq. (2.8) to have many solutions arranged in a discrete sequence, with only a discrete sequence of possibilities for $\alpha(0)$. (In a typical calculation only a small number of the solutions would be taken seriously.) The difference of the present case is evident when we display the full functional dependence of D:

$$D(t, t(j), t'(j)) = b_0 + \tfrac{1}{2}t'(j)\ln\{[t(j)-t]/t'(j)\}\,. \tag{3.1}$$

Here we have dropped all the terms that were shown to be unimportant in Sec. II, assuming that they remain unimportant if $\alpha(0)$ is sufficiently close to 1, or what is the same thing, $t(1)$ is sufficiently close to zero. Thus, the equation determining $t(j)$ is

$$D\left(\tfrac{1}{4}t(j), t\left(\frac{j+1}{2}\right), t'\left(\frac{j+1}{2}\right)\right)=0\,,$$

which is a nonlinear differential-difference equation. We expect such equations to have solutions passing through *any* $t(1)$. In fact, we expect an infinite number of solutions passing through any $t(1)$, because the differential-difference equation is equivalent to a differential equation of infinite order:

$$D\left(\tfrac{1}{4}\exp\left[\frac{j-1}{2}\,\frac{d}{dj}\right]t\left(\frac{j+1}{2}\right), t\left(\frac{j+1}{2}\right), t'\left(\frac{j+1}{2}\right)\right)=0\,. \tag{3.2}$$

Thus the problem is not why there should be solutions with any $t(1)$, but how to deal with an infinite number of possibilities. The explanation lies in the factor $j-1$ in the translation operator in Eq. (3.2); it guarantees that all the solutions having a given $t(1)$ have the same asymptotic expansion about $j=1$. This behavior is plausible because $t(j) = t(\frac{1}{2}(j+1))$ at $j=1$, and it explains why we found a unique expansion about $j=1$ in Sec. II. If it were not for the asymptotic equality of all the solutions of Eq. (3.2) at $j=1$, the functional equation would have little predictive power.

These observations about bootstrap equations are of general applicability. If one goes beyond crude approximations, bootstrap equations must generally be expected to contain differential operators that greatly reduce the predictive power of the equations.

The specific self-consistency equation we wish to study is

$$t'(1+\epsilon) = \frac{-2b_0}{\ln\{[t(1+\epsilon)-\frac{1}{4}t(1+2\epsilon)]/t'(1+\epsilon)\}} \qquad (j=1+2\epsilon)\,, \tag{3.3}$$

for parameters $b_0>0$, $t(1)>0$, $t(j)$ real for $j>1$. The existence proof for solutions of Eq. (3.3) is difficult because the argument is advanced: To calculate $t'(1+\epsilon)$ for $|\epsilon|<\rho$, one must know $t(1+\epsilon)$ for $|\epsilon|<2\rho$. Therefore, no local existence proof is possible in the neighborhood of $\epsilon=0$. Existence proofs in the literature are based on continuity assumptions that are not satisfied by Eq. (3.3).[25] We are forced to assume that Eq. (3.3) has solutions, and proceed to a qualitative discussion of their properties.

For some $\mathrm{Re}\,\epsilon_0<0$ we have $t(1+\epsilon_0)=\frac{1}{4}t(1+2\epsilon_0)$. $1+\epsilon_0$ is the angular momentum and $\frac{1}{4}t(1+2\epsilon_0)$ is the energy at which the Pomeranchukon pole and two-Pomeranchukon cut trajectories collide. From Eq. (3.3) we might expect that $t'(1+\epsilon)$ has the singular behavior $-2b_0/\ln(\epsilon-\epsilon_0)$, so that $t(1+\epsilon)$ has a singular term behaving like $-2b_0(\epsilon-\epsilon_0)/\ln(\epsilon-\epsilon_0)$. The presence of $t(1+2\epsilon)$ in Eq. (3.3), together with this singularity, implies that $t'(1+\epsilon)$ has a further singularity at $\frac{1}{2}\epsilon_0$, with leading behavior $2A_1(\epsilon-\frac{1}{2}\epsilon_0)/\ln(\epsilon-\frac{1}{2}\epsilon_0)$. This leads to a second singular term in $t(1+\epsilon)$ with leading behavior $A_1(\epsilon-\frac{1}{2}\epsilon_0)^2/\ln(\epsilon-\frac{1}{2}\epsilon_0)$. The argument can be continued with the conclusion that near $\epsilon_0/2^n$ there is a singularity in $t(1+\epsilon)$ with leading behavior

$$\frac{A_n(\epsilon-\epsilon_0/2^n)^{n+1}}{\ln(\epsilon-\epsilon_0/2^n)}\,. \tag{3.4}$$

These singularities of the inverse trajectory are successively weaker, and they accumulate at $\epsilon=0$, or $j=1$. The presence of such singularities means that $f(t,j)$ has fixed j-singularities accumulating at $j=1$, even though $\alpha(0)=1-\delta$.

