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Crossing-Symmetry Restrictions on Dispersion Relations, and Sum Rules for $\pi\pi$ Scattering Lengths*

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We study the constraints crossing symmetry imposes on fixed-variable dispersion relations for $\pi\pi$ scattering. We show that the sum rules relating $2a_0^0-5a_0^2-18a_1^1$, a_2^0, and a_2^2 to the total cross sections, which were derived by Wanders using the Mandelstam representation, follow from twice-subtracted dispersion relations. These sum rules are good physical-region constraints to supplement the unphysical-region constraints of Martin and Roskies in the study of models for low-energy $\pi\pi$ scattering. Using a restriction on the absorptive parts following from crossing symmetry, we transform Wanders's sum rule for the $I=0$, $l=2$ scattering length into a form which is manifestly positive. Keeping only the S- and P-wave contributions, we obtain a lower bound for a_2^0. If the ρ-trajectory intercept is less than 1, we show that $\lim_{s\to\infty} \operatorname{Re} T^I(s,0,4-s)/s$ is determined by the total cross sections. If, in addition, the leading isospin-2 trajectory has intercept less than zero, then even without imposing elastic unitarity, the $I=0$ S wave is determined by the absorptive parts without the freedom of adding an arbitrary constant.

I. INTRODUCTION

Martin[1] has derived rigorous inequality constraints on the $\pi\pi$ partial-wave amplitudes in the unphysical region $0 \leq s \leq 4m_\pi^2$. Roskies[2] has found sum rules involving integrals of the partial-wave amplitudes over $0 \leq s \leq 4m_\pi^2$, which follow from crossing symmetry. There have been recent attempts[3,4] to use these unphysical-region constraints to study the behavior of the $\pi\pi$ amplitudes above threshold. Within a given parametrization of the partial waves, it has been possible to make physical-region predictions.[3,4] However, Ulrich[5] has found a new parametrization of the S and P waves in which the unphysical-region constraints hardly constrain the physical-region phase shifts. He introduced the experimental ρ meson into the P wave, and found that there existed a family of S waves exactly satisfying the Martin and Roskies constraints, which had drastically different phase shifts above threshold.

In this note, we discuss several sum rules which relate the $\pi\pi$ scattering lengths to integrals of the absorptive parts over the physical region. We show that these sum rules, which were originally derived by Wanders[6] using the Mandelstam representation, are direct consequences of twice-subtracted dispersion relations. Since the integrands behave like s^{-3} at large energies, these sum rules are most sensitive to the energy region below 1 GeV. Therefore, these sum rules are good physical-region constraints to supplement the unphysical-region constraints of Martin and Roskies in the study of low-energy $\pi\pi$ models. They relate the tip of the unphysical region to the resonance region. Also, the ρ and σ enter the sum rule for $2a_0^0-5a_0^2-18a_1^1$ with opposite signs, making it very sensitive to the detailed form of the S- and P-wave phase shifts.

In Sec. II, we discuss the constraints crossing symmetry imposes upon the subtraction constants appearing in fixed-variable dispersion relations.

In Sec. III, we remark that the same analysis used to study the asymptotic behavior of the real part of an su-symmetric amplitude[7] can be applied to the definite isospin $\pi\pi$ amplitudes, if one assumes the ρ-trajectory intercept is less than 1. In Sec. IV, we present a derivation of Wanders's[6] sum rule for $2a_0^0-5a_0^2-18a_1^1$, using twice-subtracted dispersion relations. We also clarify the derivation of Olsson's[8] sum rule for a_1^1 by using subtracted dispersion relations. In Sec. V, we derive Wanders's sum rules for the D-wave scattering lengths. Using a restriction on the absorptive parts which follows from crossing symmetry,[9] we manipulate the sum rule for a_2^0 into a form which is manifestly positive. By keeping only the S- and P-wave contributions to this sum rule, we find a rigorous lower bound for a_2^0. In Sec. VI, we evaluate numerically the sum rule for $2a_0^0-5a_0^2-18a_1^1$, and the lower bound for a_2^0, using a model for the S and P waves.[4]

