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[22]A summary of the various limits may be in order.

B limit: $q_0 \to i\infty$, fixed p, $\vec{q}$.
D limit: $\nu \to \infty$, fixed $\omega \equiv -q^2/2\nu$.
Q limit: $q^2 \to \infty$, fixed ν.
R limit: $\nu \to \infty$, fixed q^2.
S limit: $\nu \to \infty$, fixed $s = q^2 + 2\nu + m^2$.

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Light-Cone Structure of Current Commutators in the Gluon–Quark Model*

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The formal structure of current commutators near the light cone (currents separated by nearly lightlike distances) is studied for quark models with interactions mediated by SU(3)×SU(3) singlet vector or scalar mesons. The idea in the end is to abstract the general structure features, and thereafter to discard the specifics of the underlying model. The approach adopted here is formal, in the sense that all manipulations are based on *canonical* equal-time commutation relations and on the corresponding canonical equations of motion. The light-cone commutator is expressed in terms of certain bilocal operators and has a definite tensor and SU(3)×SU(3) structure that forms the heart of the abstraction. The analysis is greatly facilitated by a theorem which shows that the gluons can be treated in the external-field approximation for purposes of determining the leading light-cone singularities. It is an important result that the tensor and SU(3)×SU(3) structure of the light-cone commutator turns out to be the same for the gluon models as for the free-quark model. The practical implications of this structure, already discussed by Gell-Mann, are therefore preserved in the gluon model. We review the applications. Some of the delicate points connected with current conservation are discussed, and it is shown how to write the commutator in a form where current conservation is manifest. We also discuss the structure of the vacuum matrix elements of the commutator. Finally, we investigate the commutation relations among the bilocal operators and show that the algebra closes for bilocals defined on a single lightlike ray. In an Appendix our various results are compared with those obtained by canonical light-cone quantization procedures.

I. INTRODUCTION

Theoretical interest in the properties of current commutators near the light cone has been greatly stimulated recently by the striking results of the SLAC-MIT experiments on deep-inelastic electron scattering.[1] In the Bjorken scaling limit the amplitude is controlled by the structure of the electromagnetic current commutators near the light cone; and similarly for deep-inelastic neutrino processes, where the weak semileptonic currents come into play. In varying degrees, parton models and more abstract considerations based on the algebra of equal-time current commutators all bear on this problem of light-cone structure. The current-algebra[2–4] and parton-model[5,6] arguments lead to certain sum rules on the structure functions of inelastic lepton-hadron scattering. In the most popular version, one identifies the partons with quarks; and for the analysis based on current algebra, one often adopts the naive equal-time commutators (ETC) of the gluon-quark model (currents bilinear in quark fields, strong quark interactions mediated by SU(3)-singlet vector "gluons").

To a certain extent the two approaches have led to identical results; and in no case has any conflict arisen. But until now the parton model has been the more predictive. For example, in the parton model one obtains the relation[6]

$$12[F_1^{\gamma p}(\omega)-F_1^{\gamma n}(\omega)]=[F_3^{\nu p}(\omega)-F_3^{\nu n}(\omega)],$$

whereas in the current-algebra approach the corresponding equality has so far been established only for the zeroth moments (integrals over ω) of the above structure functions.[7]

An inclusive framework for the discussion of light-cone structure is provided by Wilson's ideas on operator-product expansions,[8] when these are extended to the light cone.[9] In particular, Gell-Mann[10] has recently pointed out that all of the parton-model sum rules can be gotten by abstracting from the *free*-quark model its tensor and SU(3) $\times$SU(3) structure near the light cone. This abstraction is motivated by the indications from the SLAC-MIT experiments that the operators appearing in the light-cone expansion have canonical "dimensions," i.e., physical dimensions; also by the indication that σ_L/σ_T may be vanishing in the Bjorken scale limit. These things are consequences of the light-cone structure of the free-quark model. On the other hand they are known to be incorrect, in perturbation theory, when nonsuper renormalizable interactions are switched on.[11] In this sense, to quote Gell-Mann, it appears that Nature "reads the free-field theory books."

In this paper we consider whether the light-cone structure of the free-quark model is *formally* preserved when interactions of the neutral vector gluon sort are switched on. The qualification "formal" means that, in analyzing the light-cone structure, we make use of the canonical equal-time commutation relations and of the formal equations of motion. As already said, these formal manipulations are falsified in a renormalized perturbation treatment of the light-cone structure.[11] That is, although the formal results would obtain in a cutoff theory, they would be altered beyond recognition when the cutoff is removed and renormalization carried out. Our idea, however, is to abstract from the gluon model its formal light-cone structure, after which we discard the model. We shall find on the light cone that the tensor and SU(3)$\times$SU(3) structure of the free-quark model is preserved in the gluon model; and for the latter model we shall determine the operators that appear in the light-cone expansion. These formal procedures are motivated by the SLAC-MIT results, which suggest that Nature, whatever else she does, may not pay too much attention to perturbation theory.

In Sec. II we review some general matters pertaining to the short-distance and light-cone expansion of current commutators. The explicit light-cone structure of the free-quark model is then exhibited in Sec. III. The problem of preserving current conservation in the presence of interactions is raised here, in connection with the idea that the model be discarded, but its abstract light-cone structure retained for the real world. The slight modifications that have to be made are discussed. In Sec. IV we turn to the gluon model. We show that it is sufficient to treat the gluon as a c-number field for purposes of determining the formal light-cone structure. We are thus led to consider, near the light cone, the Green's function for a quark field interacting with an external gluon field. This problem is solved in closed form, to all orders in the coupling constant. The resulting light-cone structure of the current commutators is discussed in Sec. V, along with some of the implications which follow from this structure. Finally, in Sec. VI we discuss the light-cone commutators of the "bilocal" operators that have appeared in the preceding section.

II. OPERATOR-PRODUCT EXPANSION AT SHORT AND LIGHTLIKE DISTANCES

A convenient framework for describing the short-distance behavior of local-operator pairs has been introduced by Wilson.[8] The idea is to express the product (or commutator, say) of the local operators as an expansion in local operators, with singular c-number functions as coefficients. Thus, the commutator of two scalar operators has the short-distance expansion

$$[A(x),B(y)]=\sum_n C_n(x-y)(x-y)^{\mu_1}\cdots(x-y)^{\mu_{J_n}}\times\Theta^{(n)}_{\mu_1\cdots\mu_{J_n}}(\tfrac{1}{2}(x+y)),\qquad(1)$$

where $\Theta^{(n)}_{\mu_1\cdots\mu_{J_n}}$ is a local tensor operator of rank J_n. The singularity of the commutator is contained in the c-number functions $C_n(x-y)$. Wilson supposes that the degree of singularity is determined by the "dimensions" of the operators A, B, and $\Theta^{(n)}$, as if scale invariance were a good symmetry. Operationally this means that for each operator one can assign a dimension in such a way that C_n is a homogeneous function of $(x-y)$, of degree $-d(C_n)$, i.e.,

$$C_n(x)=(x^2-i\epsilon x_0)^{-d(C_n)/2}-(x^2+i\epsilon x_0)^{-d(C_n)/2}$$

and

$$d(C_n)=d_A+d_B-d(\Theta^{(n)})+J_n\,.\qquad(2)$$

(Note that we measure dimension in units of mass.) In a free-field theory, or in an interacting theory with cutoff, the dimension of an operator is equal to its physical dimension, so that, for example, $d[\bar{\psi}\psi]=3$, $d[\bar{\psi}\partial_\nu\psi]=4$, etc. Since our discussion of interacting theories is to be formal, we take all dimensions to be physical.[12]

The commutator at short distances receives dominant contributions only from operators $\Theta^{(n)}$ with dimension $d_n \leq d_A + d_B$. Operators with larger dimension are relevant for the short-distance behavior of derivatives of the commutator. In particular, it is evident that the short-distance expansion is fully specified by the full set of *equal-time* commutators

$$\left[\frac{\partial^k}{\partial t^k} A(\vec{x}, t), B(\vec{y}, t)\right], \quad k=0,1,2,\ldots,$$

and conversely. For the kth derivative it is the operators $\Theta^{(n)}$ with $d_n \leq d_A + d_B + k$ that contribute.

We now want to turn to the structure of the commutator near the light cone,[9] $(x-y)^2 \approx 0$. Since here it is no longer true that all four components of $(x-y)$ are small, the dominant contributions will come from operators with $d_n - J_n \leq d_A + d_B$. We understand here that J_n is the maximum spin of the operator $\Theta^{(n)}_{\mu_1 \cdots \mu_{J_n}}$. Evidently it is no longer the dimension alone that determines the importance of an operator near the light cone, but rather the difference between dimension and spin. We shall call this quantity the "*twist*" (τ) of an operator:

$$\tau_n \equiv d_n - J_n .$$

Thus, for example, a quark current $\bar{\psi}\gamma_\mu\psi$ has twist $\tau=2$, as does the operator $\bar{\psi}(x)\gamma_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_r}\psi(x)$.