These fixed j-cuts are disturbing, because there are many channels with the following properties: (1) nonvacuum quantum numbers; (2) $\alpha(0)$ well below one; (3) the amplitude receives contributions from intermediate states of two Reggeons with the quantum numbers of the channel. The discussion we have just given seems to require the existence of fixed cuts accumulating at $j=1$, unless the tra-

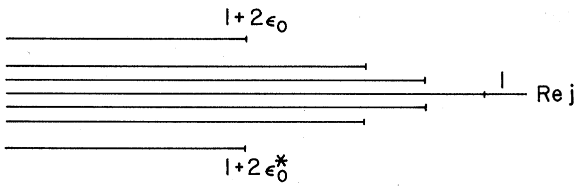

FIG. 3. Branch lines of $t(j)$. There are an infinite number of branch points that accumulate at $j=1$. After detailed calculation, these branch lines are shown to be on unphysical sheets, and $t(j)$ is analytic in the plane with a cut up to $j=1$.

jectories behave so peculiarly that the pole never collides with its cuts. However, this conclusion is not required by t-channel unitarity when $|\epsilon_0|$ is large, as it is when $\alpha(0)$ is well below one. In Eq. (3.1) we can replace the numerator of the logarithm by an arbitrary function $h(t(j)-t)$. Then, as long as $h(x)\to x$ when $x\to 0$, we satisfy t-channel unitarity. We are perfectly free to make h singular to x_0, where $x_0=t(1+\frac{1}{2}\epsilon_0)-\frac{1}{4}t(1+\epsilon_0)$, in such a way as to remove the singularities of Eq. (3.4). There is a spurious moving cut introduced into $f(t,j)$ by the singularity of h, but it lies below the leading singularities. This possibility is unavailable only when $x_0\to 0$, which corresponds to $\alpha(0)=1-\delta$, the case considered in this section.

The terms of Eq. (3.4) are complex at $\epsilon=\epsilon_0$, since $\mathrm{Re}\,\epsilon_0<0$. Real analyticity of $t(j)$ implies that there are two solutions of $t(1+\epsilon_0)=\frac{1}{4}t(1+2\epsilon_0)$, ϵ_0 and ϵ_0^*. There are two pole-cut collisions for $\epsilon_0\neq 0$, and two sequences of singularities of the form given in Eq. (3.4). These singularities of $t(j)$ are illustrated in Fig. 3. It should be emphasized that there are further sequences of cuts due to multi-Pomeranchukon intermediate states that are not illustrated in Fig. 3.

There is one further possible elaboration, which is actually realized. Since there are cuts of $t(j)$ all the way up to $j=1$, it is possible for the pole-cut collisions to occur on unphysical sheets of $t(j)$, even for very small ϵ_0. Then $t(j)$ is analytic in the j plane cut along the real axis up to $j=1$, and the branch points of Fig. 3 are on other sheets of $t(j)$.

This qualitative discussion can be put on a quantitative basis by constructing a series of approximations to $t(j)$. The sequence we construct has the following three properties: (1) It is an asymptotic sequence as $t(1)\to 0$, so we expect it to be a useful approximation to the solutions of Eq. (3.3) when $t(1)$ is small; (2) the $(n+1)$st member of the sequence has n of the singularities exhibited in Eq. (3.4), on unphysical sheets of $t(j)$, and with slightly modified properties; (3) each member of the sequence is analytic at $j=1$. Because of property (3) we know that the sequence cannot converge, since $t(j)$ is not analytic at $j=1$. To construct the sequence, we begin with a power series about $j=1$ as an ansatz for $t(j)$,

$$t(j)=t(1)+\sum_{n=1}^{\infty}a_n(j-1)^n\,. \tag{3.5}$$

We then put Eq. (3.3) in the form

$$t(1+\epsilon)=t(1+\epsilon)+\frac{2b_0a_1\epsilon}{2b_0+a_1}+\frac{a_1}{2b_0+a_1}\int_0^{\epsilon}d\bar{\epsilon}\,t'(1+\bar{\epsilon})\ln\left[\frac{t(1+\bar{\epsilon})-\frac{1}{4}t(1+2\bar{\epsilon})}{t'(1+\bar{\epsilon})}\right]. \tag{3.6}$$

We substitute Eq. (3.5) on both sides and find

$$a_1=-2b_0/[\ln\tfrac{3}{4}t(1)-\ln a_1]\,, \tag{3.7}$$

$$a_n=\frac{a_1}{(2b_0+a_1)n}\left[na_n+\sum_{p=1}^{n-1}(n-p)a_{n-p}\sum_{m=1}^{p}(-1)^{m+1}(m-1)!\right]\sideset{}{'}\sum_{\substack{q_1\cdots q_p\\(m)}}\left[\prod_{i=1}^{p}\frac{1}{q_i!}\left(\frac{(4-2^i)a_i}{3t(1)}\right)^{q_i}-\prod_{i=1}^{p}\frac{1}{q_i!}\left(\frac{(i+1)a_{i+1}}{a_1}\right)^{q_i}\right], \tag{3.8}$$

where

$$\sideset{}{'}\sum_{\substack{q_1\cdots q_p\\(m)}}$$

means sum over all sets of p non-negative integers $\{q_i\}$ that satisfy the restrictions

$$\sum_{i=1}^{p}q_i=m\,,\quad\sum_{i=1}^{p}iq_i=p\,. \tag{3.9}$$

Equation (3.7) determines a_1. When $t(1)$ is close to zero, a_1 has the leading behavior $-2b_0/\ln t(1)$, and the parameter $a_1/(2b_0+a_1)$, which appears in Eqs. (3.6) and (3.8), has the behavior $-1/\ln t(1)$.

