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Veneziano-Model Predictions of Density Matrices for Final Vector Mesons*

P. Aurenche
Physics Department, University of Illinois, Urbana, Illinois 61801
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The generalized Veneziano model is used to describe the hypothetical reaction $\sigma\sigma\to\rho\rho\sigma$. The amplitude for this process is, in this model, the residue at the double spin-1 resonance of the seven-point function. The density-matrix elements of the vector mesons produced are calculated, and their variation with the momentum transfers is studied.

I. INTRODUCTION

Ever since the appearance of the Veneziano model[1] and its generalization to N-point functions,[2] numerous attempts have been made to use it to fit strong-interaction data.[3] In particular, great emphasis has been placed upon 2-particle→3-particle processes, for which the function B_5 defined by Bardakci and Ruegg[2] is used. This model is appealing to phenomenologists for many reasons. It has most of the properties required by theory for a scattering amplitude: analyticity, crossing symmetry, resonances at positive integral values of the trajectories with the right residues, single- and double-Regge behavior, and duality between direct-channel resonances and crossed-channel Regge poles. Furthermore, using various arguments of symmetry, exchange degeneracy, and the absence of exotic resonances, the number of free parameters can be reduced to a few constants; there are no undetermined vertex functions as there are in the case of the double-Regge model. So far the fits to the invariant mass and momentum-transfer distributions have been good.[3,4]

However, the model cannot accommodate all the features of strong-interaction physics: (a) The Pomeranchukon exchange, not dual to any resonance, is not described by the model and has to be included artificially (as done by Satz and Pokorski,[5] for instance); (b) there is no satisfying way to deal with baryon trajectories and to avoid parity doubling; (c) few attempts have been made to take spin effects into account, although the model can make definite predictions in this respect.

The inclusion of spin can be accomplished in two different ways: (a) The amplitude is decomposed into its invariant amplitudes, which are then parametrized as a sum of appropriate N-point functions[6]; (b) if a process like $\sigma\sigma\to\sigma\rho$ is to be described, the amplitude is just the residue, at the spin-1 resonance, of the five-point function associated with the reaction $\sigma\sigma\to\sigma\sigma\sigma$.[7] Thus when the trajectories have been chosen, we are left with only one (normalization) parameter. In other words, the invariant amplitudes are completely determined up to an over-all constant.

In the following we shall apply the second method to the process $\sigma\sigma\to\rho\rho\sigma$, where σ is the hypothetical scalar, isoscalar particle of mass equal to that of the pion, and calculate the density-matrix elements of the vector mesons produced. We can thus gain information on the residue functions, about which no definite prediction is made in the double-Regge model. This reaction is chosen for its simplicity; furthermore, by factorization the results can be used to describe $\pi p\to\rho\rho p$ if only the π-A_1 trajectory is included.

II. THE MODEL

We consider a scattering process involving seven spinless particles. Such a reaction is de-

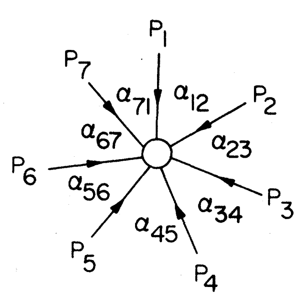

FIG. 1. Kinematics for the seven-particle reaction.

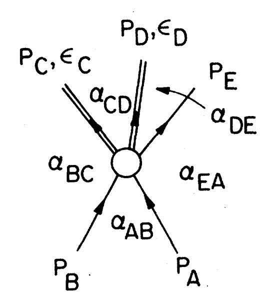

FIG. 2. Kinematics for the production process $\sigma\sigma\to\rho\rho\sigma$.

scribed by the seven-point function B_7 invariant under cyclic and anticyclic permutations of the external momenta[2] (Fig. 1). The residue of B_7 at simultaneous poles of spin 1 in the (71) and (23) channels is just the amplitude for the reaction $\sigma\sigma \to \rho\rho\sigma$ (Fig. 2). The ordering of the particles is important. After a lengthy calculation, we obtain (see Appendix A for details)