It is convenient to sum all local operators of equal twist into a bilocal operator. Thus, with

$$\Delta = \tfrac{1}{2}(x-y), \quad X = \tfrac{1}{2}(x+y),$$

we write the bilocal operator

$$\Theta^{(\tau)}(X,\Delta) = \sum_n \Delta^{\mu_1}\cdots\Delta^{\mu_{J_n}}\Theta^{(n)}_{\mu_1\cdots\mu_{J_n}}(X), \tag{3}$$

where the sum is over all the local operators $\Theta^{(n)}$ with common twist τ. In this way we expand the commutator as a sum over bilocal operators of different twist

$$[A(x), B(y)] = \sum_\tau C^{(\tau)}(\Delta)\Theta^{(\tau)}(X,\Delta). \tag{4}$$

The most singular contributions near the light cone come from the bilocal operators with smallest twist τ, with $C^{(\tau)}(\Delta) \sim (\Delta^2)^{-(d_A+d_B-\tau)/2}$. Of course the bilocal operator of lowest twist does not contain the equal-time commutator of $A(x)$ and all its derivatives with $B(y)$. Only the highest-spin component of each of the equal-time commutators is contained in the most singular term on the light cone. Conversely, the highest-spin components of the equal-time commutators completely determine the lowest-twist operators and, therefore, the leading light-cone singularity.[13] In the following discussion we will make much use of this equivalence between the light-cone structure and the formal properties of equal-time commutators.

It is instructive to recall what the SLAC-MIT experiments indicate for the light-cone commutator structure of electromagnetic currents. What is measured is the diagonal nucleon matrix element of the commutator

$$\int dx\, e^{iq\cdot x}\langle p|[J_\mu^{\text{em}}(\tfrac{1}{2}x), J_\nu^{\text{em}}(-\tfrac{1}{2}x)]|p\rangle = 2\pi\frac{p_\mu p_\nu}{m^2}W_2(q^2, q\cdot p)+\cdots. \tag{5}$$

The evidence[1] appears to support Bjorken's scaling hypothesis, according to which $q\cdot p\, W_2$ approaches a nontrivial limit as $-q^2\to\infty$, $q\cdot p\to\infty$, with $\omega=-q^2/2q\cdot p$ fixed; i.e.,

$$q\cdot p\, W_2 \to mF_2(\omega). \tag{6}$$

This implies that, near the light cone, the matrix element has the structure[13,14]

$$\langle p|[J_\mu^{\text{em}}(\tfrac{1}{2}x), J_\nu^{\text{em}}(-\tfrac{1}{2}x)]|p\rangle = -\frac{2i}{\pi}\epsilon(x_0)\delta(x^2)\frac{p_\mu p_\nu}{m}\int_0^1 d\omega\cos[\omega x\cdot p]F_2(\omega)+\cdots. \tag{7}$$

In turn, this suggests that the commutator has a light-cone expansion dominated by a twist-2 bilocal operator $\Theta_{\mu\nu}(X,\Delta)$, whose matrix element is

$$\langle p|\Theta_{\mu\nu}(0,x)|p\rangle = p_\mu p_\nu\int_0^1 d\omega\cos[\omega x\cdot p]F_2(\omega)+\cdots. \tag{8}$$

The local operators $\Theta_{\mu\nu,\alpha_1\cdots\alpha_{2m}}(0)$ appearing in the short-distance expansion of the bilocal $\Theta_{\mu\nu}(0,x)$ have matrix elements given by

$$\langle p|\Theta_{\mu\nu,\alpha_1\cdots\alpha_{2m}}(0)|p\rangle = p_\mu p_\nu p_{\alpha_1}\cdots p_{\alpha_{2m}}\frac{(-1)^m}{(2m)!}\int_0^1 d\omega\,\omega^{2m}F_2(\omega)+\cdots. \tag{9}$$

We may immediately remark that the free-quark model has such a family of twist-2 operators

$$\Theta_{\mu_1 \cdots \mu_n}(x) = \overline{\psi}(x)\gamma_{\mu_1}\overleftrightarrow{\partial}_{\mu_2} \cdots \overleftrightarrow{\partial}_{\mu_n}\psi(x) .$$

With respect to isospin, the experiments indicate that the leading light-cone operators have both isoscalar and isovector parts. Finally, the experiments are compatible with the vanishing of σ_L/σ_T in the scaling limit. This accords with the light-cone tensor structure of the free-quark model.

III. LIGHT-CONE STRUCTURE IN THE FREE-QUARK MODEL

The commutator of two local operators can of course be calculated exactly in a free-field theory. In the free-quark model, let us define the currents

$$J_\mu^a(r;x) = \overline{\psi}(x)(1+r\gamma_5)\gamma_\mu \tfrac{1}{2}\lambda^a \psi(x), \quad r=\pm 1, \quad a=0,1,\ldots,8, \tag{10}$$

and abstract the SU(3)×SU(3) and tensor structure of current commutators near the light cone, following Gell-Mann. In carrying out the computation we encounter the anticommutator $\{\psi(x), \overline{\psi}(y)\} = -iS(x-y)$. Since the quark mass terms do not affect the leading light-cone singularity, we take

$$-iS(x-y) \approx \gamma_\rho \frac{\partial}{\partial x_\rho} D(x-y), \qquad D(x) = \frac{1}{2\pi}\epsilon(x_0)\delta(x^2). \tag{11}$$

Then

$$[J_\mu^a(r;x), J_\nu^b(r';y)] = 0, \quad r \neq r' \tag{12}$$

and

$$\begin{aligned}[J_\mu^a(r,x), J_\nu^b(r;y)] = {} & i f_{abc}\{J_\mu^c(r,S;X,\Delta)g_{\nu\alpha} + J_\nu^c(r,S;X,\Delta)g_{\mu\alpha} - J_\alpha^c(r,S;X,\Delta)g_{\mu\nu} - ir\epsilon_{\mu\nu}^{\lambda\alpha}J_\lambda^c(r,A;X,\Delta)\}\frac{1}{8}\frac{\partial}{\partial\Delta_\alpha}D(\Delta) \\ & + d_{abc}\{J_\mu^c(r,A;X,\Delta)g_{\nu\alpha} + J_\nu^c(r,A;X,\Delta)g_{\mu\alpha} - J_\alpha^c(r,A;X,\Delta)g_{\mu\nu} - ir\epsilon_{\mu\nu}^{\lambda\alpha}J_\lambda^c(r,S;X,\Delta)\}\frac{1}{8}\frac{\partial}{\partial\Delta_\alpha}D(\Delta).\end{aligned} \tag{13}$$

We have defined here the symmetric Hermitian $J(S)$ and antisymmetric anti-Hermitian $J(A)$ bilocal operators of twist 2:

$$J_\mu^c\left(r, {S \atop A}; X, \Delta\right) = \tfrac{1}{2}[\overline{\psi}(x)(1+r\gamma_5)\gamma_\mu \tfrac{1}{2}\lambda^c\psi(y) \pm \overline{\psi}(y)(1+r\gamma_5)\gamma_\mu\tfrac{1}{2}\lambda^c\psi(x)], \tag{14}$$

and we have defined $X = \frac{1}{2}(x+y)$, $\Delta = \frac{1}{2}(x-y)$. In terms of local-operator expansions it is evident, for example, that

$$J_\mu^a(r,S;X,\Delta) = \sum_{m=0,2,4,\ldots} \frac{1}{m!}\Delta^{\alpha_1}\cdots\Delta^{\alpha_m}\overline{\psi}(X)\tfrac{1}{2}\lambda^a(1+r\gamma_5)\overleftrightarrow{\partial}_{\alpha_1}\cdots\overleftrightarrow{\partial}_{\alpha_m}\psi(X), \tag{15}$$

where $\overleftrightarrow{\partial}_\alpha \equiv \overleftarrow{\partial}_\alpha - \overrightarrow{\partial}_\alpha$.

The idea now is to abstract from the model the SU(3)×SU(3) and tensor structure in Eq. (13), relinquishing the specific form of the bilocal operators given in Eq. (14) and relinquishing also any use of the free-quark equations of motion. The temptation, that is, is to take the light-cone structure embodied in Eq. (13) as applying to the real world, with interactions. Although we cannot then claim to know the matrix elements of the bilocal operators appearing in this equation, the tensor and SU(3)×SU(3) structure is nevertheless predictive. It allows one to derive all those results of the (quark) parton model which are independent of detailed assumptions on the distributions of the partons in a hadron. We shall enumerate these results later on, after we have shown that the light-cone structure of Eq. (13) also obtains in models with interactions switched on.