Equations (3.8) are a set of nonlinear recursion relations that determine the a_n for $n\geqslant 2$. The right-hand side depends only on the a_i with $i<n$, since the $p=n=1$, $m=1$ term of the sum contains a contribution $-na_n$.

It is easy to verify that a_n has the form

$$a_n = \frac{(a_1)^n}{[t(1)]^{n-1}} \sum_{p=1} a_{np} \left(\frac{a_1}{2b_0 + a_1} \right)^p , \tag{3.10}$$

where the a_{np} are numerical constants independent of $t(1)$. For $t(1)$ close to zero the terms with small p dominate the terms with large p because of the limiting behavior of the parameter $a_1/(2b_0+a_1)$. In particular, the leading contribution to a_n is given by the constants

$$a_{n1} = \frac{(-1)^n}{n(n-1)} (\tfrac{2}{3})^{n-1} . \tag{3.11}$$

It is easy to verify that the recursion relation, Eq. (3.8), determines the a_{np} in terms of the $a_{n'p'}$ with $n' < n$, $p' < p$. The condition $p' < p$ is crucial for the construction of our sequence of approximations. It means that *any* approximation that has the correct $a_{np'}$ for $p' < p$ generates an approximation with correct $a_{np'}$ for $p' < p+1$ when the former is inserted on the right-hand side of either of Eqs. (3.6) or (3.8). This suggests the following iteration procedure. We begin with

$$t_0(1+\epsilon) = t(1) + a_1\epsilon , \tag{3.12}$$

where a_1 is determined by Eq. (3.7). We then generate a sequence of approximations by turning Eq. (3.6) into an iteration formula, i.e.,

$$t_{p+1}(\epsilon+1) = t_p(1+\epsilon) + \frac{2b_0 a_1 \epsilon}{2b_0 + a_1} + \frac{a_1}{2b_0+a_1} \int_0^\epsilon d\bar{\epsilon}\, t_p'(1+\bar{\epsilon}) \ln\left(\frac{t_p(1+\bar{\epsilon}) - \frac{1}{4} t_p(1+2\bar{\epsilon})}{t_p'(1+\bar{\epsilon})} \right) . \tag{3.13}$$

The foregoing discussion establishes that the sequence has the property

$$t_{p+1}(1+\epsilon) - t_p(1+\epsilon) = [a_1/(2b_0+a_1)]^{p+1} t(1) g_p(a_1\epsilon/t(1)) . \tag{3.14}$$

This is precisely the criterion that the sequence be an asymptotic sequence in $t(1)$ for fixed $a_1\epsilon/t(1)$.[26] We therefore expect the sequence to form a good sequence of approximations to the solutions of Eq. (3.3) for $0 \leqslant |\epsilon| \lesssim t(1)/a_1$. Although this interval shrinks as $t(1)$ approaches zero, it always includes the points at which $t(1+\epsilon)$ is singular, as we shall see.

The first iteration is

$$t_1(1+\epsilon) = t(1) + \frac{2a_1 b_0 \epsilon}{2b_0 + a_1} + \frac{3a_1 t(1)}{2(2b_0+a_1)} \left(1 + \frac{2a_1\epsilon}{3t(1)} \right) \ln\left(1 + \frac{2a_1\epsilon}{3t(1)} \right) . \tag{3.15}$$

It is easily verified that the power-series expansion of this approximation is given by Eqs. (3.10) and (3.11) with $a_{np} = 0$ for $p > 1$. The trajectory is singular at $\epsilon_1 = -3t(1)/2a_1$, but the singularity does not have the character indicated by Eq. (3.4), nor does ϵ_1 quite satisfy the collision equation $t_1(1+\epsilon_0) = \frac{1}{4} t_1(1+2\epsilon_0)$.

To understand these deficiencies, we first note that we can replace $t_1(1+\epsilon)$ by $\bar{t}_1(1+\epsilon)$ without harm:

$$\bar{t}_1(1+\epsilon) = -\tfrac{1}{2} t(1) - \frac{a_1^2 \epsilon}{2b_0 + a_1} + \tfrac{3}{2} t(1) \left[1 + \frac{2a_1\epsilon}{3t(1)} \right] \left[1 - \frac{a_1}{2b_0+a_1} \ln\left(1 + \frac{2a_1\epsilon}{3t(1)} \right) \right]^{-1} . \tag{3.16}$$

The expansion of $\bar{t}_1(1+\epsilon)$ has the same a_{np} for $p \leqslant 1$ as $t_1(1+\epsilon)$, so $\bar{t}_1(1+\epsilon)$ is as suitable for further iterations as $t_1(1+\epsilon)$. This shows that the appearance of the first power of the logarithm in Eq. (3.15) does not indicate a real discrepancy with Eq. (3.4). Higher iterations will provide the corrections, although the corrections might be smaller if one iterates $\bar{t}_1(1+\epsilon)$.