$$\begin{aligned}
A = {} & \alpha'^2 p_B\cdot\epsilon_C(p_A-p_E)\cdot\epsilon_D B_5(-\alpha_{AB},-\alpha_{BC},-\alpha_{CD},-\alpha_{DE},-\alpha_{EA}) \\
& -\alpha'^2 p_B\cdot\epsilon_C p_A\cdot\epsilon_D B_5(-\alpha_{AB},-\alpha_{BC},-\alpha_{CD},-\alpha_{DE},-\alpha_{EA}+1) \\
& +[\alpha'^2 p_B\cdot\epsilon_C p_B\cdot\epsilon_D+\alpha'^2(p_A-p_E)\cdot\epsilon_C(p_A-p_E)\cdot\epsilon_D+\tfrac{1}{2}\alpha'^2\epsilon_C\cdot\epsilon_D]B_5(-\alpha_{AB},-\alpha_{BC}+1,-\alpha_{CD},-\alpha_{DE},-\alpha_{EA}) \\
& +[\alpha'^2(p_A-p_E)\cdot\epsilon_C p_A\cdot\epsilon_D-\alpha'^2 p_A\cdot\epsilon_C(p_A-p_E)\cdot\epsilon_D]B_5(-\alpha_{AB},-\alpha_{BC}+1,-\alpha_{CD},-\alpha_{DE},-\alpha_{EA}+1) \\
& -\alpha'^2 p_A\cdot\epsilon_C p_A\cdot\epsilon_D B_5(-\alpha_{AB},-\alpha_{BC}+1,-\alpha_{CD},-\alpha_{DE},-\alpha_{EA}+2) \\
& +\alpha'^2(p_A-p_E)\cdot\epsilon_C p_B\cdot\epsilon_D B_5(-\alpha_{AB},-\alpha_{BC}+2,-\alpha_{CD},-\alpha_{DE},-\alpha_{EA}) \\
& +\alpha'^2 p_A\cdot\epsilon_C p_B\cdot\epsilon_D B_5(-\alpha_{AB},-\alpha_{BC}+2,-\alpha_{CD},-\alpha_{DE},-\alpha_{EA}+1)\,. \qquad (2.1)
\end{aligned}$$

ϵ_C and ϵ_D are the polarization vectors of particles C and D, B_5 is the five-point function of Bardakci and Ruegg, and α' is the common slope of the trajectories.

Several comments must be made about this amplitude:

(a) The residue of B_7 at the simultaneous poles is the amplitude for the production of the ρ mesons together with all their daughters. In (2.1) we have kept only the terms which are linear in each polarization vector; these correspond to the production of the two vector mesons alone.

(b) The model allows only trajectories of natural parity with respect to the external particles.[8] In particular, the particles lying on the trajectories α_{BC} and α_{AE} will have parity $(-1)^J$.

(c) The trajectories in every channel are exchange-degenerate. To allow for nonexchange degeneracy in one channel, we have to add to the amplitude other terms coming from a different ordering of the external momenta in the initial B_7 function and thus introduce a signature factor which will eliminate every second resonance on the leading trajectory. For simplicity this will not be done here.

(d) Notice that we automatically get the invariant amplitudes in terms of B_5 functions (this is a general property of the B_N functions) which give the characteristic single- and double-Regge behavior in the high-energy limit.[9]

(e) For simplicity we choose identical σ trajectories in all channels.[7] This choice will approximate the double π exchange in the real reaction $\pi p \to \rho\rho p$. In order to get the proper Regge behavior and to give nonzero widths to the resonances, it is necessary to put some imaginary part in the trajectories. Therefore, we write

$$\begin{aligned}
\alpha_{i,i+1} &= \alpha'_R(s_{i,i+1}-m_\sigma^2) \\
&\quad + i\alpha'_I\theta(s_{i,i+1}-4m_\sigma^2)(s_{i,i+1}-4m_\sigma^2)^{1/2}, \\
i &= A,B,C,D,E;\quad m_\sigma = 0.138\text{ GeV}; \qquad (2.2)\\
\alpha'_R &= 1\text{ GeV}^{-2};\quad \alpha'_I = 0.1\text{ GeV}^{-2}.
\end{aligned}$$

This particular choice of the imaginary parts gives equal width to the particles lying on the same trajectory as is experimentally observed for meson resonances.

(f) If we set $\alpha_{AE}=0$ or $\alpha_{BC}=0$, the amplitude (2.1) reduces to that of the reactions $\sigma\sigma \to \rho\rho$ or $\sigma\sigma \to \sigma\rho$ obtained directly from the B_6 and B_5 functions, respectively (bootstrap consistency). If α_{AE} and α_{BC} vanish simultaneously, we get just the Feynman amplitude.

We proceed, now, to calculate the density-matrix elements of the mesons produced. To simplify the algebra, this will be done in the double-Regge limit – i.e., when the subenergies s_{CD} and s_{DE} are large compared to the masses or the momentum transfers, with the ratio $s_{CD}s_{DE}/s_{AB}$ kept constant.[9]

III. THE DENSITY-MATRIX ELEMENTS

We now have to define the frames in which we measure the spins of the particles C and D.

For particle C, a convenient choice is the Gottfried-Jackson frame,[10] denoted I, where in the rest frame of particle C the z axis lies along the incident momentum $\vec{p}_B$ and the y axis along $(\vec{p}_A-\vec{p}_E)\times\vec{p}_D=\vec{p}_D\times\vec{p}_B$ [Fig. 3(a)]. In the double-Regge limit, if we keep only the terms of leading order, the momenta of all particles lie in the xz plane.