However, there is one minor complication that has to be dealt with before this light-cone structure can be adopted. One has to ensure that the leading light-cone behavior is consistent with current conservation. In the real world, of course, all of the SU(3)×SU(3) currents are in fact not conserved. But where the symmetry breaking is due to mass terms in the Lagrangian, as in the quark-gluon model, one can treat all currents as conserved insofar as the leading light-cone singularity is at issue. This is because, formally, the symmetry breaking will give rise to terms explicitly proportional to the quark-mass matrix, and these will necessarily have larger twist than the SU(3)×SU(3)-symmetric operators. Indeed, if we are going to abstract the SU(3)×SU(3) light-cone structure of the *free*-quark model, we must insist that all the currents be conserved to leading order on the light cone.

In the (literally) free-quark model, Eq. (13) is indeed consistent with current conservation, although not manifestly so. To confirm the conservation properties, we have to make use of the specifics of Eq. (14) and of the equations of motion for free quarks. To see how this goes, let us write Eq. (13) in the more compact form

$$[J^a_\mu(r,x), J^b_\nu(r,y)] = \Theta^{ab}_{\mu\nu\alpha}(X,\Delta)\partial^\alpha D(\Delta)\,, \tag{16}$$

where the structure of $\Theta^{ab}_{\mu\nu\alpha}$ can be inferred from Eq. (13). Without making any use of equations of motion, or use of Eq. (14), we can directly verify that

$$\Theta^{ab}_{\mu\nu\alpha}\,\partial^\mu\partial^\alpha D(\Delta) = \Theta^{ab}_{\mu\nu\alpha}\partial^\nu\partial^\alpha D(\Delta) = 0\,. \tag{17}$$

To verify current conservation, it is only necessary then to show that

$$\frac{\partial}{\partial x_\mu}\Theta^{ab}_{\mu\nu\alpha} = \frac{\partial}{\partial y_\nu}\Theta^{ab}_{\mu\nu\alpha} = 0\,. \tag{18}$$

Here we make use of Eq. (14) and of the quark field equation of motion. It is then easy to show that

$$\frac{\partial}{\partial X^\mu}J^a_\mu\left(r, {S \atop A}; X, \Delta\right) = \frac{\partial}{\partial \Delta^\mu}J^a_\mu\left(r, {S \atop A}; X, \Delta\right) = 0\,, \tag{19}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial X^\mu}J^a_\nu\left(r, {S \atop A}\right) - \frac{\partial}{\partial X^\nu}J^a_\mu\left(r, {S \atop A}\right) = -ir\epsilon^{\sigma\eta}_{\mu\nu}\,\frac{\partial}{\partial \Delta^\eta}J^a_\sigma\left(r, {A \atop S}\right),\\
&\frac{\partial}{\partial \Delta^\mu}J^a_\nu\left(r, {S \atop A}\right) - \frac{\partial}{\partial \Delta^\nu}J^a_\mu\left(r, {S \atop A}\right) = -ir\epsilon^{\sigma\eta}_{\mu\nu}\,\frac{\partial}{\partial X^\eta}J^a_\sigma\left(r, {A \atop S}\right).
\end{aligned} \tag{20}$$

Equation (18) follows readily from Eqs. (19) and (20). But these latter will no longer remain true when we abstract the light-cone structure to a nontrivial interacting theory. Indeed, if $(\partial/\partial X^\mu)J_\mu(X,\Delta) = 0$ were true, this would imply the existence of an infinite number of local conserved operators, with arbitrarily large spins. But this is possible only in a trivial theory, without interactions.

We propose to overcome this difficulty in the following way. We stick with the (literally) free-quark model a while longer, using its equation of motion to re-express the leading light-cone structure in a form where current conservation is manifest, without further reference to the equations of motion. To this end, let us rewrite Eq. (16) in the form

$$[J^a_\mu(r;x), J^b_\nu(r;y)] = g_{\mu\mu'}\Box_\Delta g_{\nu\nu'}\Box_\Delta \Box_\Delta^{-2}[\Theta^{ab}_{\mu'\nu'\alpha}\partial^\alpha D(\Delta)]\,, \tag{21}$$

where

$$\Box_\Delta = \frac{\partial}{\partial\Delta^\mu}\,\frac{\partial}{\partial\Delta_\mu}\,.$$

Now define a new bilocal operator $\tilde\Theta^{ab}_{\mu\nu\alpha}$, such that

$$\Box_\Delta^2[\tilde\Theta^{ab}_{\mu\nu\alpha}\,\partial^\alpha\tilde D(\Delta)] = \Theta^{ab}_{\mu\nu\alpha}\partial^\alpha D(\Delta)\,, \tag{22}$$

where

$$\begin{aligned}
&\Box_\Delta^2\partial^\alpha\tilde D(\Delta) = \partial^\alpha D(\Delta)\,,\\
&\tilde D(\Delta) = (1/64\pi)\epsilon(\Delta_0)\Delta^2\Theta(\Delta^2)\,.
\end{aligned} \tag{23}$$

Since $\Theta^{ab}_{\mu\nu\alpha}(X,\Delta)$ is a power series in Δ, with local operators at X as coefficients, this defines $\tilde\Theta^{ab}_{\mu\nu\alpha}$ to be a bilocal operator of the same nature. Using

$$\Box_\Delta^2 = 16[\Box_x\Box_y - (\partial_x\cdot\partial_X)(\partial_y\cdot\partial_X)] + 3\Box_X^2 - 2\Box_X\Box_\Delta\,, \tag{24}$$

we can replace Eq. (21) by

$$\begin{aligned}
[J^a_\mu(r;x), J^b_\nu(r;y)] \underset{\Delta^2\to 0}{\simeq} \Big\{&16\left(g_{\mu\mu'}\Box_x - \frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial x^{\mu'}}\right)\left(g_{\nu\nu'}\Box_y - \frac{\partial}{\partial y^\nu}\,\frac{\partial}{\partial y^{\nu'}}\right)\\
&- 16\left(g_{\mu\mu'}\partial_x\cdot\partial_X - \frac{\partial}{\partial X^\mu}\,\frac{\partial}{\partial x^{\mu'}}\right)\left(g_{\nu\nu'} - \partial_{y'}\partial_X - \frac{\partial}{\partial X^\nu}\,\frac{\partial}{\partial y^\nu}\right)\Big\}\tilde\Theta^{\mu'\nu'\alpha}_{ab}\,\partial_\alpha\tilde D(\Delta)\,.
\end{aligned} \tag{25}$$

We have added terms which are identically zero in the free-quark model [Eq. (19)] and terms which are less singular on the light cone than the leading terms. In the free-quark model, therefore, Eq. (25) is equivalent on the light cone to Eq. (13); but it is manifestly consistent with current conservation and can therefore be abstracted as a model for the light-cone commutator in the real world.

It would be possible, by use of Eqs. (22) and (23), to derive explicit expressions for the bilocal operators appearing in $\tilde\Theta^{ab}_{\mu\nu\alpha}$, but there is little merit in this. The net effect of the projection operators in Eq. (25) is easily taken into account in momentum space. One simply multiplies the Fourier transforms of the right-hand side of Eq. (16),

$$\int dx\,dy\,e^{iq_1\cdot x-iq_2\cdot y}\,\Theta^{ab}_{\mu\nu\alpha}(X,\Delta)\partial^\alpha D(\Delta)\,,$$

by the projection operator

$$(g_{\mu\mu'}q_1^2-q_{1,\mu}q_{1,\mu'})(g_{\nu\nu'}q_2^2-q_{2,\nu}q_{2,\nu'})(Q^2)^{-2}+(g_{\mu\mu'}q_1\cdot T-T_\mu q_{1,\mu'})(g_{\nu\nu'}q_2\cdot T-T_\nu q_{2,\nu'})(Q^2)^{-2},\tag{26}$$

where

$$Q=\tfrac{1}{2}(q_1+q_2),\quad T=q_1-q_2\,.$$

IV. FORMAL LIGHT-CONE STRUCTURE OF INTERACTING QUARK MODELS

We turn now to the light-cone structure of current commutators in the presence of interactions; and in particular, we take up the gluon-quark model. Here the currents are exactly as in the free-quark model. But the quark fields now couple to a neutral [SU(3) singlet] vector-boson field B_μ, according to the interaction

$$\mathcal{L}_{\rm int}(x)=g\overline{\psi}(x)\gamma_\mu\psi(x)B_\mu(x)\,.\tag{27}$$

It is important for our purposes to define the model in such a way that the gluon-field components and their first time derivatives have scale invariant equal-time commutators, i.e., in such a way that the gluon mass does not appear in these commutators. How this is achieved is spelled out in Appendix A. Although we focus on the gluon model, it will be evident that our results will apply to the more general case in which the quarks couple to SU(3) singlet scalar and pseudoscalar fields:

$$\mathcal{L}'_{\rm int}=g_P\overline{\psi}\gamma_5\psi\phi+g_S\overline{\psi}\psi\sigma\,.\tag{28}$$

As emphasized in the Introduction, our manipulations on the model are deliberately formal in character, based on the naive equations of motion and the canonical equal-time commutation relations. These manipulations are valid in a cutoff version of the model, but they are not justified in a perturbation treatment with infinite renormalization.