II. CROSSING-SYMMETRY RESTRICTIONS ON SUBTRACTION CONSTANTS

Roskies[2] has shown that crossing symmetry requires the definite isospin $\pi\pi$ scattering amplitudes to have the form:

$$T^0(s,t,u)=5f(s,t,u)+2(2s-t-u)g(s,t,u)+2(2s^2-t^2-u^2)h(s,t,u),$$

$$T^1(s,t,u)=3(t-u)g(s,t,u)+3(t^2-u^2)h(s,t,u),$$

$$T^2(s,t,u)=2f(s,t,u)+(t+u-2s)g(s,t,u)+(t^2+u^2-2s^2)h(s,t,u),$$

where f, g, and h are totally symmetric functions of their arguments and the isospin is measured in the s channel. It has been shown from axiomatic considerations[10] that $T^I(s,t,u)$ satisfies a twice-subtracted dispersion relation for t fixed, $0\le t<4$ $(m_\pi{}^2=1)$. Therefore, $h(s,t,u)$ satisfies an unsubtracted dispersion relation. When evaluated at $s=4$, $t=u=0$, this relation becomes a sum rule for the linear combination of S- and P-wave scattering lengths $2a_0^0-5a_0^2-18a_1^1$, previously derived by Wanders.[6] With two subtractions, there exists the freedom of adding a constant c to $f(s,t,u)$ and a constant b to $g(s,t,u)$, without violating the analyticity, asymptotic behavior, or crossing symmetry of the $\pi\pi$ amplitudes. For this reason, the lowest partial waves are determined by the absorptive parts of the $\pi\pi$ amplitudes, only up to the ambiguity[2]

$$f_0^0(s)\to f_0^0(s)+5c+2b(3s-4),$$

$$f_1^1(s)\to f_1^1(s)+b(s-4),$$

$$f_0^2(s)\to f_0^2(s)+2c-b(3s-4).$$

The freedom to add a constant b to $g(s,t,u)$ implies that $\lim_{s\to\infty}\mathrm{Re}T^I(s,0,4-s)/s$ is not determined by the absorptive parts.

If we assume that $T^1(0,t,4-t)$ is of order $t^{1-\epsilon}$ $(\epsilon>0)$ when $t\to\infty$, then it follows that one no longer has the freedom to add an arbitrary constant b to $g(s,t,u)$. In this case, a_1^1, $2a_0^0-5a_0^2$, and $\lim_{s\to\infty}\mathrm{Re}T^I(s,0,4-s)/s$ are all fixed by the absorptive parts of the $\pi\pi$ scattering amplitudes. Under the further assumption that $T^2(0,t,4-t)$ is of order $t^{-\epsilon}$ $(\epsilon>0)$ when $t\to\infty$, we see that one no longer has the freedom of adding a constant c to $f(s,t,u)$. In this case, even without imposing elastic unitarity, there does not exist the freedom of adding an arbitrary constant to the $\pi^0\pi^0$ amplitude. Also, the S-wave scattering lengths a_0^0 and a_0^2 are individually determined by the absorptive parts.[11]

III. ASYMPTOTIC BEHAVIOR OF $\mathrm{Re}T^I(s,0,4-s)$

The forward scattering amplitude for the process $A+B\to A+B$, where $B=\overline{B}$, is symmetric in s and u. The asymptotic behavior of the real part of such an amplitude has been previously studied.[7] We wish to note that if one assumes that the leading $I=1$ trajectory has an intercept less than 1, then the same considerations apply to the definite isospin $\pi\pi$ amplitudes. The reason for this is simple. There are two su-symmetric amplitudes $\frac{1}{3}[T^0(s,t,u)+T^2(s,t,u)]$ and $\frac{1}{2}[T^1(s,t,u)+T^2(s,t,u)]$ corresponding to $\pi^0\pi^0\to\pi^0\pi^0$ and $\pi^+\pi^0\to\pi^+\pi^0$, respectively. The assumption that

$$\mathrm{Re}[\tfrac{1}{3}T^0(s,0,4-s)+\tfrac{1}{2}T^1(s,0,4-s)-\tfrac{5}{6}T^2(s,0,4-s)]/s$$

tends to zero as s tends to infinity, allows one to determine $\lim_{s\to\infty}\mathrm{Re}T^I(s,0,4-s)/s$ by considering the two su-symmetric amplitudes exclusively.