For particle D there are two possible choices of a Gottfried-Jackson-type frame depending on which of $(\vec{p}_B - \vec{p}_C)$ or $(\vec{p}_A - \vec{p}_E)$ is considered to be the "incident momentum." We define frame II to be the rest frame of particle D, where the z axis is along $(\vec{p}_B - \vec{p}_C)$ and the y axis along $\vec{p}_A \times \vec{p}_E$ [Fig. 3(b)]; whereas in frame II′ the z axis is defined by $(\vec{p}_A - \vec{p}_E) = -(\vec{p}_B - \vec{p}_C)$ and the y axis by $\vec{p}_B \times \vec{p}_C$ [Fig. 3(c)]. Therefore, when t_{BC} is small ρ^{II} (the density matrix of particle D in frame II) will look like the density matrix of the ρ produced in the reaction $\sigma\sigma \rightarrow \rho\sigma$, where t_{AE} is the momentum transfer squared; on the contrary, when t_{AE} is small we expect $\rho^{\text{II}'}$ to have the same behavior as ρ^{I}.

In frames II and II′ the kinematics is more complicated since, in general, the momenta of the particles are not coplanar. Indeed the plane defined by $(\vec{p}_A, \vec{p}_E)$ makes an angle ω, the Toller angle, with the plane $(\vec{p}_B, \vec{p}_C)$. In our approximation, we have[11]

$$\cos\omega = \frac{1}{2\sqrt{-t_{AE}}\sqrt{-t_{BC}}}\left[(t_{BC} + t_{AE} - m_D^2) + \frac{1}{\xi}(t_{BC}^2 + t_{AE}^2 + m_D^4 - 2t_{BC}t_{AE} - 2t_{BC}m_D^2 - 2t_{AE}m_D^2)\right], \tag{3.1}$$

where $\xi = s_{CD}s_{DE}/s_{AB}$. The condition $|\cos\omega| \leq 1$ requires

$$(\sqrt{-t_{AE}} - \sqrt{-t_{BC}})^2 \leq \xi - m_D^2 \leq (\sqrt{-t_{AE}} + \sqrt{-t_{BC}})^2, \tag{3.2}$$

which shows that if one momentum transfer is small, the other has to be large enough in order to satisfy the second inequality. It is easy to get the following relation between ρ^{II} and $\rho^{\text{II}'}$:

$$\rho^{\text{II}'}_{\lambda_D\mu_D} = (-1)^{\lambda_D - \mu_D} e^{-i(\lambda_D - \mu_D)\omega} \rho^{\text{II}}_{-\lambda_D - \mu_D}. \tag{3.3}$$

IV. PARITY TRANSFORMATION

The amplitude can be written in a condensed form as

$$A_{\lambda_C\lambda_D} = \sum_{i,j=1}^{3} A_{ij} p_i \cdot \epsilon_{\lambda_C} q_j \cdot \epsilon_{\lambda_D}, \tag{4.1}$$

where p_i, q_j are any three independent external momenta. The quantity $p_i \cdot \epsilon_{\lambda_C}$ is calculated in I and $q_j \cdot \epsilon_{\lambda_D}$ in II or II′. The numbers A_{ij} are functions of the five invariants $(s_{AB}, s_{CD}, s_{DE}, t_{BC}, t_{AE})$ denoted (inv), whereas the polarization products depend on (inv) and $\sin\varphi_i$, φ_i being the azimuthal angle of p_i or q_i. Since the azimuthal angles are pseudoscalar, they can be expressed in terms of one of them, say $\sin\omega$. (This is true because there is only one independent pseudoscalar constructed out of the external momenta of the problem, namely $\epsilon^{\alpha\beta\gamma\delta} p_{A\alpha} p_{B\beta} p_{C\gamma} p_{D\delta}$. The value of $\sin\omega$ is equal to this, up to some invariant factors.) Writing explicitly the arguments in the amplitude, we have

$$A_{\lambda_C\lambda_D}(\text{inv}, \sin\omega) = \sum_{i,j=1}^{3} A_{ij}(\text{inv}) p_i \cdot \epsilon_{\lambda_C}(\text{inv}, \sin\omega) q_j \cdot \epsilon_{\lambda_D}(\text{inv}, \sin\omega). \tag{4.2}$$

Applying the parity operator, it becomes

$$A_{\lambda_C\lambda_D}(\text{inv}, \sin\omega) = \eta_A\eta_B\eta_C\eta_D\eta_E(-1)^{s_C - \lambda_C}(-1)^{s_D - \lambda_D} A_{-\lambda_C - \lambda_D}(\text{inv}, -\sin\omega). \tag{4.3}$$

The η's are the intrinsic parities of the particles.