The collision equation, $t_1(1+\epsilon_0) = \frac{1}{4} t_1(1+2\epsilon_0)$, has solutions near ϵ_1 at ϵ_0 and ϵ_0^*, where

$$\epsilon_0 = -\frac{3t(1)}{2a_1} - \frac{3t(1)}{4b_0} - \tfrac{3}{8} i\pi t(1) + O(t(1)a_1) . \tag{3.17}$$

The remainder cannot be evaluated from t_1, but the three terms exhibited will be the same in all orders of iteration. These collisions are very near ϵ_1 – showing that t_1 does the best it can – but they are on sheets of $t_1(1+\epsilon)$ reached by passing through the branch cut that terminates at ϵ_1. We take this to signify that $t(1+\epsilon)$ is analytic in the ϵ plane cut along the negative axis. There will be a sequence of branch points on the negative real axis, with the branch point at $\epsilon_1/2^n$ appearing in the $(n+1)$st and higher iterations. The collision singularities shown in Fig. 3 will be on unphysical sheets entered by passing through these real cuts. There will be singularities at $\epsilon_0/2^n$ and $\epsilon_0^*/2^n$ in the $(n+2)$nd and higher iterations.

We can use Eqs. (2.5) and (3.15) to obtain the first of an asymptotic sequence of approximations to the

Pomeranchukon trajectory:

$$\alpha_1(t) = 1 - \frac{t(1)}{2a_1} + \frac{2t(1)}{4(2b_0 + a_1)}\left[\left(1 + \frac{8t}{t(1)}\right)\ln\left(\frac{1}{3} + \frac{8t}{3t(1)}\right) + 2 - \frac{8t}{t(1)}\right] + O\left(a_1 t(1) f\left(\frac{t}{t(1)}\right)\right). \tag{3.18}$$

The relation between $\alpha(0)$ and $t(1)$ given by this equation verifies that $t(1)\gtrsim 0$ corresponds to $\alpha(0)\lesssim 1$.

IV. POMERANCHUKON SINGULARITIES WITH $\alpha(0)=1$ AND MANDELSTAM CUT SIGN

In Sec. II we pointed out that the Pomeranchukon trajectory and partial-wave amplitude found there are not acceptable because the discontinuity across the two-Pomeranchukon cut contributes positively to total cross sections. In this section we shall investigate what alterations must be made to change the sign.

The reason the cut is forced to contribute positively in Sec. II is that the pole residue vanishes at $t=0$ so that the cut leads. The vanishing of the pole residue at $t=0$ can be traced to the logarithmic singularity of $\alpha'(t)$ there. The only way to change the behavior of $\alpha'(t)$ is to make $B(t,j)$ singular at $t=0$ and $j=1$, so that the logarithmic singularity in Eq. (2.9) is compensated by B rather than $t'(\frac{1}{2}(j+1))$.

There are two ways to make B singular at $t=0$ and $j=1$. One is to attribute to B a fixed j-cut that diverges at $j=1$. Unfortunately, the constraints imposed on fixed j-cuts by t-channel unitarity are ambiguous, as discussed in Sec. II, so it is not feasible to pursue the possibility of a fixed cut. The second possibility is some sort of moving pole that passes through $t=0$ and $j=1$. We can easily treat this possibility.

By trial we find that the simplest behavior of A and B near $t=0$ and $j=1$ is provided by the expressions

$$\begin{aligned} A(t,j) &= \frac{a_0 + \cdots}{[c_1 t + c_2(j-1) + \cdots]^2}, \\ B(t,j) &= \frac{b_1 t + b_2(j-1) + b_3 t^2 + b_4(j-1)^2 + b_5 t(j-1)}{[c_1 t + c_2(j-1) + \cdots]^2}. \end{aligned} \tag{4.1}$$

Note that A and B must share a second-order pole if the Pomeranchukon-pole residue is to be finite at $t=0$. This behavior is also desirable on two further grounds. First, the amplitude for particle + particle $\to$ Pomeranchukon + Pomeranchukon is $\sqrt{A}/D$, so a first-order pole in A would introduce a spurious moving branch point into this amplitude. With Eq. (4.1), the Pomeranchukon-Pomeranchukon production amplitude has a moving first-order zero, so its analyticity is unmodified. Second, the fact that A and B share the zero means that the elastic amplitude has no zero that moves through $t=0$ and $j=1$, and there is no violation of the Herglotz property in taking a second-order pole. The sole effect of the moving pole is to weaken the two-Pomeranchukon cut enough to permit $\alpha'(t)$ to remain finite at $t=0$.