IV. DISCUSSION OF SUM RULES FOR $\pi\pi$ SCATTERING

On general axiomatic grounds,[10] one can write twice-subtracted dispersion relations for the $\pi\pi$ scattering amplitude $T^I(t,s,u)$ with t fixed, $0\le t<4$. We adopt the convention that the first variable denotes the channel in which the isospin is measured.

$$T^I(t,s,u)=a^I(t)+\frac{s^2}{\pi}\int_4^\infty ds'\frac{A^{[I]}(s',t)}{s'^2(s'-s)}+\frac{u^2}{\pi}\int_4^\infty du'\frac{A^{[I]}(u',t)}{u'^2(u'-u)},\quad (I=0,2) \tag{1a}$$

$$T^1(t,s,u)=b(t)(s-u)+\frac{s^2}{\pi}\int_4^\infty ds'\frac{A^{[1]}(s',t)}{s'^2(s'-s)}-\frac{u^2}{\pi}\int_4^\infty du'\frac{A^{[1]}(u',t)}{u'^2(u'-u)}. \tag{1b}$$

We define

$$A^I(s',t)=16\pi\sum_l(2l+1)\mathrm{Im}f_l^I(s')P_l\left(1+\frac{2t}{s'-4}\right), \tag{2a}$$

$$A^{[I]}(s',t)=\sum_{I'}C_{II'}A^{I'}(s',t), \tag{2b}$$

$$A^{\{I\}}(s',t)=\sum_{I'}D_{II'}A^{I'}(s',t)=A^I(s',t)-2C_{I1}A^{[1]}(s',t), \tag{2c}$$

where $C_{II'}$ and $D_{II'}$ are the $s\leftrightarrow t$ and $s\leftrightarrow u$ isospin crossing matrices, respectively.

$$C_{II'}=\begin{bmatrix}\frac{1}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{bmatrix},\quad D_{II'}=\begin{bmatrix}\frac{1}{3} & -1 & \frac{5}{3}\\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6}\\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6}\end{bmatrix}.$$

Crossing symmetry implies

$$T^I(s,t,u)=\sum_{I'}C_{II'}T^{I'}(t,s,u). \tag{3}$$

We would like to comment on some constraints crossing symmetry imposes on Eqs. (1a) and (1b). Equation (3), together with Eqs. (1a) and (1b), implies

$$T^1(s,t,u)=\tfrac{1}{3}a^0(t)+\tfrac{1}{2}b(t)(s-u)-\tfrac{5}{6}a^2(t)+\frac{s^2}{\pi}\int_4^\infty ds'\frac{A^1(s',t)}{s'^2(s'-s)}+\frac{u^2}{\pi}\int_4^\infty du'\frac{A^{\{1\}}(u',t)}{u'^2(u'-u)}, \tag{4}$$

where from Eq. (2c) we find $A^{\{1\}}(u',t)=A^1(u',t)-A^{[1]}(u',t)$. The antisymmetry of $T^1(s,t,u)$ in t and u implies $T^1(s,t,u)=0$ for $s=4-2t$. Then Eq. (4) becomes

$$\frac{T^1(s,t,u)}{t-u}=b(t)+\frac{1}{\pi}\int_4^\infty ds'A^1(s',t)\left[\frac{1}{(s'-s)(s'-4+2t)}-\frac{1}{s'^2}\right]+\frac{1}{\pi}\int_4^\infty du'A^{\{1\}}(u',t)\left[\frac{1}{u'^2}-\frac{1}{(u'-u)(u'-t)}\right]. \tag{5a}$$

Dividing Eq. (1b) by $u-s$ yields

$$\frac{T^1(t,s,u)}{u-s}=-b(t)+\frac{1}{\pi}\int_4^\infty ds'A^{[1]}(s',t)\left[\frac{1}{s'^2}-\frac{1}{(s'-s)(s'-u)}\right]. \tag{5b}$$

The unknown function $b(t)$ disappears when Eqs. (5a) and (5b) are added.

$$\frac{T^1(t,s,u)}{u-s}+\frac{T^1(s,t,u)}{t-u}=\frac{t-s}{\pi}\int_4^\infty ds'\frac{\frac{1}{3}A^0(s',t)-\frac{5}{6}A^2(s',t)+A^1(s',t)[(4-3s')/2(s'+2t-4)]}{(s'-s)(s'-t)(s'-u)}. \tag{6}$$

Setting $t=0$ and letting $s\to 4$, we obtain a sum rule previously derived by Wanders,[6]

$$2a_0^0-5a_0^2-18a_1^1=\frac{1}{2\pi^2}\int_4^\infty ds'\frac{2A^0(s',0)-5A^2(s',0)+3A^1(s',0)[(4-3s')/(s'-4)]}{s'^2(s'-4)}. \tag{7}$$

This sum rule follows from what has been established from axiomatic considerations. It is a good constraint to apply to low-energy $\pi\pi$ models, since it depends chiefly on the energy region below 1 GeV. Furthermore, the ρ and σ contributions enter with opposite signs, making the sum rule very sensitive to the detailed form of the S- and P-wave phase shifts.