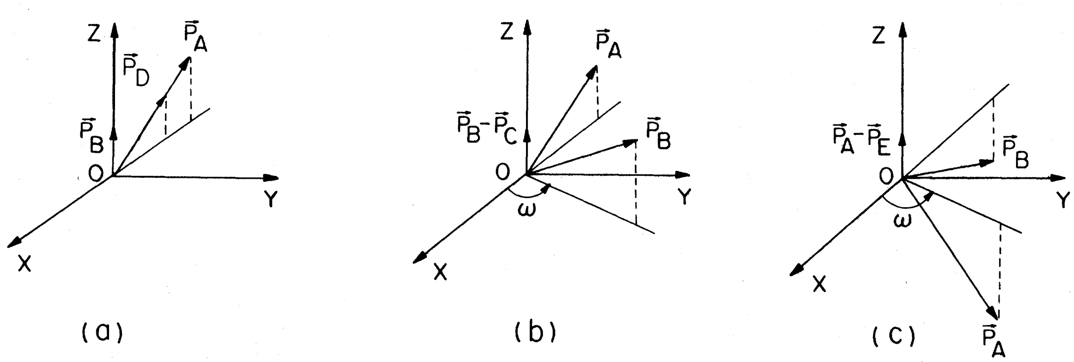

FIG. 3. (a) Definition of frame I where the spin of particle C is measured. (b) Definition of frame II where the spin of particle D is measured. (c) Definition of frame II′.

Since in frame I all the momenta are coplanar, $p_i \cdot \epsilon_{\lambda_1}$ does not depend on $\sin\omega$ and, therefore, we get the well-known relations

$$\rho^{\mathrm{I}}_{\lambda_C \mu_C}(\mathrm{inv}) = (-1)^{\lambda_C - \mu_C} \rho^{\mathrm{I}}_{-\lambda_C - \mu_C}(\mathrm{inv}) . \tag{4.4}$$

For particle D we have (in both frame II and frame II′)

$$\rho^{\mathrm{II}}_{\lambda_D \mu_D}(\mathrm{inv}, \sin\omega) = (-1)^{\lambda_D - \mu_D} \rho^{\mathrm{II}}_{-\lambda_D - \mu_D}(\mathrm{inv}, -\sin\omega) . \tag{4.5}$$

V. RESULTS

We study, first, the variations of the density matrices with t_{AE} when t_{BC} is fixed. The matrix elements of particle C are independent of t_{AE} for constant t_{BC} [Fig. 4 (a)]. This does not come as a surprise since the amplitude factorizes and, therefore, the left vertex does not depend on the right momentum transfer. Because we have the additional symmetry $\rho^{\mathrm{I}}_{-11} = -\rho^{\mathrm{I}}_{11}$, due to the fact that ρ^{I} does not depend on the Toller angle and that only natural parity can be exchanged in the (BC) channel, we have plotted only $\mathrm{Re}\rho_{10}$ and ρ_{11}. In frame II′ notice the change in $\mathrm{Re}\rho_{10}$, when $-t_{BC}$ passes through the value $\xi - m_D{}^2$, for small values of $-t_{AE}$ [Fig. 5 (b)].

If we fix now t_{AE} and vary t_{BC}, we get the curves of Fig. 5. The density-matrix elements of particle C have the same behavior as those of the vector meson produced in the reaction[7] $\sigma\sigma \to \rho\sigma$ (this is a consequence of the bootstrap consistency of the model). The elements of ρ^{II} are similar to those of $\rho^{\mathrm{II}'}$, where t_{AE} and t_{BC} have been interchanged. Note also that, as expected, for small t_{AE}, $\mathrm{Re}\rho^{\mathrm{II}'}_{10}$ and $\rho^{\mathrm{II}'}_{11}$ have the same variations as $\mathrm{Re}\rho^{\mathrm{I}}_{10}$ and ρ^{I}_{11}, which shows that the vertex functions are slowly varying when the external masses are continued off the mass shell.

The change which appears in $\mathrm{Re}\rho^{\mathrm{II}}_{10}$ ($\mathrm{Re}\rho^{\mathrm{II}'}_{10}$) when t_{BC} (t_{AE}) goes through $-(\xi - m_D{}^2)$ is due to kinematical effects. More precisely, the trouble arises because, for certain values of the pair ($-t_{BC}$, $-t_{AE}$), the frames II or II′ are not well defined. Figure 6 shows, for fixed ξ, the range of variation of the momentum transfers for which the reaction $\sigma\sigma \to \rho\rho\sigma$ is kinematically allowed. In frame II, when $-t_{AE}$ goes through $(\xi - m_D{}^2)$ and $-t_{BC}$ is kept equal to its minimum value (which corresponds to $\sin\omega = 0$), the momenta of the external particles and therefore all the polarization products $p \cdot \epsilon_D$ vary continuously, although $\cos\omega$ does not: ρ^{II} has then a smooth variation [the products