Just as in the free-quark model, discussion of the light-cone structure of the current commutators begins with the problem of finding the leading light-cone singularities of the quark-field anticommutator

$$\{\psi(x),\overline{\psi}(y)\}=-iS(x,y)\,.\tag{29}$$

In the expansion of $S(x,y)$ in terms of bilocal operators, the most singular terms can be reconstructed from the equal-time anticommutators of the quark fields and their time derivatives. This follows from the general discussion in Sec. II. In fact, the local operators appearing in the expansion of these dominant bilocal operators are just the lowest-twist components of the equal-time anticommutators of the quark fields and their time derivatives. We shall use this fact to prove the following theorem. *In order to construct the leading light-cone singularity of $S(x,y)$, it is sufficient to treat the gluon as an external c-number field.* Namely, for the equal-time anticommutator

$$\left\{\frac{\partial^n}{\partial t^n}\psi(\vec{x},0),\ \overline{\psi}(0)\right\}_{t=0},$$

we assert that the form of the lowest-twist operator is the same whether B_μ is treated as a quantized field or as a given c-number field. In evaluating this anticommutator, we make use of the equations of motion to reduce everything to *canonical* equal-time commutators and anticommutators (involving zeroth-time derivatives of the quark fields and zeroth and first derivatives of the gluon field). To see how this goes, let us consider the equal-time anticommutator

$$\left\{\frac{\partial^3}{\partial t^3}\psi(\vec{x},t),\overline{\psi}(0)\right\}_{t=0}.\tag{30}$$

Using the quark-field equations of motion, we express this as a sum of terms of which only the following one involves anything higher than a first time derivative of B_μ and, therefore, depends on the quantized nature of the gluon field:

$$g\left\{\gamma_0\gamma_\alpha\psi(\vec{x},t)\frac{\partial^2B_\alpha(\vec{x},t)}{\partial t^2},\overline{\psi}(0)\right\}_{t=0}.\tag{31}$$

In turn we can decompose this into two pieces,

$$g\{\gamma_0\gamma_\alpha\psi(\vec{x},t),\overline{\psi}(0)\}_{t=0}\frac{\partial^2B_\alpha}{\partial t^2}(\vec{x},0)+g\gamma_0\gamma_\alpha\psi(\vec{x},0)\left[\frac{\partial^2B_\alpha(\vec{x},t)}{\partial t^2},\overline{\psi}(0)\right]_{t=0}.\tag{32}$$

The first term does not involve the quantized nature of the gluon field. The second term does. For it we use the equation of motion

$$(\Box+m_B^2)B_\mu=g\overline{\psi}\gamma_\mu\psi\,,\tag{33}$$

and therefore, in computing the commutator in the second term of Eq. (32), we replace $\partial_0^2B_\alpha$ by $g\overline{\psi}\gamma_\alpha\psi$. Both of course have the same physical dimension, 3; but whereas the former operator has maximum spin 3 (hence twist 0), the latter has spin 1 (hence twist 2). That is, the lowest-twist part of Eq. (30) comes from terms which do not contain any equal-time commutators involving the gluon field, i.e., do not involve the quantized nature of the gluon field. Evidently this is a general feature of any of the equal-time anticommutators of Eq. (29). Thus, insofar as we are interested only in the lowest-twist operators that dominate the light-cone behavior of $S(x, y)$, we can treat B_μ as an external field.

In this approximation the anticommutator $S(x, y)$ is a c-number function and its leading light-cone singularity is rather easily discussed. For convenience it will be simplest to determine the light-cone structure in the first instance for the causal propagator

$$S_F(x, y) = -iT(\psi(x)\bar{\psi}(y)), \tag{34}$$

where the symbol T denotes Wick time ordering; the corresponding result for the anticommutator function $S(x, y)$ will be evident. In the presence of the external field, S_F satisfies the equations

$$\gamma_\mu\left(i\frac{\partial}{\partial x^\mu} - gB^\mu(x)\right)S_F(x, y) = \delta^4(x-y) = S_F(x, y)\left[i\frac{\overleftarrow{\partial}}{\partial y^\mu} + gB^\mu(y)\right]\gamma_\mu. \tag{35}$$

(To leading order on the light cone, we are entitled to neglect the quark masses.) Expanding $S_F(x, y)$ as a power series in the coupling constant g, we have for the nth-order term

$$S_F^{(n)}(x, y) = g^n\int dz_1 \cdots dz_n S_F^{(0)}(x-z_1)\gamma\cdot B(z_1) \times S_F^{(0)}(z_1-z_2)\cdots\gamma\cdot B(z_n)S_F(z_n-y), \tag{36}$$

where $S_F^{(0)}(x-y)$ is the free-field propagator function:

$$S_F^{(0)}(x-y) = i\gamma\cdot\partial D_F(x-y) = \int\frac{d^4p}{(2\pi)^4}\frac{\gamma\cdot p}{p^2+i\epsilon}e^{-ip\cdot(x-y)}. \tag{37}$$

Regarding $S_F^{(n)}$ as a function of $X = \frac{1}{2}(x+y)$ and $\Delta = \frac{1}{2}(x-y)$, we Fourier-transform Eq. (36), writing

$$S_F^{(n)}(X, \Delta) = \int d^4P\, d^4Q\, e^{2i(P\cdot\Delta + Q\cdot X)}S_F^{(n)}(P, Q). \tag{38}$$

We then observe that the leading light-cone singularity is determined by the behavior of $S_F^{(n)}(P, Q)$ in the limit $P_0 \approx |\vec{P}| \to \infty$. To leading order on the light cone $(\Delta^2 \to 0)$, we therefore have

$$S_F^{(n)}(X, \Delta) = \frac{g^n}{(2\pi)^4}\int d^4Q\, d^4P \prod_{i=1}^n d^4q_i\,\delta\left(Q - \tfrac{1}{2}\sum_{i=1}^n q_i\right)e^{-2i(P\cdot\Delta+Q\cdot X)}\frac{\gamma\cdot P\gamma\cdot B(q_1)\gamma\cdot P\gamma\cdot B(q_2)\cdots\gamma\cdot B(q_n)\gamma\cdot P}{[P^2+2P\cdot Q][P^2+2P\cdot(Q-q_1)]\cdots[P^2-2P\cdot Q]}, \tag{39}$$

where

$$B_\mu(q) = \frac{1}{(2\pi)^4}\int d^4x\, e^{iq\cdot x}B_\mu(x).$$

In the numerator of Eq. (39) we encounter the product

$$\gamma\cdot P\gamma\cdot B\gamma\cdot P = 2P\cdot B\gamma\cdot P - \gamma\cdot BP^2,$$

and we observe that for probing the leading light-cone singularity $(P_0 \approx |\vec{P}| \approx P^2 \to \infty)$, we can neglect the second term on the right-hand side. Thus

$$\gamma\cdot P\gamma\cdot B(q_1)\cdots\gamma\cdot B(q_n)\gamma\cdot P \to \gamma\cdot P\, 2^n\prod_{i=1}^n P\cdot B(q_i).$$

We now express the denominator in Eq. (39) in Feynman parametric form, and after some further algebra obtain the following expression for the leading light-cone behavior:

$$S_F^{(n)}(x,y) = (2ig)^n S_F^{(0)}(x-y)\int d^4q_1\cdots d^4q_n\tilde{F}(q_1)\cdots\tilde{F}(q_n) \times\int d\alpha_0 d\alpha_1\cdots d\alpha_n\,\delta\left(1-\sum_0^n\alpha_i\right)\exp\left\{-i\sum_{i=1}^n q_i\cdot\left[X+\Delta\left(1-2\sum_{j=1}^i\alpha_j\right)\right]\right\}, \tag{40}$$

where

$$\tilde{F}(q) = \Delta\cdot B(q). \tag{41}$$

On suitably redefining the Feynman parameters, we find that this can be written

$$S_F^{(n)}(x,y)=\frac{(2ig)^n}{n!}\,S_F^{(0)}(x-y)\left[\int_0^1 da\,F(X+(1-2a)\Delta)\right]^n, \tag{42}$$

where we have used

$$F(z)=\int d^4q\,e^{-iq\cdot z}\tilde{F}(q). \tag{43}$$

All of this is for the causal propagator. For the anticommutator $S(x,y)$, we merely replace $S_F^{(0)}$ by $S^{(0)}$, the free anticommutator function. Summing over all orders of g, we finally obtain

$$S(x,y)\underset{\Delta^2\to 0}{\longrightarrow}\exp\left(2ig\int_0^1 da\,\Delta^\mu B_\mu(X+(1-2a)\Delta)\right)S^{(0)}(x-y)=\exp\left(-ig\int_y^x dz^\mu B_\mu(z)\right)S^{(0)}(x-y), \tag{44}$$

where the integral in the second expression is taken along a straight lightlike path from y to x.