Let us note that we have defined the scattering lengths in the following manner:

$$T^I(s,t,u)=16\pi\sum_l(2l+1)f_l^I(s)P_l\left(1+\frac{2t}{s-4}\right), \tag{8a}$$

$$f_l^I(s)=(\sqrt{s}/k)e^{i\delta_l^I(s)}\sin\delta_l^I(s),\quad \text{where } k^2=(s-4)/4, \tag{8b}$$

$$2a_l^I=\lim_{k\to 0} f_l^I(s)/k^{2l}. \tag{8c}$$

If we now make the physically reasonable assumption that the intercept of the leading $I=1$ trajectory is less than 1, then the following superconvergence relation is obtained from Eqs. (5b):

$$b(0)=\frac{1}{\pi}\int_4^\infty ds'\,\frac{A^{[1]}(s',0)}{s'^2}. \tag{9}$$

This implies that Eq. (5b) takes the form

$$\frac{T^1(0,s,4-s)}{4-2s}=-\frac{1}{\pi}\int_4^\infty ds' A^{[1]}(s',0)\frac{1}{(s'-s)(s'+s-4)}. \tag{10a}$$

Equation (10a) is well known; when evaluated at $s=4$, $t=0$, it becomes the Adler sum rule for $2a_0^0-5a_0^2$. The purpose of our discussion is to point out that Eq. (9), together with Eq. (5a), implies

$$\frac{T^1(s,0,4-s)}{s-4}=\frac{1}{\pi}\int_4^\infty ds'\left[\frac{A^1(s',0)}{(s'-s)(s'-4)}-\frac{A^{\{1\}}(s',0)}{(s'+s-4)s'}\right]. \tag{10b}$$

Note that no superconvergence relation for $b(0)$ could have been determined from Eq. (5a) directly. It was necessary to use crossing symmetry and determine $b(0)$ from Eq. (5b). Equation (10b) shows that the P-wave scattering length a_1^1 is determined by knowledge of the total cross sections. Olsson[8] has used Eq. (10b) to obtain information about the P-wave scattering length from the experimental $\pi\pi$ phase shifts. His derivation was formal since he used unsubtracted dispersion relations. It was not obvious from his derivation that one could not add an arbitrary constant to the right-hand side. This would not violate the analyticity or asymptotic behavior of the left-hand side of Eq. (10b). Crossing symmetry rules out the possibility of such an arbitrary additive constant.

Morgan and Shaw[12] have used forward dispersion relations to perform a comprehensive phenomenological study of low-energy $\pi\pi$ scattering. They point out that one of their solutions (denoted DUI in Table I of their paper) has a P-wave scattering length $a_1^1=0.031$, differing from Olsson's result of $a_1^1=0.040\pm0.005$. Morgan and Shaw comment that Olsson's sum rule is automatically satisfied in their work, because their amplitudes satisfy forward dispersion relations. We have shown that Olsson's sum rule follows not from analyticity alone; one must also impose crossing symmetry. Therefore, it is not automatic that the amplitudes of Morgan and Shaw satisfy Olsson's sum rule.

V. SUM RULES FOR D-WAVE SCATTERING LENGTHS

Let us define the function $F_n^I(s,t,u)$ which is symmetric in s and t.

$$F_n^I(s,t,u)=\frac{1}{t-s}\left[\frac{(\partial/\partial t)^n T^I(s,t,u)}{t-u}+\frac{(\partial/\partial s)^n T^I(t,s,u)}{u-s}\right], \tag{11}$$

where $n=1,3,5\ldots$ for I even, and $n=0,2,4,\ldots$ for $I=1$. The t derivative is taken with s fixed, and the s derivative with t fixed. Because of the Bose symmetry of $T^I(s,t,u)$, we have introduced no spurious singularities into $F_n^I(s,t,u)$. Since $T^I(s,t,u)$ satisfies a twice-subtracted dispersion relation, $F_n^I(s,t,u)$ satisfies an unsubtracted dispersion relation.