The self-consistent equation for $t(j)$ is

$$\begin{aligned} 0 = {} & \tfrac{1}{4}b_1 t(j) + \tfrac{1}{2}b_2(j-1) + \tfrac{1}{16}b_3 t^2(j) + \tfrac{1}{4}b_4(j-1)^2 + \tfrac{1}{8}b_5 t(j)(j-1) \\ & + \tfrac{1}{2}t'\left(\frac{j+1}{2}\right)[\tfrac{1}{4}c_1 t(j) + \tfrac{1}{2}c_2(j-1)]^2 \ln\left\{\left[t\left(\frac{j+1}{2}\right) - \tfrac{1}{4}t(j)\right]\Big/ t'\left(\frac{j+1}{2}\right)\right\}. \end{aligned} \tag{4.2}$$

For $t(j)$ we write the expansion

$$t(j) = \lambda_0(j-1) + \lambda_1(j-1)^2 + \lambda_2(j-1)^2\ln(j-1) + \cdots. \tag{4.3}$$

Substituting into Eq. (4.2) we find

$$\begin{aligned} &\lambda_0 = -2b_2/b_1, \quad \lambda_1 = -b_4/b_1 + b_2 b_5/b_1^2 - b_2^2 b_3/b_1^3 - 2b_2(b_1c_2 - b_2c_1)^2(\ln 2)/b_1^4, \\ &\lambda_2 = b_2(b_1c_2 - b_2c_1)^2/b_1^4. \end{aligned} \tag{4.4}$$

The corresponding Pomeranchukon trajectory is obtained from Eq. (2.5):

$$\alpha(t) = 1 - \frac{b_1 t}{b_2} - (b_1^2 b_4 - b_1 b_2 b_5 + b_2^2 b_3)\frac{t^2}{b_2^3} + (b_1c_2 - b_2c_1)^2 \frac{t^2}{b_1 b_2^2}\ln\left(-\frac{b_1 t}{4b_2}\right) \tag{4.5}$$

In this formulation of the self-consistency equation, there would be a pole trajectory passing through $\alpha(0) = 1$ in the absence of cuts; it is given by the first three terms of Eq. (4.5). The two-Pomeranchukon cut provides a modest modification to this trajectory. b_1/b_2 must be negative so the trajectory rises.

The partial-wave amplitude near $t=0$ and $j=1$ has the form

$$f(t,j) = a_0\left\{b_1 t + b_2(j-1) + b_3 t^2 + b_4(j-1)^2 + b_5 t(j-1) - \frac{b_2}{b_1}[c_1 t + c_2(j-1)]^2 \ln\left(j - 1 + \frac{tb_1}{2b_2}\right)\right\}^{-1}. \tag{4.6}$$

The cuts and poles produced by this amplitude are shown in Figs. 1 and 2. Note that in Eq. (4.6) there is no fixed cut at $j=1$. Its absence is due only to the fact that we have kept the minimal number of terms required to expose the branch point in $t(j)$. In higher approximations the fixed cut will be present. At $t=0$ we can evaluate the Sommerfeld-Watson integral. In this instance the predictions are

$$\sigma_{\text{tot}} \xrightarrow[s\to\infty]{} (\text{const})^2 \frac{a_0}{b_2}\left[1+\frac{c_2^{\,2}}{b_1 \ln(s/s_0)}\right], \qquad \frac{\operatorname{Re} F(s,0)}{\operatorname{Im} F(s,0)} \xrightarrow[s\to\infty]{} -\frac{\pi c_2^{\,2}}{2b_1 \ln^2(s/s_0)}. \tag{4.7}$$

Note that if we insist on Mandelstam's sign of the two-Pomeranchukon cut contribution to σ_{tot} (which is the motivation of this section), then $\operatorname{Re}F/\operatorname{Im}F$ disagrees in sign with experiment in the case of pion-nucleon scattering at Brookhaven energies. Indeed, at present energies both the sign of $\operatorname{Re}F/\operatorname{Im}F$ and the approach to constant cross section from above agree with the Amati-Fubini-Stanghellini sign of the cut. Thus we have a situation where on the one hand we have good theoretical reasons for choosing $b_1<0$, and on the other hand experimental data suggest $b_1>0$ at presently accessible energies. The simplest rationalization is to assert that we have not yet reached asymptotic energies.

One point that bears on this discussion is the following. There is a general connection between the signs of $\operatorname{Re}F/\operatorname{Im}F$ and the behavior of σ_{tot} at high energy if at $t=0$ the partial-wave amplitude has a branch point at $j=1$. We begin with the Sommerfeld-Watson integral:

$$F(s,0) \xrightarrow[s\to\infty]{} \frac{s}{2i}\int_C dj\, s^{\,j-1} f(0,j)\left[i+\tfrac{1}{2}\pi(j-1)\right]. \tag{4.8}$$

Here C is a contour around the cut of $f(0,j)$, and we have approximated $-\cot\frac{1}{2}\pi j$ by $\frac{1}{2}\pi(j-1)$. Since $f(0,j)$ is real analytic, for high s we find

$$\operatorname{Re}F(s,0) \xrightarrow[s\to\infty]{} \frac{\pi s}{2}\frac{d}{d(\ln s)}\frac{\operatorname{Im}F(s,0)}{s} = (\text{const})^2 s \frac{d\sigma_{\text{tot}}}{d(\ln s)}. \tag{4.9}$$