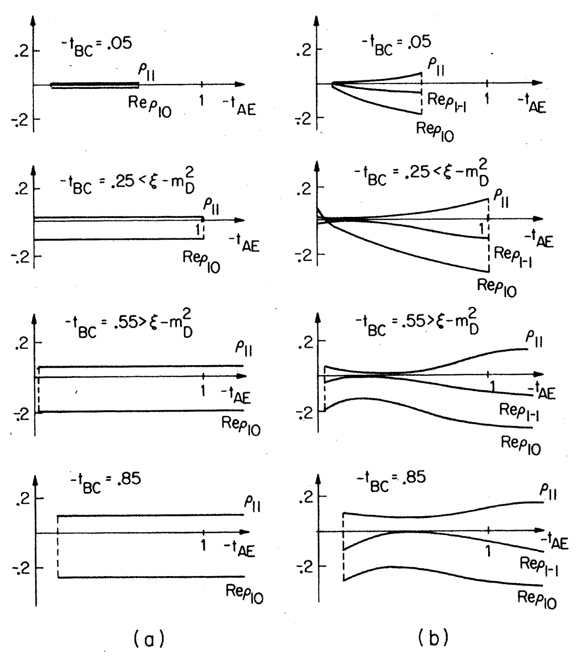

FIG. 4. (a) Variation of the density-matrix elements of particle C in frame I at fixed momentum transfer t_{BC} when t_{AE} varies. (b) Variation of the density-matrix elements of particle D in frame II at fixed momentum transfer t_{BC} when t_{AE} varies.

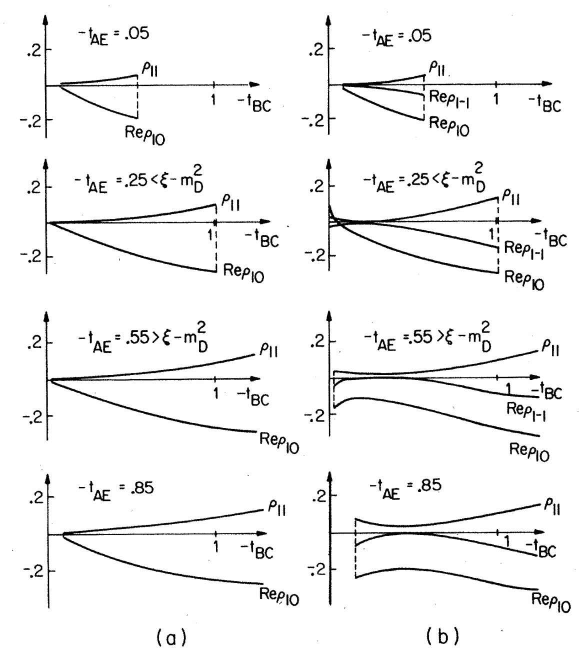

FIG. 5. (a) The same as Fig. 4. (a) When t_{AE} is fixed and t_{BC} varies. (b) Variation of the density-matrix elements of particle D in frame II′ at fixed t_{AE} when t_{BC} varies.

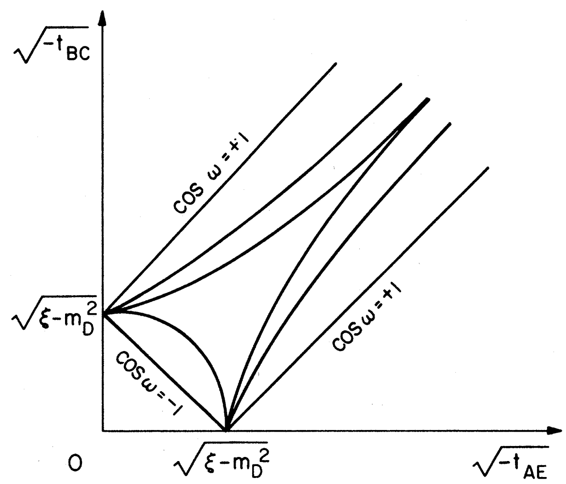

FIG. 6. The physical region of scattering for fixed ξ, is the domain enclosed by the lines $\cos\omega=-1$ and $\cos\omega=+1$. Curves of constant $\cos\omega$ are also shown.

$p\cdot\epsilon_D$ are, of course, continuous, since they depend only on (inv)]. But $\vec{p}_B\times\vec{p}_C$ goes through zero and changes sign; in frame II′, where the y axis is along $\vec{p}_B\times\vec{p}_C$, this causes the momenta of the particles nonparallel to Oz to be discontinuous. Using now (3.3) which states in particular

$$\mathrm{Re}\rho_{10}^{\mathrm{II}'}=-(\cos\omega\,\mathrm{Re}\rho_{-10}^{\mathrm{II}}+\sin\omega\,\mathrm{Im}\rho_{-10}^{\mathrm{II}}) \tag{5.1}$$

and (4.5) which, for $\sin\omega=0$, yields

$$\mathrm{Re}\rho_{-10}^{\mathrm{II}}=-\mathrm{Re}\rho_{10}^{\mathrm{II}}, \tag{5.2}$$

we get

$$\mathrm{Re}\rho_{10}^{\mathrm{II}'}=\cos\omega\,\mathrm{Re}\rho_{10}^{\mathrm{II}}\quad(\text{for }\sin\omega\simeq0\text{ only}). \tag{5.3}$$

This explains the change of $\mathrm{Re}\rho_{10}^{\mathrm{II}'}$ seen in Fig. 5(b).