From Eq. (6), it follows that

$$F_0^1(s,t,u)=\frac{1}{6\pi}\int_4^\infty ds'\frac{2A^0(s',t)-5A^2(s',t)+3A^1(s',t)[(4-3s')/(s'+2t-4)]}{(s'-s)(s'-t)(s'-u)}. \tag{6'}$$

In Sec. II, we saw that $F_0^1(s,t,u)$ was totally symmetric. In Eq. (6′), the symmetry in s and u is manifest, but the symmetry in s and t imposes constraints upon the absorptive parts. Since

$$\frac{\partial}{\partial t}F_0^1(s,t,u)\bigg|_{s=t}=0,$$

we find

$$0=\frac{1}{\pi}\int_4^\infty \frac{ds'}{(s'-t)^2(s'+2t-4)}\left[2\frac{\partial A^0(s',t)}{\partial t}-5\frac{\partial A^2(s',t)}{\partial t}+3\frac{\partial A^1(s',t)}{\partial t}\frac{4-3s'}{s'+2t-4}+6A^1(s',t)\frac{3s'-4}{(s'+2t-4)^2}\right]. \qquad (12)$$

Equation (12) has been previously derived by Roskies.[9] The algebra involved in the derivation presented here is considerably reduced.

We shall use the symmetry of $F_1^I(s,t,u)$ to derive sum rules for the D-wave scattering lengths a_2^I. These relations have been previously derived by Wanders[6] using the Mandelstam representation. Our derivation shows they follow from what has been established by Jin and Martin.[10] Using Eq. (12) together with Wanders's sum rule (15), we obtain an equation relating a_2^0 to the total cross sections which is manifestly positive. Martin[13] has shown that a_l^0 $(l\geq 2)$ is positive. He writes a Froissart-Gribov representation for $f_l^0(t)$, $l\geq 2$, $0\leq t<4$. Since the absorptive part $A^{[0]}(s',t)$ is positive, the limit of $f_l^0(t)/(t-4)^l$, as $t\to 4$, is positive. Our Eq. (17) is a manifestly positive sum rule relating a_2^0 to the total cross sections.

It follows directly from Eq. (1a) that

$$\frac{(\partial/\partial s)T^I(t,s,u)}{u-s}=-\frac{1}{\pi}\int_4^\infty ds' A^{[I]}(s',t)\frac{2s'-s-u}{(s'-s)^2(s'-u)^2}\quad (I=0,2). \qquad (13a)$$

Using Eqs. (1a) and (1b) together with the crossing relations (3) and the condition

$$\frac{\partial}{\partial t}T^I(s,t,u)\bigg|_{t=u}=0,$$

we find

$$\frac{(\partial/\partial t)T^I(s,t,u)}{t-u}=2C_{I1}\frac{\partial b}{\partial t}+\frac{1}{\pi}\int_4^\infty ds'\frac{\partial A^I(s',t)}{\partial t}\left[\frac{1}{(s'-s)(s'+2t-4)}-\frac{1}{s'^2}\right]$$
$$+\frac{1}{\pi}\int_4^\infty ds'\frac{\partial A^{\{I\}}(s',t)}{\partial t}\left[\frac{1}{s'^2}-\frac{1}{(s'-u)(s'-t)}\right]+\frac{1}{\pi}\int_4^\infty ds' A^{\{I\}}(s',t)\frac{2s'-t-u}{(s'-t)^2(s'-u)^2}\quad (I=0,2). \qquad (13b)$$

The unknown function $\partial b/\partial t$ can be determined from the condition that

$$\frac{(\partial/\partial s)T^I(t,s,u)}{u-s}+\frac{(\partial/\partial t)T^I(s,t,u)}{t-u}$$

vanishes when $s=t$. Inserting the resulting expression for $\partial b/\partial t$ into Eq. (13b), we obtain

$$\frac{(\partial/\partial t)T^I(s,t,u)}{t-u}=\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'+2t-4}\frac{\partial A^I(s',t)}{\partial t}\left[\frac{1}{s'-s}-\frac{1}{s'-t}\right]+\frac{1}{\pi}\int_4^\infty\frac{ds'}{s'-t}\frac{\partial A^{\{I\}}(s',t)}{\partial t}\left[\frac{1}{s'+2t-4}-\frac{1}{s'-u}\right]$$
$$+\frac{1}{\pi}\int_4^\infty\frac{ds'}{(s'-t)^2}A^{\{I\}}(s',t)\left[\frac{2s'-t-u}{(s'-u)^2}-\frac{2s'+t-4}{(s'-4+2t)^2}\right]+\frac{1}{\pi}\int_4^\infty ds' A^{[I]}(s',t)\frac{2s'+t-4}{(s'-t)^2(s'+2t-4)^2}. \qquad (14)$$