The question arises how Eq. (44) is to be interpreted for the actual gluon model, where B_μ is a quantized field which does not commute with itself at lightlike distances. In this case the exponential seems to be ambiguous with respect to ordering of factors in a power-series expansion. According to our previous arguments, however, this ordering should not matter. The resolution of this apparent paradox is contained in the statement that

$$[(x-y)^\mu B_\mu(x),(x-y)^\nu B_\nu(y)]\to 0,\quad (x-y)^2\to 0. \tag{45}$$

This is established in Appendix A. We may also remark here that the light-cone structure of Eq. (44) continues to hold if one switches on added interactions with scalar or pseudoscalar gluons, in the manner of Eq. (28). From Eq. (39) we can see that the singularity is reduced by a factor Δ^2 if one replaces $g\gamma\cdot B(q)$ by $g_P\gamma_5\phi(q)+g_S\sigma(q)$. Alternatively, with the vector-gluon coupling switched off, the only twist-2 local operator of spin n that can appear in the short-distance expansion of current commutators is $\bar\psi(x)\gamma_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\times\overleftrightarrow{\partial}_{\mu_n}\psi(x)$ and this is independent of the fields ϕ and σ. We conclude that Eq. (44) is *formally* valid for all renormalizable quark models which involve SU(3)×SU(3) singlet gluon fields.

It must be emphasized that Eq. (44) does not represent an exact result even in the case of an external gluon field. It expresses only the most singular terms on the light cone, i.e., those that behave like $\delta'(\Delta^2)$ as $\Delta^2\to 0$ [or for S_F, terms that behave like $(\Delta^2)^{-2}$]. In fact if we apply Eq. (35) to our approximate solution S_F, we find

$$\gamma_\mu\left[i\frac{\partial}{\partial x^\mu}-gB_\mu(x)\right]S_F(x,y)-\delta^4(x-y)=-2ig\int_0^1 da\,a\gamma^\mu\Delta^\nu F_{\mu\nu}(X+(1-2a)\Delta)S_F^{(0)}(x-y), \tag{46}$$

where $F_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu$. Only in the trivial case where B_μ is curl-less does our approximate solution satisfy the equation of motion. Indeed, the right-hand side of Eq. (46) is again of order $(\Delta^2)^{-2}$, even though S_F has been determined correctly up to (but not including) terms of order $(\Delta^2)^{-1}$. This is because the $(\Delta^2)^{-1}$ corrections to S_F will contribute additional $(\Delta^2)^{-2}$ terms to the right-hand side of Eq. (46). It will be useful for later purposes to add these corrections to S_F, in order that Eq. (35) shall be satisfied to order $(\Delta^2)^{-1}$. We will want this when we come to write the light-cone current commutators in a way where current conservation is manifest. We can get the correction terms by retaining the next leading terms in the Born series for S_F; or equivalently by adding to the leading approximation a correction which guarantees the validity of Eqs. (35) up to order $(\Delta^2)^{-1}$. In this way, for the anticommutator function, we find

$$S(x,y)=\exp\left(-ig\int_y^x dz^\mu B_\mu(z)\right)S^{(0)}(x-y)+T(x,y)D(x,y)+\cdots, \tag{47}$$

where

$$T(x,y)=-ig\int_0^1 da[(2a-1)\gamma^\mu\Delta^\nu-\tfrac{1}{2}\gamma^\mu\gamma^\rho\Delta_\rho\gamma^\nu]\times F_{\mu\nu}(X+(1-2a)\Delta)\exp\left(-ig\int_y^x dz^\mu B_\mu\right). \tag{48}$$

The correction term explicitly indicated here behaves like $\delta(\Delta^2)$ as $\Delta^2\to 0$ [or for S_F, behaves like $(\Delta^2)^{-1}$]; but when one applies the operator $\gamma_\mu(\partial/\partial x^\mu)$ from the left, or $\gamma_\mu(\partial/\partial y_\mu)$ from the right, it generates terms which behave like $\delta'(\Delta^2)$ [or for S_F, like $(\Delta^2)^{-2}$] and which cancel the right-hand side of Eq. (46). There are additional terms of order $\delta(\Delta^2)$ for $S(x,y)$, not indicated in Eq. (47). Some of these already occur in the external-field problem, e.g.,

$$g\int da\,a(1-a)\gamma\cdot\Delta\,\Delta^\alpha\Box B_\alpha(X+(1-2a)\Delta)D(x-y);$$

and some arise from the quantized nature of the B field and are operators, such as

$$[\gamma \cdot \Delta \psi(X)][\bar{\psi}(X)\gamma \cdot \Delta] D(x-y)\,.$$

However, these additional terms do not contribute to $\delta'(\Delta^2)$ singularities in Eqs. (35).

V. LIGHT-CONE CURRENT COMMUTATOR

Having derived the leading terms in the quark-field anticommutator $S(x, y)$, it is now easy for us to write down the leading light-cone commutator terms in the vector-gluon model. In deriving the leading light-cone singularity, we can treat $\psi(x)$ and $\psi(y)$ as anticommuting operators. The terms which receive contributions from their anticommutator will necessarily be less singular on the light cone, since they will arise only if the equation of motion for B_μ is used at least once. From the fact that $S(x, y)$ is proportional to the free-quark-model anticommutator $S^{(0)}(x, y)$, with a coefficient function that is a scalar and an SU(3) $\times$SU(3) singlet, we immediately see that the tensor and SU(3)$\times$SU(3) structure of the light-cone commutator is exactly as in the free-quark model. That is, Eq. (13) continues to hold for the gluon model, to leading order on the light cone, where now

$$J^a_\mu\left(r, \genfrac{}{}{0pt}{}{S}{A}; X, \Delta\right) = \tfrac{1}{2}\bar{\psi}(x)(1 + r\gamma_5)\gamma_\mu \tfrac{1}{2}\lambda^a \times \exp\left(-ig\int_y^x dz^\mu B_\mu(z)\right)\psi(y) \pm (x \leftrightarrow y)\,. \tag{49}$$

It is instructive to cast this bilocal operator into a form which exhibits explicitly the local operators that appear in a power-series expansion in Δ. To this end we make the replacements

$$\psi(y) = e^{-\Delta \cdot \vec{\partial}_X}\psi(X)\,,$$
$$\bar{\psi}(x) = \bar{\psi}(X)e^{\Delta \cdot \overleftarrow{\partial}_X}\,.$$

The shift operators may now be combined with the exponential in Eq. (49) by use of the Hausdorff-Baker theorem and the identity

$$\int_X^x dz^\mu B_\mu(z) = \int_0^1 d\lambda\, \Delta \cdot B(X + \lambda\Delta)$$
$$= \left(\frac{e^{\Delta \cdot \partial_X} - 1}{\Delta \cdot \partial_X}\right)\Delta \cdot B(X)\,.$$

We then have

$$J^a_\mu\left(r, \genfrac{}{}{0pt}{}{S}{A}; X, \Delta\right) = \tfrac{1}{2}\bar{\psi}(X)(1 + r\gamma_5)\gamma_\mu \tfrac{1}{2}\lambda^a e^{\Delta \cdot \overleftarrow{\partial}_X + ig\Delta \cdot B(X)}\, e^{-i\Delta \cdot \vec{\partial}_X + ig\Delta \cdot B(X)}\psi(X) \pm (\Delta \to -\Delta)\,. \tag{50}$$

This form allows us to identify the *local* operators of twist 2 that appear in the short-distance expansion of the light-cone commutator (or equivalently, the highest-spin components of the relevant equal-time commutators):

$$J^a_\mu\left(r, \genfrac{}{}{0pt}{}{S}{A}\right) = \sum_{n\binom{\text{even}}{\text{odd}}} \frac{\Delta_{\alpha_1} \cdots \Delta_{\alpha_n}}{n!}\bar{\psi}(X)(1 + r\gamma_5)\gamma_\mu \tfrac{1}{2}\lambda^a \sum_{i=0}^n \binom{n}{i} \overleftarrow{\mathrm{d}}_{\alpha_1} \cdots \overleftarrow{\mathrm{d}}_{\alpha_i} \vec{\mathrm{d}}_{\alpha_{i+1}} \cdots \vec{\mathrm{d}}_{\alpha_n}\psi(X)\,, \tag{51}$$

where

$$\vec{\mathrm{d}}_\mu = -\frac{\vec{\partial}}{\partial X^\mu} + igB_\mu(X)\,,$$
$$\overleftarrow{\mathrm{d}}_\mu = \frac{\overleftarrow{\partial}}{\partial X^\mu} + igB_\mu(X)\,. \tag{52}$$

It may be noted here that the bilocal as well as the local operators are explicitly invariant under gauge transformations of the second kind.