Therefore, any distribution of Pomeranchukon singularities that has σ_{tot} bounded by a constant at high energy will have the signs of $\operatorname{Re}F/\operatorname{Im}F$ and the leading cut contribution to σ_{tot} linked as in Eq. (4.7). On the other hand, the Froissart bound allows σ_{tot} to grow as fast as $(\ln s)^2$ at high energies. This corresponds to a third-order pole at $j=1$, and in this case the pole leads the cut in both $\operatorname{Re}F$ and $\operatorname{Im}F$.

A pole-cut relationship similar to the one we have found has been obtained by Gribov and Migdal,[14] starting from Gribov's Reggeon field theory.[13] One conclusion of these authors is that their Pomeranchukon pole must be "quasi-stable," meaning that the three-Pomeranchukon coupling vanishes when one of the Reggeons has $j=1$ and $t=0$. Quasi-stability is required if the two-Pomeranchukon cut is to have Mandelstam's sign. It is not difficult to see that our assumption that $B(t,j)$ has a second-order pole is equivalent to quasi-stability. The amplitude for Pomeranchukon + Pomeranchukon → Pomeranchukon + Pomeranchukon is $1/D(t,j)$. There is a second-order zero in the residue of the Pomeranchukon pole at $t=0$ in this amplitude, which corresponds to a linear vanishing of the three-Pomeranchukon vertex at $t=0$.[27] Gribov *et al.* have no more insight than we do about the dynamical origin of the pole in B, or quasi-stability. Quasi-stability is simply necessary if we simultaneously demand $\alpha(0)=1$ and the Mandelstam cut sign.

For $t\neq 0$, there is a part of the diffraction cone that is adequately described by the Pomeranchukon pole and two-Pomeranchukon cut. It is specified by Eq. (2.17), with b_0 replaced by $-b_2/b_1$.

V. SUMMARY OF RESULTS

Two main insights have resulted from our investigation. The first is that near $t=0$ the multi-Pomeranchukon cuts are suppressed by the phase-space factor exhibited in Eq. (2.3). This suppression is necessary in order to achieve a self-consistency condition on $\alpha(t)$ near $t=0$ in terms of a finite number of parameters. Likewise, the suppression allows us to give expressions for the partial-wave amplitude near $t=0$ and $j=1$ in terms of a few parameters, and to determine the asymptotic form of the s-channel amplitudes within the momentum transfer interval given in Eq. (2.17).

The second insight is that the two-Pomeranchukon cut must itself be suppressed if a partial-wave amplitude is to be constructed that has the pole leading at $t=0$, and allows the Mandelstam cut sign. This was achieved in Sec. IV by means of a moving pole in B. This additional suppression has dynamical origin that does not lie in t-channel unitarity, and it is clearly of interest to discover its origin. The suppression makes the Pomeranchukon quasi-stable in the terminology of Gribov.

In Sec. IV we have constructed the simplest partial-wave amplitude that is consistent with t-channel unitarity, and which yields the Mandelstam cut sign. Although we have not listed all the possibilities investigated, we did try all the simpler forms of A and B before settling on Eq. (4.1) as the simplest parametrization consistent with our objective. Here we wish only to point out that one important

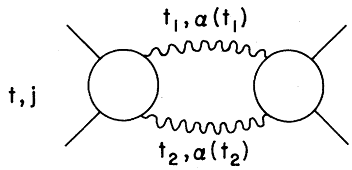

FIG. 4. Unitarity diagram for the contribution of two Pomeranchukon poles to the discontinuity across the four-particle cut. The energy and angular momentum are indicated.

case was left out. Its nature can be indicated by reviewing the mechanism by which Regge cuts are generated. In Fig. 4 we show the contribution of intermediate states of two Pomeranchukon poles to the discontinuity of $f(t,j)$ across the four-particle cut. A Regge pole occurs at $j=\alpha(t_1)+\alpha(t_2)-1$, and since t_1 and t_2 take on continuous values in the integration over phase space, the pole is drawn out into a cut. The branch point occurs where $\alpha(t_1)+\alpha(t_2)-1$ achieves its extremum, with t_1 and t_2 constrained by energy conservation, $t_1^{1/2}+t_2^{1/2}=t^{1/2}$. This yields the familiar trajectory $2\alpha(\frac{1}{4}t)-1$. The area of the (t_1, t_2) phase-space plane that contributes to the discontinuity across the Regge cut near its tip is inversely proportional to the second derivative of $\alpha(t_1)+\alpha((t^{1/2}-t_1^{1/2})^2)$ with respect to t_1 at $t=\frac{1}{4}t_1$, which is $2\alpha''(\frac{1}{4}t)+4\alpha'(\frac{1}{4}t)/t$. This expression vanishes identically if $\alpha(t)=\alpha(0)+\gamma t^{1/2}$, which is just the Schwarz cut if $\alpha(0)=1$. For such trajectories multi-Pomeranchukon phase space no longer vanishes as rapidly near the tip of the cut as we have assumed, and our investigation must be repeated. Near $t=0$ there are phase-space modifications even if there are corrections to the Schwarz cut; it is only necessary that the leading terms in $\alpha(t)$ be $\alpha(0)+\gamma t^{1/2}$. As mentioned in Sec. I, the Schwarz trajectory is interesting because it generates only two Regge cuts, and because it arises from the eikonalization of a pole with $\alpha(0)>1$. We shall investigate this remaining possibility in another paper.