The model described above is interesting because it contains a minimum number of parameters, namely, the trajectories and an over-all normalization constant, but it is limited in its applications by the fact that it allows only natural parity for the parent trajectories. Also the spin of the nucleons is not properly taken into account.

VI. ACKNOWLEDGMENT

I would like to thank Professor L. M. Jones for help and encouragement during the course of this work.

APPENDIX A

Consider a scattering process involving seven identical particles (Fig. 1). All momenta are taken to be incoming. Define the invariant subenergies

$$s_{i,i+1}=(p_i+p_{i+1})^2\quad\text{and}\quad s_{j,j+1,j+2}=(p_j+p_{j+1}+p_{j+2})^2,\qquad i,j=1,2,\ldots,7\,.$$

Two types of channels exist in this reaction, namely, 2 particles → 5 particles and 3 particles → 4 particles. Consequently, we introduce seven trajectories associated with the two-body channels:

$$\alpha_{i,i+1}=\alpha^0_{i,i+1}+\alpha' s_{i,i+1},\quad i=1,2,\ldots,7$$

and seven trajectories associated with the three-body channels:

$$\alpha_{j,j+1,j+2}=\alpha^0_{j,j+1,j+2}+\alpha' s_{j,j+1,j+2},\quad j=1,2,\ldots,7\,.$$

We define the amplitude for this process by[2]

$$B_7(-\alpha_{i,i+1};-\alpha_{j,j+1,j+2})=\int_0^1\cdots\int_0^1 du_7du_2dv_6dv_2\frac{v_6v_2}{u_1^3v_7v_1}\prod_{i=1}^7 u_i^{-\alpha_{i,i+1}-1}\prod_{j=1}^7 v_j^{-\alpha_{j,j+1,j+2}-2}, \tag{A1}$$

where the integration variables u_i and v_j satisfy the following relations (necessary to prevent the occurrence of coincident poles in the amplitude which do not correspond to Feynman graphs):

$$\begin{aligned}
&u_1=1-u_7u_2v_6v_2,\quad v_1=\frac{1-v_2v_6u_7}{1-u_7u_2v_6v_2},\\
&u_3=\frac{1-u_2}{1-u_2v_2},\quad v_3=\frac{1-u_2v_2}{1-u_2v_2v_6},\\
&u_4=\frac{(1-v_2)(1-u_2v_2v_6)}{(1-u_2v_2)(1-v_2v_6)},\quad v_4=\frac{(1-v_2v_6)(1-u_2u_7v_2v_6)}{(1-u_2v_2v_6)(1-u_7v_2v_6)},\\
&u_5=\frac{(1-v_6)(1-u_7v_2v_6)}{(1-v_2v_6)(1-u_7v_6)},\quad v_5=\frac{1-u_7v_6}{1-u_7v_2v_6},\\
&u_6=\frac{1-u_7}{1-u_7v_6},\quad v_7=\frac{1-u_2v_2v_6}{1-u_2u_7v_2v_6}.
\end{aligned} \tag{A2}$$

The factor $v_6v_2/u_1^3v_7v_1$ is necessary to ensure the invariance of B_7 under cyclic and anticyclic permutations of the external momenta. Written under this form, B_7 already displays poles at zero values of the trajectories. We need now to continue the function to the region where the α's are positive. For this purpose we expand the integrand in powers of u_2 and u_7 and then integrate term by term in these two variables. It comes out

$$B_7=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty}\frac{(-1)^{l+m+n+p}}{l+m+n+p-\alpha_{23}}\,\frac{(-1)^{l'+m'+n'+p}}{l'+m'+n'+p-\alpha_{71}}$$
$$\times\frac{(a)_l}{l!}\frac{(b)_m}{m!}\frac{(c)_n}{n!}\frac{(d)_p}{p!}\frac{(a')_{l'}}{l'!}\frac{(b')_{m'}}{m'!}\frac{(c')_{n'}}{n'!}B_5(m+n+p+n'-\alpha_{234},-\alpha_{45},-\alpha_{56},n+p+m'+n'-\alpha_{671},-\alpha_{456}),\tag{A3}$$

where B_5 is the five-point function of Bardakci and Ruegg.

$$\begin{aligned}&(a)_n=a(a-1)\cdots(a-n+1),\\&(a)_0=1,\\&a=-\alpha_{34}-1,\quad a'=-\alpha_{67}-1,\\&b=\alpha_{34}+\alpha_{45}-\alpha_{345},\quad b'=\alpha_{56}+\alpha_{67}-\alpha_{567},\\&c=\alpha_{345}+\alpha_{456}-\alpha_{45}-\alpha_{712},\quad c'=\alpha_{456}+\alpha_{567}-\alpha_{56}-\alpha_{123},\\&d=\alpha_{712}+\alpha_{123}-\alpha_{12}-\alpha_{456}.\end{aligned}\tag{A4}$$

This formula shows explicitly the pole structure of B_7 and allows the continuation of the initial integral representation to positive values of the α's, except for poles at positive integers.