At this point it is incumbent upon us to specify what we mean by the products, at a single space-time point X, of field operators that appear in the above expansion of the bilocal operator. These products are themselves singular. But again we take refuge in the fact that we are formally dealing with a cutoff field theory. In this case the only singularities are contained in vacuum matrix elements of the local operators. The connected matrix elements, that is, we shall suppose to be finite; so for the purposes of computing connected matrix elements, we can discard the c-number part of the bilocal operator $J^a_\mu(X, \Delta)$. However, this vacuum expectation value part is of interest in its own right, since it is this part that is probed experimentally in electron-positron annihilation experiments. In a cutoff version of the gluon model, the vacuum matrix element of $J^a_\mu(X, \Delta)$ is given by the free-field-theoretic expression

$$\left\langle J^a_\mu\left(r, \genfrac{}{}{0pt}{}{S}{A}; X, \Delta\right)\right\rangle_0 = \tfrac{1}{2}\langle\bar{\psi}(x)\gamma_\mu \tfrac{1}{2}\lambda^a \psi(y) \pm \bar{\psi}(y)\gamma_\mu \tfrac{1}{2}\lambda^a \psi(x)\rangle_0$$
$$= 4i\delta_{a0}\partial_\mu[D^{(0)}_+(x-y) \pm D^{(0)}_-(x-y)]\,. \tag{53}$$

We therefore abstract for the vacuum matrix element of the light-cone commutator

$$\langle [J^a_\mu(r,x), J^b_\nu(r,y)]\rangle_0 = \text{const}\, \delta_{ab}[\partial_\mu D_1(\Delta)\partial_\nu D(\Delta) + \partial_\nu D_1(\Delta)\partial_\mu D(\Delta) - g_{\mu\nu}\partial_\lambda D_1(\Delta)\partial^\lambda D(\Delta)], \tag{54}$$

where

$$D_1(\Delta) = \frac{1}{2\pi^2} P\left(\frac{1}{\Delta^2}\right).$$

This structure leads to SU(3)×SU(3)-symmetric, quadratically divergent, c-number Schwinger terms in the space-time equal-time commutators; and therefore to the prediction that the electron-positron annihilation cross section scales[15]:

$$\sigma(s) \xrightarrow[s\to\infty]{} \text{const}/s\,.$$

The Weinberg sum rules,[16] generalized to SU(3) ×SU(3),[17] emerge as an additional consequence of our formal analysis. This is because the only bilocal operators which have nonvanishing vacuum expectation values and which are not invariant under SU(3)×SU(3) must be of twist 6, or higher, i.e., must involve four or more quark fields in an SU(3)×SU(3) singlet combination. For the local currents, therefore, the vacuum expectation values of the equal-time commutators $[J_0^a, J_i^b]$ and $[\partial_0 J_i^a, J_j^b]$ are invariant under SU(3)×SU(3); symmetry breaking arises only for equal-time commutators involving third and higher time derivatives.

Returning to the light-cone commutator structure of Eq. (13), we still face the task of casting it into a form where current conservation is manifest. This is necessary for applications, where we have no theoretical knowledge concerning the matrix elements of the bilocal operators. For the free-quark model, recall, we made use of the equations of motion to extract projection operators that served to ensure current conservation. For the gluon model we follow what is essentially the same procedure. Suppressing all but Lorentz indices, let us again write

$$[J_\mu(x), J_\nu(y)] = \Theta_{\mu\nu\lambda}(X,\Delta)\partial^\lambda D(\Delta) \tag{55}$$

as in Eq. (16), where $\Theta_{\mu\nu\lambda}$ is defined by Eqs. (13) and (50). For the gluon model we no longer have the conservation equations

$$\frac{\partial}{\partial x^\mu}\Theta_{\mu\nu\lambda} = \frac{\partial}{\partial y^\nu}\Theta_{\mu\nu\lambda} = 0\,.$$

Although strict conservation cannot be achieved by our approximation method, we *can* ensure conservation to leading order on the light cone by incorporating the correction term in Eq. (47). That is, we now take

$$[J_\mu(x), J_\nu(y)] = [\Theta_{\mu\nu\lambda}(X,\Delta)\partial^\lambda + \Theta_{\mu\nu}(X,\Delta)]D(\Delta), \tag{56}$$

where

$$\Theta_{\mu\nu} = \bar\psi(x)\gamma_\mu T(X,\Delta)\gamma_\nu\psi(y) \pm (\mu \leftrightarrow \nu,\ x\leftrightarrow y) \tag{57}$$

[see Eq. (48) and recall that we are suppressing all but Lorentz indices]. To leading order of singularities on the light cone, we then have

$$\frac{\partial}{\partial x^\mu}[J_\mu(x), J_\nu(y)] = \frac{\partial}{\partial y^\nu}[J_\mu(x), J_\nu(y)] \approx 0\,. \tag{58}$$

We can now follow the procedure outlined in Sec. III to rewrite Eq. (56) in a manifestly current-conserving form, by adding to the commutator nonleading terms [of order $\delta(\Delta^2)$]. We find

$$[J_\mu(x), J_\nu(y)] = P_{\mu\mu';\nu\nu'}\{[\tilde{Q}_{\mu'\nu'\lambda}\partial^\lambda + \tilde\Theta_{\mu'\nu'}]\tilde{D}(\Delta)\}, \tag{59}$$

where $P_{\mu\mu';\nu\nu'}$ is the projection operator appearing within the curly brackets of Eq. (25) and $\tilde{D}$ is defined by Eq. (23).

Notice that the $\tilde\Theta_{\mu\nu}$ term in Eq. (59), although it is less singular on the light cone than the other term appearing with it, cannot be ignored, since both terms give equally singular contributions when they are acted on by the projection operator.

As the remaining item in this section, let us examine some of the results that follow from the structure of Eq. (13), when this is applied to deep-inelastic lepton-hadron scattering. Let p be the momentum of the hadron and consider the spin-averaged matrix element

$$\frac{1}{2\pi}\int dx\, e^{iq\cdot x}\langle p|[J_\mu{}^a(r;\tfrac{1}{2}x), J_\nu^b(r;-\tfrac{1}{2}x)]\,|p\rangle$$

$$= \frac{p_\mu p_\nu}{m\, q\cdot p}F_2^{(a,b)}(r;q\cdot p,p^2) - \frac{g_{\mu\nu}}{m}F_1^{(a,b)}(r;q\cdot p,p^2) \mp ir\,\frac{\epsilon_{\mu\nu\sigma\lambda}p_\sigma q_\lambda}{2m\,q\cdot p}F_3^{(a,b)}(r;q\cdot p,p^2) + \cdots\,, \tag{60}$$

where we suppress terms proportional to q_μ or q_ν. In dealing with these diagonal matrix elements, we can work directly with Eq. (13) and need not resort to Eq. (59). Passing to the Bjorken limit $-q^2\to\infty$, $q\cdot p\to\infty$, $\omega=-q^2/2(q\cdot p)$ fixed, one finds from Eqs. (13) and (60)

$$\frac{1}{\omega}F_2^{(a,b)}(r,\omega)=if_{abc}G_S^c(r;\omega)+d_{abc}G_A^c(r;\omega)\,, \tag{61a}$$

$$2\omega F_1^{(a,b)}(r;\omega)=F_2^{(a,b)}(r;\omega)\,, \tag{61b}$$

$$F_3^{(a,b)}(r;\omega)=r[if_{abc}G_A^c(r;\omega)+d_{abc}G_S^c(r;\omega)]\,, \tag{61c}$$

where we have defined the matrix element of the bilocal operators to be

$$\langle p|J_\mu^a\left(r,{S\atop A};0,\Delta\right)|p\rangle =\frac{p_\mu}{m}\int_{-1}^{1}d\omega\, e^{2i\omega p\cdot\Delta}\begin{Bmatrix}G_S^a(r;\omega)\\ G_A^a(r;\omega)\end{Bmatrix}+\cdots\,. \tag{62}$$