---

*This work is supported in part through funds provided by the Atomic Energy Commission under Contract No. AT(30-1)-2098.

[1]M. Froissart, Phys. Rev. 123, 1053 (1961).

[2]J. V. Allaby *et al.*, Phys. Letters 30B, 500 (1969).

[3]H. Cheng and T. T. Wu, Phys. Rev. Letters 24, 1456 (1970).

[4]S. Mandelstam, Nuovo Cimento 30, 1127 (1963); 30, 1148 (1963).

[5]V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. 139, B184 (1965).

[6]A. A. Anselm, Ya. I. Azimov, G. S. Danilov, I. T. Dyatlov, and V. N. Gribov, Ann. Phys. (N.Y.) 37, 227 (1966); J. B. Bronzan and C. E. Jones, Phys. Rev. 160, 1494 (1967).

[7]If one assumes there is a simple pole with $\alpha(0)=1$, the measurement of $\mathrm{Re}F(s,0)/\mathrm{Im}F(s,0)$ is a particularly easy way to isolate the contribution of Regge cuts, and to estimate their strength relative to the pole. This is discussed in Sec. IV.

[8]T. T. Wu and C. N. Yang, Phys. Rev. 137, B708 (1965); N. Byers and C.N. Yang, *ibid.* 142, 976 (1966); T. T. Chou and C. N. Yang, in *Proceedings of the Second International Conference on High-Energy Physics and Nuclear Structure, Rehovoth, Israel, 1967,* edited by G. Alexander (North-Holland, Amsterdam, The Netherlands, 1967), pp. 348–359; Phys. Rev. 170, 1591 (1968); Phys. Rev. Letters 20, 1213 (1968); Phys. Rev. 175, 1832 (1968); J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, *ibid.* 188, 2159 (1969).

[9]G. F. Chew and A. Pignotti, Phys. Rev. 176, 2112 (1968); G. F. Chew, M. L. Goldberger, and F. E. Low, *ibid.* 180, 1577 (1969); M. L. Goldberger, C.-I Tan, and J. M. Wang, *ibid.* 184, 1920 (1969).

[10]R. C. Hwa, Phys. Rev. D1, 1790 (1970).

[11]J. H. Schwarz, Phys. Rev. 167, 1342 (1968); T. Sawada, *ibid.* 165, 1848 (1968); R. E. Mickens, Nuovo Cimento 56A, 799 (1968).

[12]V. N. Gribov, I. Yu. Kobsarev, V. D. Mur, L. B. Okun, and V. S. Popov, Phys. Letters 32B, 129 (1970); A. A. Anselm, G. S. Danilov, I. T. Dyatlov, and E. M. Levin, Yadern. Fiz. 11, 896 (1970) [Soviet J. Nucl. Phys. 11, 500 (1970)]; J. Finkelstein, Phys. Rev. Letters 24, 172 (1970); R. Oehme, Phys. Rev. D 3, 3217 (1971).

[13]V. N. Gribov, Zh. Eksperim. i Teor. Phys. 53, 654 (1967) [Soviet Phys. JETP 26, 414 (1968)].

[14]V. N. Gribov and A. A. Migdal, Yadern. Fiz. 8, 1002 (1968) [Soviet J. Nucl. Phys. 8, 583 (1969)]; 8, 1213 (1968) [8, 703 (1969)]; Zh. Eksperim. i Teor. Phys. 55, 1498 (1968) [Soviet Phys. JETP 28, 784 (1969)]; V. N. Gribov, Yadern. Fiz. 9, 424 (1969) [Soviet J. Nucl. Phys. 9, 246 (1969)].

[15]R. Oehme, in *Strong Interactions and High Energy Physics,* edited by R. G. Moorhouse (Oliver and Boyd, London, 1964), p. 148.

[16]R. Oehme, Phys. Letters 30B, 414 (1969).

[17]R. Oehme, Phys. Letters 32B, 573 (1970); Phys. Rev. D2, 801 (1971); J. S. Ball and F. Zachariasen, Phys. Rev. Letters 23, 346 (1969); P. Kaus and F. Zachariasen, Phys. Rev. D1, 2962 (1970); J. S. Ball, G. Marchesini, and F. Zachariasen, Phys. Letters 32B, 583 (1970).