From Eq. (8), we see that

$$\lim_{s\to 4}\frac{\partial T^I(s,0,4-s)/\partial t}{s-4}=60\pi a_2^I.$$

Evaluating Eq. (14) at $t=0$, and letting $s\to 4$, we find

$$180\pi a_2^I = \frac{4}{\pi}\int_4^\infty ds' \frac{3s'\partial A^I(s',0)/\partial t + 3(s'-4)\partial A^{\{I\}}(s',0)/\partial t}{s'^2(s'-4)^2}$$

$$+\frac{1}{\pi}\int_4^\infty ds' \frac{6(s'-4)^2 A^{\{I\}}(s',0) + 3s'(2s'-4)[A^{[I]}(s',0) - A^{\{I\}}(s',0)]}{s'^3(s'-4)^2} \quad (I=0,2). \tag{15}$$

These sum rules for a_2^I were derived previously by Wanders.[6] By using Eq. (12) we can improve the sum rule for a_2^0 by finding a sum rule which is manifestly positive.

The sum rule for a_2^0 can be written

$$180\pi a_2^0 = \frac{4}{\pi}\int_4^\infty ds \frac{(4s-4)\partial A^0(s,0)/\partial t - 3(s-4)\partial A^1(s,0)/\partial t + 5(s-4)\partial A^2(s,0)/\partial t}{s^2(s-4)^2}$$

$$+\frac{1}{\pi}\int_4^\infty ds \frac{2(s-4)^2 A^0(s,0) + 6(s^2+4s-16)A^1(s,0) + 10(s-4)^2 A^2(s,0)}{s^3(s-4)^2}. \tag{16}$$

Equation (12) evaluated at $t=0$ becomes

$$0 = \frac{4}{\pi}\int_4^\infty ds \frac{(s-4)[2\partial A^0(s,0)/\partial t - 5\partial A^2(s,0)/\partial t] + 3(4-3s)\partial A^1(s,0)/\partial t}{(s-4)^2 s^2} + \frac{4}{\pi}\int_4^\infty ds \frac{6(3s-4)A^1(s,0)}{(s-4)^3 s^2}. \tag{12$'$}$$

Subtracting Eq. (12′) from Eq. (16), we obtain

$$180\pi a_2^0 = \frac{4}{\pi}\int_4^\infty ds \frac{(2s+4)\partial A^0(s,0)/\partial t + 10(s-4)\partial A^2(s,0)/\partial t}{s^2(s-4)^2} + \frac{1}{\pi}\int_4^\infty ds \frac{2A^0(s,0) + 10A^2(s,0)}{s^3}$$

$$+\frac{6}{\pi}\int_4^\infty \frac{ds}{s^2(s-4)^2}\left[\left(\frac{s^2+4s-16}{s} - \frac{4(3s-4)}{s-4}\right)A^1(s,0) + 4s\frac{\partial A^1(s,0)}{\partial t}\right]. \tag{17}$$

The first two integrals on the right-hand side of Eq. (17) are obviously positive. In order to show that the third integral is positive, we expand the absorptive parts into partial waves:

$$A^1(s,0) = 16\pi\sum_{l=1}^\infty (2l+1)\mathrm{Im} f_l^1(s), \qquad \frac{\partial}{\partial t} A^1(s,0) = 16\pi\sum_{l=1}^\infty (2l+1)\mathrm{Im} f_l^1(s)\frac{l(l+1)}{s-4}.$$