The functions G_S and G_A are, respectively, symmetric and antisymmetric in ω, and real. From the above equations we can easily derive various sum rules. For example, the standard current-algebra results based on equal-time commutators are recovered by passing to the limit $\Delta=0$, where the bilocal operator $J_\mu^a(0,\Delta=0)$ reduces to the local current operator, and where the right-hand side of Eq. (62) reduces to an integral over the structure functions. In this way we obtain the Adler[2] sum rule and the Gross–Llewellyn-Smith[4] sum rule. Moreover, the very structure of Eq. (13) guarantees that $\sigma_L/\sigma_T\to 0$ in the scaling limit,[3] as we see directly from Eq. (61b). Finally, we recover a number of the relations which follow from the parton model by noting that for deep-inelastic scattering of photons, neutrinos, and antineutrinos on neutrons and protons there are six independent structure functions (for the neutrino cases, we consider only $\Delta S=0$ transitions), whereas these are expressed in terms of only five bilocal operators, $J_A^{3,V}$, $J_A^{8,V}$, $J_S^{8,V}$, $J_A^{0,V}$, $J_S^{0,V}$ {we define $J^{a,V}=\frac{1}{2}[J^a(r=1)+J^a(r=-1)]$}. This leads to the relation[6]

$$12[F_1^{\gamma p}(\omega)-F_1^{\gamma n}(\omega)]=F_3^{\nu p}(\omega)-F_3^{\nu n}(\omega)\,. \tag{63}$$

Moreover, from the positivity of $F_1^{(ab)}+F_1^{(ba)}$ we can establish from Eqs. (61a) and (61b) the inequality[6]

$$F_1^{\gamma p}(\omega)+F_1^{\gamma n}(\omega)\geq\tfrac{5}{18}[F_1^{\nu p}(\omega)+F_1^{\nu n}(\omega)]\,. \tag{64}$$

VI. THE BILOCAL LIGHT-CONE COMMUTATORS

In our discussion of the light-cone commutator of local currents, we have encountered certain bilocal operators. It seems natural, following Gell-Mann,[10] to investigate the commutation relations among the bilocal operators themselves. In the free-quark model this is easily done. For example [suppressing SU(3) matrices], we have that

$$[\bar\psi(x)\gamma_\mu\psi(y),\bar\psi(z)\gamma_\nu\psi(t)]=\bar\psi(x)\gamma_\mu\gamma_\alpha\gamma_\nu\psi(t)\partial^\alpha D(y-z) -\bar\psi(z)\gamma_\nu\gamma_\alpha\gamma_\mu\psi(y)\partial^\alpha D(x-t)\,. \tag{65}$$

On the basis of such relations one easily verifies that the commutator of $J_\mu^a(r,{S\atop A};X,\Delta)$ with another bilocal operator closes on the same set of bilocal operators. However, it is not clear what features of this algebra can be safely extracted beyond the free-quark model. One is examining the structure of

$$[[J_\mu^a(x),J_\nu^b(y)],[J_\lambda^c(z),J_\eta^d(t)]] \tag{66}$$

initially for $(x-y)^2\approx 0$, $(z-t)^2\approx 0$. But should one abstract from a model those features which hold wherever any other pair of coordinates has a lightlike separation [say $(y-z)^2\approx 0$]; or wherever *all* separations are lightlike; or wherever x, y, z, t are collinear points on a lightlike ray? In the free-quark model, the algebra closes under all these conditions so far as leading singularities are concerned. But this is no longer the case when interactions are switched on, as in the gluon model. In general the commutators of the original bilocal operators generate *new* bilocal operators of twist 2; and in their short-distance expansion these generate new local operators of twist 2, of the form, e.g.,

$$\bar\psi(X)\overleftarrow{\partial}_{\alpha_1}\cdots\overleftarrow{\partial}_{\alpha_i}\left[\sum_{K=1}^{n}\binom{n}{K}\overleftarrow{\partial}_{\mu_1}\cdots\overleftarrow{\partial}_{\mu_K}\overrightarrow{\partial}_{\mu_{K+1}}\cdots\overrightarrow{\partial}_{\mu_n}\right]\overrightarrow{\partial}_{\beta_1}\cdots\overrightarrow{\partial}_{\beta_j}\psi(X)\,. \tag{67}$$

The only situation in which the algebra of our bilocal operators closes is when all four points in Eq. (66) are collinear on a lightlike ray. For this case we can again treat the gluon as an external field. We are concerned with the commutator

$$\left[\bar\psi(x)\gamma_\mu\exp\left(-ig\int_y^x du^\mu B_\mu(u)\right)\psi(y),\bar\psi(z)\gamma_\nu\exp\left(-ig\int_t^z dv^\mu B_\mu(v)\right)\psi(t)\right], \tag{68}$$

and our theorem applies when x, y, z, t are collinear because the integrands appearing in the line integrals then commute with each other. Employing Eq. (44) we then find for this collinear situation

$$\big[[J^a_\mu(r;x),J^b_\nu(r;y)],[J^c_\lambda(r;z),J^d_\eta(r;t)]\big]$$
$$\to \tfrac{1}{8}\bar\psi(x)\lambda^a\lambda^b\lambda^c\lambda^d\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\gamma(1+r\gamma_5)\gamma_\lambda\gamma^\beta\gamma_\eta\exp\left(-ig\int_t^x du^\mu B_\mu(u)\right)\psi(t)\partial^\alpha D(x-y)\partial^\beta D(z-t)\partial^\gamma D(z-y)$$
$$-\tfrac{1}{8}\bar\psi(y)\lambda^b\lambda^a\lambda^c\lambda^d\gamma_\nu\gamma^\alpha\gamma_\mu\gamma^\gamma(1+r\gamma_5)\gamma_\lambda\gamma^\beta\gamma_\eta\exp\left(-ig\int_t^y du^\mu B_\mu(u)\right)\psi(t)\partial^\alpha D(x-y)\partial^\beta(z-t)\partial^\gamma D(z-x)$$
$$-[\lambda\leftrightarrow\eta,\ z\leftrightarrow t,\ c\leftrightarrow d]\,. \tag{69}$$

The right-hand side of Eq. (69) can be reduced to a sum of terms involving the original bilocal operators, but we will not bother here to carry out the reduction. The important point, formally, is that the algebra of the bilocal operators closes in the gluon model for collinear points. The requirement of collinearity is very restrictive. No doubt the commutators of our bilocal operators have interesting singularities in other regions of configuration space, but we are unable to say anything about these other regions. Even so, the structure described in Eq. (69) for the collinear situation has observable consequences. We shall return to these matters in a future publication.

APPENDIX A

In this Appendix we shall prove an assertion made in Sec. IV, namely, that

$$(x-y)^\mu(x-y)^\nu[B_\mu(x),B_\nu(y)]=0\,,\quad (x-y)^2=0\,. \tag{A1}$$

Before doing so, however, we give a more precise specification of our gluon model in such a way that the canonical equal-time commutators of the gluon field are scale-invariant. This can be achieved if we perform a Stueckelberg-like transformation on the traditional gluon model. Let us introduce into the Lagrangian a free, negative-metric scalar field ϕ, of mass m_B; and let us redefine the gluon and quark fields by

$$B_\mu\to B_\mu-(1/M)\partial_\mu\phi\,,\qquad \psi\to e^{i(g/M)\phi}\psi\,. \tag{A2}$$

In terms of the transformed fields, the Lagrangian is given by

$$\mathcal{L}=-\tfrac14F_{\mu\nu}F^{\mu\nu}+\tfrac12 m_B^2B_\mu B^\mu-\tfrac12 m_B(B_\mu\partial^\mu\phi+\partial^\mu\phi B_\mu)-\tfrac12 m_B^2\phi^2+\bar\psi[-\tfrac12\gamma\cdot\overleftrightarrow{\partial}+g\gamma\cdot B-M]\psi\,. \tag{A3}$$

The canonical variables are ψ and $\pi_\psi=\psi^\dagger$, B_i and $\partial_0 B_i$, ϕ and $\pi_\phi=MB_0$. The equations of motion are

$$\begin{aligned}&(\gamma\cdot\partial+iM)\psi=ig\gamma\cdot B\psi\,,\\&\partial^\mu B_\mu=M\phi\,,\\&(\Box+m_B^2)B_\mu=-g\bar\psi\gamma_\mu\psi\,,\\&(\Box+m_B^2)\phi=0\,.\end{aligned} \tag{A4}$$

The canonical equal-time commutators of the gluon field are now scale-invariant:

$$\begin{aligned}&[B_\mu(\vec x,0),B_\nu(0)]=0\,,\\&[\partial_0B_\mu(\vec x,0),B_\nu(0)]=ig_{\mu\nu}\delta^3(\vec x)\,.\end{aligned} \tag{A5}$$