We concentrate now on the residue at $\alpha_{23}=1$ and $\alpha_{71}=1$. It corresponds to the production and decay of two spin-1 resonances together with their daughters. If we write the residue in terms of the external momenta p_i's, the amplitude for the production and decay of the vector mesons will be made up of the terms which are linear in both (p_2-p_3) and (p_7-p_1), where p_2 and p_3 are the momenta of the decay products in the (23) channel and p_1 and p_7 those in the (71) channel. We make the following change of notation:

$$\begin{aligned}&p_7+p_1=-p_C,\quad \alpha_{456}=\alpha_{CD},\\&p_2+p_3=-p_D,\quad \alpha_{234}=\alpha_{DE},\\&p_4=-p_E,\quad \alpha_{45}=\alpha_{EA},\\&p_5=p_A,\quad \alpha_{56}=\alpha_{AB},\\&p_6=p_B,\quad \alpha_{671}=\alpha_{BC},\end{aligned}\tag{A5}$$

and

$$\begin{aligned}&p_7-p_1=\epsilon_C,\quad p_C\cdot\epsilon_C=0,\quad \epsilon_C{}^2<0,\\&p_3-p_2=\epsilon_D,\quad p_D\cdot\epsilon_D=0,\quad \epsilon_D{}^2<0.\end{aligned}\tag{A6}$$

We can interpret ϵ_C and ϵ_D as the polarization vectors of two spin-1 particles C and D produced in the channels (23) and (71), respectively, and conclude that the production amplitude for these two vector mesons is made of the terms in the residue which are linear in both ϵ_C and ϵ_D. This is legitimate since the amplitude for the decay of a vector meson into two scalar particles of momenta q and q' is unique and of the form $\epsilon\cdot(q-q')$. Using, now, the recursion relation

$$\begin{aligned}B_5(-\alpha_{AB},-\alpha_{BC},-\alpha_{CD},-\alpha_{DE}+1,-\alpha_{EA})=&B_5(-\alpha_{AB},-\alpha_{BC},-\alpha_{CD},-\alpha_{DE},-\alpha_{EA})\\&-B_5(-\alpha_{AB},-\alpha_{BC},-\alpha_{CD}+1,-\alpha_{DE},-\alpha_{EA}+1),\end{aligned}\tag{A7}$$

we get amplitude (2.1).

APPENDIX B

The polarization vector ϵ_{λ_i} will be defined in the rest frame of particle i by

$$\epsilon_{\pm 1} = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \pm 1\\ i\\ 0\end{pmatrix}, \quad \epsilon_0 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{B1}$$

We denote

$$\Lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx .$$

The following results hold in the double-Regge limit only. In frame I we have

$$\begin{aligned}
p_C &= (m_C, 0, 0, 0),\\
p_D &= \left(\frac{s_{CD}}{2m_C}, -\frac{s_{CD}}{2m_C}\sin\theta, 0, \frac{s_{CD}}{2m_C}\cos\theta\right),\\
p_E \simeq p_A &= \left(\frac{s_{AB}}{2m_C}, -\frac{s_{AB}}{2m_C}\sin\theta, 0, \frac{s_{AB}}{2m_C}\cos\theta\right),\\
p_B &= \left(\frac{m_B{}^2 + m_C{}^2 - t_{BC}}{2m_C}, 0, 0, \frac{\Lambda^{1/2}(m_B{}^2, m_C{}^2, t_{BC})}{2m_C}\right),
\end{aligned} \tag{B2}$$

where

$$\cos\theta = -\frac{t_{BC} + m_C{}^2 - m_B{}^2}{\Lambda^{1/2}(m_B{}^2, m_C{}^2, t_{BC})},$$

$$\sin\theta = \frac{2m_C t_{BC}}{\Lambda^{1/2}(m_B{}^2, m_C{}^2, t_{BC})}.$$

In frame II

$$\begin{aligned}
p_D &= (m_D, 0, 0, 0),\\
p_B - p_C &= \left(\frac{t_{BC} - t_{AE} + m_D{}^2}{2m_D}, 0, 0, \frac{\Lambda^{1/2}(t_{BC}, t_{AE}, m_D{}^2)}{2m_D}\right),\\
p_B &= \left(\frac{s_{CD}}{2m_D}, \frac{s_{CD}\sin\theta_B\cos\omega}{2m_D}, \frac{s_{CD}\sin\theta_B\sin\omega}{2m_D}, \frac{s_{CD}\cos\theta_B}{2m_D}\right),\\
p_E \simeq p_A &= \left(\frac{s_{DE}}{2m_D}, -\frac{s_{DE}\sin\theta_A}{2m_D}, 0, -\frac{s_{DE}\cos\theta_A}{2m_D}\right).
\end{aligned} \tag{B3}$$