[18]According to Ref. 5, $\alpha'(\frac{1}{4}t)$ should be replaced by $\{\alpha'(\frac{1}{4}t)[\alpha'(\frac{1}{4}t)+(\frac{1}{2}t)\alpha''(\frac{1}{4}t)]\}^{1/2}$. This leads to trivial modifications, so we use the simpler term.

[19]The factor $t'(j)$ in the argument of the logarithm can be replaced by a constant, or various other expressions, with trivial changes in our results. This shows that our results are insensitive to fixed j-cuts in B that induce such replacements. Gribov's unitarity study does not distinguish among the choices.

[20]In Ref. 16, Oehme points out that α becomes singular only when two or more poles collide, but not, for example, when a pole and cut collide. We satisfy Oehme's stipulation because an infinite number of poles collide at $t=0$.

[21]D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento 26, 896 (1962).
[22]G. F. Chew and W. R. Frazer, Phys. Rev. 181, 1914 (1969).
[23]J. Finkelstein and M. Jacob, Nuovo Cimento 56A, 681 (1968).
[24]K. J. Foley, R. S. Jones, S. J. Lindenbaum, W. A. Love, S. Ozaki, E. D. Plattner, C. A. Quarles, and E. H. Willen, Phys. Rev. 181, 1775 (1969).
[25]S. Sugiyama, in *Proceedings, United States-Japan Seminar on Differential and Functional Equations,* edited by W. A. Harris, Jr. and Y. Sibuya (Benjamin, New York, 1967).
[26]A. Erdélyi, *Asymptotic Expansions* (Dover, New York, 1956), p. 8.
[27]A vanishing of the three-Pomeranchukon vertex at $t=0$ should be distinguished from the identical vanishing of this vertex. The latter possibility was discussed in Sec. II when we discussed whether the Pomeranchukon pole and cuts interact.

PHYSICAL REVIEW D VOLUME 4, NUMBER 4 15 AUGUST 1971

Quark Model of Dual Pions*

A. Neveu† and John H. Schwarz
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 27 April 1971)

Interacting pseudoscalar pions are incorporated into Ramond's model of free dual fermions. By considering the emission of $N-1$ pions and factorizing in the quark-antiquark channel, we recover the same N-pion amplitudes as were proposed in a previous paper.

Ramond[1] has recently proposed a model of free dual fermions related to the Dirac equation by a correspondence principle[2] analogous to one relating the conventional dual-resonance model to the Klein-Gordon equation. This fermion model possesses an infinite set of Ward identities that probably provides for the cancellation of all ghosts. Another recent development was the discovery of a dual model of pions[3] having a number of realistic features not shared by the conventional dual-resonance model. Subsequently, the algebraic properties responsible for the successes of this model (including the apparent absence of ghosts) have been obtained.[4]

In this paper we show that there is a deep connection between the fermion and pion models. Specifically, we construct the amplitude for emitting $N-1$ pions from a fermion line [Fig. 1(a)]. Requiring the gauge algebra of the fermion sector to hold in the presence of interaction imposes the condition $m_\pi^2=-\frac{1}{2}$, the same condition required for the gauges in the meson sector. By factorizing at the first pole in the quark-antiquark channel [Fig. 1(b)], we obtain the same N-pion amplitude as in Ref. 3.

Let us first review the algebra of Ramond's fermion model. In addition to the usual harmonic-oscillator operators[5] satisfying

$$[\alpha_m^\mu, \alpha_n^\nu] = -mg^{\mu\nu}\delta_{m,-n},$$

Ramond introduces anticommuting operators satisfying

$$\{d_m^\mu, d_n^\nu\} = -g^{\mu\nu}\delta_{m,-n}$$

and

$$[d_m^\mu, \alpha_n^\nu] = 0,$$

where m and n are integers, $d_{-m}^\mu = d_m^{\mu\dagger}$, and $d_0^\mu = -(i/\sqrt{2}\,)\gamma_5\gamma^\mu$. Then, introducing

$$P^\mu(\tau) = (\tfrac{1}{2})^{1/2}\sum_{n=-\infty}^{\infty}\alpha_n^\mu e^{-in\tau}$$

and

$$\Gamma^\mu(\tau) = i\sqrt{2}\,\gamma_5\sum_{n=-\infty}^{\infty} d_n^\mu e^{-in\tau},$$

one finds that the operators

$$L_n = -\langle e^{in\tau}\colon P^2(\tau)\colon\rangle + \tfrac{1}{4}i\left\langle e^{in\tau}\colon\Gamma(\tau)\cdot\frac{d}{d\tau}\Gamma(\tau)\colon\right\rangle$$

satisfy the Virasoro algebra[6]

$$[L_m, L_n] = (m-n)L_{m+n}.$$

The wave equation for a fermion state is

$$(F_0 - m)|\psi\rangle = 0,$$

where m is the mass of the spin-$\frac{1}{2}$ ground-state fermion and

$$F_n = \langle e^{in\tau}\Gamma(\tau)\cdot P(\tau)\rangle.$$

Ramond's subsidiary ghost-eliminating conditions are

$$F_n|\psi\rangle = 0, \quad n=1,2,3,\ldots \tag{1}$$

or, equivalently,