In the absence of interactions ($g=0$), we would have for arbitrary times

$$[B_\mu(x),B_\nu(y)]=g_{\mu\nu}D(x-y;m_B)\,, \tag{A6}$$

and Eq. (A1) would hold trivially. We now ask, in the presence of interaction, whether Eq. (A6) is modified by terms which violate Eq. (A1). As before, we are discussing the formal structure of the commutator; and we can therefore resolve the issue by examining the *equal-time* commutators involving time derivatives of the gluon field. The first equal-time commutator in which the interaction of B_μ with the SU(3) singlet current J_μ figures is

$$[\partial_0^2B_\mu(\vec x,0),\partial_0^2B_\nu(0)]=g^2[J_\mu(\vec x,0)J_\nu(\vec y,0)]\,.$$

This implies that the correction to Eq. (A6) arises from a twist-2 operator, which contributes a term of the form

$$[E_\mu(X,\Delta)\Delta_\nu+E_\nu(X,\Delta)\Delta_\mu+F_\alpha(X,\Delta)\Delta^\alpha g_{\mu\nu}+\epsilon_{\mu\nu\sigma\lambda}G_\sigma(X,\Delta)\Delta^\lambda]\epsilon(\Delta^0)\Theta(\Delta^2)\,. \tag{A7}$$

But such a correction does not affect the validity of Eq. (A1). Additional corrections to Eq. (A6) will necessarily be proportional to Δ^2 and will therefore vanish on the light cone.

APPENDIX B

In a recent paper Kogut and Soper[18] have discussed an interesting new procedure for quantizing field theories in an infinite-momentum frame. They have identified the independent canonical fields and their canonical commutation relations in the case of spinor electrodynamics; and they show that the resulting theory is formally identical with the usual theory, boosted to the infinite-momentum frame. These methods have been utilized by Cornwall and Jackiw[19] to derive canonical light-cone commutators for the electromagnetic currents in spinor electrodynamics. The relation between the approach adopted in this paper and that of Kogut and Soper's *canonical* light-cone commutators is much the same as the relation between Wilson's short-distance expansion and equal-time commutation relations. Consider, for example, the anticommutator $\{\psi(x), \overline{\psi}(y)\}$. The light-cone expansion gives a manifestly covariant expression for the leading term when $(x-y)^2 \approx 0$, and it determines the lowest-twist operators appearing in the short-distance expansion. The "infinite momentum" methods, on the other hand, yield frame-dependent expressions for the anticommutators of canonically independent spinor fields at lightlike separations. These anticommutators contain information on higher-twist operators.

It is amusing to see how the canonical anticommutator of Kogut and Soper emerges, as it does, from our *leading* light-cone expression for $S(x,y)$. Their independent, canonical spinor field variables on the light cone are ψ_+ and its adjoint ψ_+^*, where

$$\psi_+ = P_+\psi = \tfrac{1}{2}(\gamma^0-\gamma^3)(\gamma^0+\gamma^3)\psi\,, \tag{B1}$$

and the canonical anticommutator is

$$\{\psi_+(x), \psi_+^*(0)\}\,\delta(x^0+x^3) = P_+\delta(x^0+x^3)\delta(x^0-x^3)\delta^2(\vec{x}_\perp)\,. \tag{B2}$$

We can recover this result immediately from our leading light-cone expression for $S(x,0) = i\{\psi(x), \overline{\psi}(0)\}$ by applying the projection P_+ to $S(x,0)\gamma^0$ and evaluating at $x^0+x^3=0$. We have

$$\{\psi_+(x), \psi_+^*(0)\}\,\delta(x^0+x^3) = \delta(x^0+x^3)\exp\left[-ig\int_0^x dz^\mu B_\mu(z)\right]P_+\left(\frac{\partial}{\partial x^0}-\frac{\partial}{\partial x^3}\right)\frac{\delta(x^2)\epsilon(x_0)}{2\pi}$$
$$= P_+\delta(x^0+x^3)\delta(x^0-x^3)\delta^2(\vec{x}_\perp)\,. \tag{B3}$$

We have used

$$P_+\gamma^3\gamma^0 = -P_+\,,\quad P_+\gamma^\perp\gamma^0P_+ = 0\,.$$

Consider now the current $J_+^a \equiv J_0^a + J_3^a$. From Eq. (B2) we immediately determine the light-cone commutator[19]

$$[J_+^a(x), J_+^b(0)]\delta(x^0+x^3) = if_{abc}J_+^c(0)\delta(x^0+x^3)\delta(x^0-x^3)\delta^2(\vec{x}_\perp)\,. \tag{B4}$$

It is easy to see that the same result emerges from our Eqs. (13) and (49).

For the case of electrodynamics (where B_μ is the electromagnetic field) Cornwall and Jackiw[19] have recently discussed also the commutator

$$[J_+^a(x), J_-^b(0)]\delta(x^0+x^3)\,,$$

where

$$J_-^a \equiv J_0^a - J_3^a\,.$$

This now involves the equations of motion. Cornwall and Jackiw[19] choose a particular gauge, $B^0+B^3=0$, as they can do in electrodynamics. We remark that the same results emerge from the methods of the present paper, provided that we include the correction term $T(x,y)$ in Eqs. (47) and (48). We shall not go through the details. The interesting point, however, is that our methods can be applied with equal ease for any choice of gauge and are applicable to massive gluon fields.

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Gradient Terms in the Scalar-Density–Charge-Density Commutator*

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We compute, in the perturbative model of Johnson and Low, the one-meson-to-vacuum matrix element of the equal-time commutator of the charge density of a vector current and a scalar density. A gradient term is found. Consequences of this result for the dimensionality of the current components are discussed.

I. INTRODUCTION

Recently Bég *et al.*[1] showed that the necessary and sufficient conditions for the hadronic vector currents J_μ^α to have scale dimension independent of the Minkowski index μ, imply the absence of gradient terms in the commutator[2]

$$[J_0^\alpha(\vec{x}, 0), \Theta(\vec{y}, 0)], \tag{1}$$

where Θ is the trace of the "new improved energy-momentum tensor."[3] In the model of current algebra with underlying quark structure, the currents J_μ^α are bilinear in Fermi fields, and Θ is expected to be a sum of mass terms of the form

$$S^0 + S^8 = \epsilon_0 \bar{\psi}\psi + \epsilon_8 \bar{\psi}\lambda_8\psi. \tag{2}$$

In this paper we compute, in the perturbative model of Johnson and Low,[4] the one-meson-to-vacuum matrix element of the equal-time commutator (ETC) of the fourth component J_4^α of a quark vector current and a quark scalar density S^β. Our results [Eq. (52)] show that for an unconserved current, gradient terms are present, contrary to the naive expectation.[5] Thus, while it may be possible for the components of a conserved current to have a unique scale dimension, "anomalies" will, in general, prevent the spatial and temporal components of an unconserved current from having the same dimension.

II. COMMUTATOR BY NAIVE METHODS

In this section and in Sec. III we will consider the following two densities:

$$A(x) = \bar{\psi}(x)\gamma_4\lambda_\alpha\psi(x),$$
$$B(y) = \bar{\psi}(y)\lambda_\beta\psi(y), \tag{3}$$

where λ_α and λ_β are the usual $SU(3)$ generators and $\psi(x)$ is a spin-$\frac{1}{2}$ field of three different quarks, with possibly three different masses (4×3 components).

The naive ETC, obtained by straightforward application of the canonical commutation relations, is simply

$$[A(\vec{x}, t), B(\vec{y}, t)] = \bar{\psi}(x)[\lambda_\alpha, \lambda_\beta]\psi(x)\delta(\vec{x} - \vec{y}). \tag{4}$$

As is well known, the manipulations which lead to this equation are in general not valid and the ETC must be defined in a proper way. One solution to the problem is to split the spatial depen-

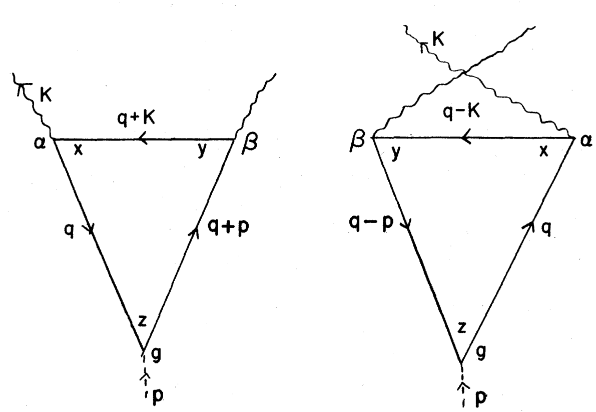

FIG. 1. Graphs contributing to the matrix element of the time-ordered product in Eq. (8), to first order in g.