The third integral becomes

$$16\pi\times\frac{6}{\pi}\int_4^\infty \frac{ds}{s^2(s-4)^2}\left\{3\,\mathrm{Im} f_1^1(s)\frac{s^2-16}{s} + \sum_{l=3}^\infty (2l+1)\mathrm{Im} f_l^1(s)\left[\frac{s^2+4s-16}{s} + \frac{4[l(l+1)-3]s+16}{s-4}\right]\right\}$$

which is manifestly positive. From Eq. (17), we can derive the following lower bound for a_2^0 by keeping only the contribution from the S and P waves:

$$180\pi a_2^0 > 32\int_4^\infty \frac{ds}{s^3}\left[\mathrm{Im} f_0^0(s) + 5\,\mathrm{Im} f_0^2(s) + 9\left(\frac{s+4}{s-4}\right)\mathrm{Im} f_1^1(s)\right]. \tag{18}$$

VI. APPLICATION OF WANDERS'S SUM RULE TO A MODEL

In a previous paper,[4] we applied the crossing-symmetry sum rules[2] to Brown and Goble's model[14] of $\pi\pi$ scattering. The P wave was considered to be given by experiment and by Brown and Goble, with $m_\rho = 750$ MeV, $\Gamma_\rho = 115$ MeV, $2a_1^1 = 0.059$. We then used the crossing-symmetry sum rules to constrain the S waves. The resulting S waves can be characterized by $a_0^0 = 0.105$, $a_0^2 = -0.051$, $\delta_0^0(m_K) - \delta_0^2(m_K) = 44°$, $\delta_0^0(810\text{ MeV}) = 90°$ and $\delta_0^2(750\text{ MeV}) = -11°$. The phase shift δ_0^0 passes through 90° very slowly, and remains near 90° well above 800 MeV.

We wish to test these model S and P waves against Wanders's[6] sum rule for $2a_0^0 - 5a_0^2 - 18a_1^1$. If we approximate the absorptive parts on the right by the lowest contributing partial waves, Eq. (7) becomes

$$2a_0^0 - 5a_0^2 - 18a_1^1 \cong \frac{8}{\pi}\int_4^L ds \frac{2\,\mathrm{Im} f_0^0(s) - 5\,\mathrm{Im} f_0^2(s) - 9[(3s-4)/(s-4)]\,\mathrm{Im} f_1^1(s)}{s^2(s-4)}. \tag{19}$$

We numerically evaluate the right-hand side for several values of the upper limit of integration L. Our results are given in Table I. The sum rule is satisfied to about 80%. The sum rule is very delicate since the ρ and σ resonances enter with opposite signs. Equation (19) provides a direct test of our unitarization procedure, since both sides are identically zero for Weinberg's current-algebra amplitude,[15] which we have taken to be the K matrix. The disagreement between the left- and right-hand sides of Eq. (19) is not due to our neglect of higher partial waves, since the factor of s^{-3} in the integrand makes the high-energy behavior of the absorptive parts unimportant. Even if the f^0 contribution were not negligible, it would enter with the wrong sign to improve the agreement.

From Eq. (18), we have the following lower bound for a_2^0:

$$a_2^0 > \frac{32}{180\pi}\int_4^\infty \frac{ds}{s^3}\left[\mathrm{Im} f_0^0(s) + 5\,\mathrm{Im} f_0^2(s) + 9\left(\frac{s+4}{s-4}\right)\mathrm{Im} f_1^1(s)\right].$$

Using the model phase shifts, we numerically evaluate the right-hand side and find $a_2^0 > 0.0013$. This can be compared with the D-wave scattering length resulting from the tail of the f^0 parametrized as

TABLE I. Numerical evaluation of sum rule for $2a_0^0 - 5a_0^2 - 18a_1^1$. In the model considered, $2a_0^0 - 5a_0^2 - 18a_1^1 = -0.066$.

L	$\frac{8}{\pi}\int_4^L ds \frac{2\,\mathrm{Im} f_0^0(s) - 5\,\mathrm{Im} f_0^2(s) - 9[(3s-4)/(s-4)]\,\mathrm{Im} f_1^1(s)}{s^2(s-4)}$
12	0.019
20	0.024
28	−0.012
36	−0.045
44	−0.050
84	−0.053

$$f_2^0(s) = \frac{-(k/\sqrt{s})^4 M\Gamma}{s - M^2 + iM\Gamma(k/\sqrt{s})^5},$$

where $M = 1250$ MeV, $\Gamma = 120$ MeV. In this case

$$a_2^0 = \lim_{k\to 0} \frac{f_2^0(s)}{2k^4} \cong \frac{1}{32}\frac{\Gamma}{M} \approx 0.003,$$

which is of the same order of magnitude.

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