In frame II′

$$\begin{aligned}
p_D &= (m_D, 0, 0, 0),\\
p_A - p_E &= \left(\frac{t_{AE} - t_{BC} + m_D{}^2}{2m_D}, 0, 0, \frac{\Lambda^{1/2}(t_{BC}, t_{AE}, m_D{}^2)}{2m_D}\right),\\
p_B &= \left(\frac{s_{CD}}{2m_D}, -\frac{s_{CD}\sin\theta_B}{2m_D}, 0, -\frac{s_{CD}\cos\theta_B}{2m_D}\right),\\
p_E \sim p_A &= \left(\frac{s_{DE}}{2m_D}, \frac{s_{DE}\sin\theta_A\cos\omega}{2m_D}, \frac{s_{DE}\sin\theta_A\sin\omega}{2m_D}, \frac{s_{DE}\cos\theta_A}{2m_D}\right),
\end{aligned} \tag{B4}$$

where

$$\cos\theta_B = \frac{t_{BC} - t_{AE} + m_D{}^2}{\Lambda^{1/2}(t_{BC}, t_{AE}, m_D{}^2)}, \quad \sin\theta_B = \frac{2m_D\sqrt{-t_{BC}}}{\Lambda^{1/2}(t_{BC}, t_{AE}, m_D{}^2)},$$

$$\cos\theta_A = \frac{t_{AE} - t_{BC} + m_D{}^2}{\Lambda^{1/2}(t_{BC}, t_{AE}, m_D{}^2)}, \quad \sin\theta_A = \frac{2m_D\sqrt{-t_{AE}}}{\Lambda^{1/2}(t_{BC}, t_{AE}, m_D{}^2)},$$

$$\cos\omega = \frac{1}{2\sqrt{-t_{BC}}\,\sqrt{-t_{AE}}}\left(t_{BC} + t_{AE} - m_D{}^2 + \frac{1}{\xi}\Lambda(t_{BC}, t_{AE}, m_D{}^2)\right),$$

$$\xi = \frac{s_{CD}s_{DE}}{s_{AB}}.$$

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Crossing-Symmetry Restrictions on Dispersion Relations, and Sum Rules for $\pi\pi$ Scattering Lengths*

Samuel Krinsky†
Yale University, New Haven, Connecticut 06520
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We study the constraints crossing symmetry imposes on fixed-variable dispersion relations for $\pi\pi$ scattering. We show that the sum rules relating $2a_0^0-5a_0^2-18a_1^1$, a_2^0, and a_2^2 to the total cross sections, which were derived by Wanders using the Mandelstam representation, follow from twice-subtracted dispersion relations. These sum rules are good physical-region constraints to supplement the unphysical-region constraints of Martin and Roskies in the study of models for low-energy $\pi\pi$ scattering. Using a restriction on the absorptive parts following from crossing symmetry, we transform Wanders's sum rule for the $I=0$, $l=2$ scattering length into a form which is manifestly positive. Keeping only the S- and P-wave contributions, we obtain a lower bound for a_2^0. If the ρ-trajectory intercept is less than 1, we show that $\lim_{s\to\infty} \mathrm{Re}\, T^I(s,0,4-s)/s$ is determined by the total cross sections. If, in addition, the leading isospin-2 trajectory has intercept less than zero, then even without imposing elastic unitarity, the $I=0$ S wave is determined by the absorptive parts without the freedom of adding an arbitrary constant.

I. INTRODUCTION

Martin[1] has derived rigorous inequality constraints on the $\pi\pi$ partial-wave amplitudes in the unphysical region $0\le s\le 4m_\pi^2$. Roskies[2] has found sum rules involving integrals of the partial-wave amplitudes over $0\le s\le 4m_\pi^2$, which follow from crossing symmetry. There have been recent attempts[3,4] to use these unphysical-region constraints to study the behavior of the $\pi\pi$ amplitudes above threshold. Within a given parametrization of the partial waves, it has been possible to make physical-region predictions.[3,4] However, Ulrich[5] has found a new parametrization of the S and P waves in which the unphysical-region constraints hardly constrain the physical-region phase shifts. He introduced the experimental ρ meson into the P wave, and found that there existed a family of S waves exactly satisfying the Martin and Roskies constraints, which had drastically different phase shifts above threshold.

In this note, we discuss several sum rules which relate the $\pi\pi$ scattering lengths to integrals of the absorptive parts over the physical region. We show that these sum rules, which were originally derived by Wanders[6] using the Mandelstam representation, are direct consequences of twice-subtracted dispersion relations. Since the integrands behave like s^{-3} at large energies, these sum rules are most sensitive to the energy region below 1 GeV. Therefore, these sum rules are good physical-region constraints to supplement the unphysical-region constraints of Martin and Roskies in the study of low-energy $\pi\pi$ models. They relate the tip of the unphysical region to the resonance region. Also, the ρ and σ enter the sum rule for $2a_0^0-5a_0^2-18a_1^1$ with opposite signs, making it very sensitive to the detailed form of the S- and P-wave phase shifts.

In Sec. II, we discuss the constraints crossing symmetry imposes upon the subtraction constants appearing in fixed-variable dispersion relations.