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Analytic Properties of Three-Body Decay Amplitudes*

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We study the nature and location of the physical singularities of a weak three-body decay amplitude in perturbation theory (nonrelativistic). We find only the threshold singularities one would expect. These are most easily expressed if final momentum variables are scaled by the total energy.

I. INTRODUCTION

In this paper we study the location and nature of the physical singularities of a weak three-body decay amplitude. The purpose of such a study is to sharpen our formal understanding of this simplest of the three-body scattering problems with an eye both to improved parametrization and approximation schemes for three-body decays and to more complex three-body problems. We also hope to shed light on the perennial problem of variable choice, by finding the variables in terms of which the singularities are most naturally expressed.

The model we consider is that of the weak three-body decay of a scalar structureless object into three interacting scalar particles. For simplicity we take them all to have the same mass, but one can easily see that unequal masses will not change any of our results significantly. The assumption that the decay interaction is structureless is also made for simplicity. Since the amplitude for weak decay is linear in that interaction (that is what we mean by weak decay), any more complex decay interaction form is easily studied. Forms that just give the weak-decay volume a finite size[1] will not affect our results. We assume the decay products interact via Yukawa potentials, but since we are concentrating only on the physical singularities, the detailed form of the interaction is irrelevant. We use nonrelativistic methods throughout; since all our singularities are thresholds, this seems appropriate. We analyze the decay amplitude in perturbation theory, but presumably our results are not dependent on the convergence of the perturbation expansion in the final-state interaction, particularly as every perturbation term (beyond the first few) has the same *leading* singularity.

As one would expect, the physical singularities of the decay amplitude are all threshold singularities. There are two types of thresholds in the problem. One is the two-body subenergy threshold of a given pair. The second is the threshold in the total energy released in the decay, which we call E. In the decay rest frame, E is related to the momenta of the three decay products $(\vec{q}_i)$ by

$$E = q_1^2 + q_2^2 + q_3^2 \tag{1.1}$$

$(\hbar = 2m = 1)$. We are interested here not just in the location of the singularities, but in their forms. Clearly we see from (1.1) that we cannot consider the behavior of the amplitude at the $E \to 0$ threshold for fixed $(\vec{q}_i)$ and keep the amplitude on the energy shell. It is therefore more convenient to label the decay products with scaled momenta,

$$\begin{aligned} \vec{q}_i &= \vec{y}_i \sqrt{E}\,, \\ 1 &= y_1^2 + y_2^2 + y_3^2 . \end{aligned} \tag{1.2}$$

We can then study $E \to 0$ for fixed y_1 and stay on shell. We shall see that the singularities of the amplitude are simpler even off shell if we consider y_i fixed rather than q_i.

In the decay rest frame the two-body subenergy threshold of a given pair (lm) is at $E = \frac{3}{2} q_n^2$ $(l \neq m \neq n)$. We find the singularity associated with that threshold is of the square-root type $[(E - \frac{3}{2} q_n^2)^{1/2}]$, as one would expect. For scaled momenta this gives $[E(1 - \frac{3}{2} y_n^2)]^{1/2}$. The leading *three-body* singularity for scaled momenta is of the form $E \ln(-E)$, occurring in every order of perturbation theory beyond the first, and all singularities are of the form

$$E^{m/2} \ln^n(-E), \quad n = 0, 1, 2, \ldots ; \quad m = 2n, 2n+1, \ldots .$$

In Sec. II we show by using the standard techniques of Feynman-diagram analysis how singularities arise in perturbation theory. In Sec. III we analyze the singularities from the point of view of unitarity, showing in particular why one gets $E\ln(-E)$ rather than the phase-space result $E^2\ln(-E)$. In Sec. IV, we present some conclusions and directions for application of our results.

II. PERTURBATION ANALYSIS

In this section we take the individual perturbation diagrams and find the singularity structure for each of them. This is done in two steps. The first is to find the points at which the respective integrals are singular, by means of the usual pinch analysis.[2] Having found the region of integration that contributes to the singularity, we approximate the integrand in a suitable way and find the nature of the singularity in the external variables. The treatment follows that of Eden *et al.*,[2] but our problem is considerably more complicated than the cases that they discuss. This is in part because our nonrelativistic treatment puts particle propagators and potentials on a different footing, and in part because we must investigate the superposition of several singularities. We restrict our attention to singularities occurring for physical values of the external variables. In the work below, our calculations frequently employ a determinant formed from the coefficients of the momenta over which we integrate. When this determinant does not vanish, a simple power-counting argument, as given in Eden *et al.*,[2] suffices to display the behavior of the singularity. Thus, the complications indicated above are associated with the vanishing of this determinant; this in turn appears to correspond to what Eden *et al.*[2] call "non-Landauian" or second-type singularities. Since we do not know how to exploit this classification for the purpose of obtaining the nature of such singularities, we resort to the special methods given below.

A. Skeleton Graphs

Consider the nth-order diagram (Fig. 1) in which the potential does not act successively on the same pair of particles. Since the potential will be seen below to serve only to give convergent integrals,

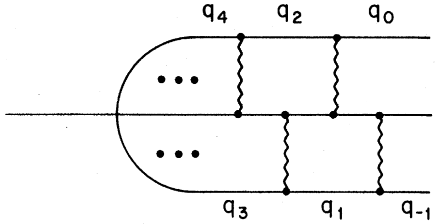

FIG. 1. General nth-order skeleton graph in which the potential does not act successively on the same pair of particles, and in which the upper and lower particle do not interact.

and does not affect the analytic properties that we derive (so long as it is short ranged and nonsingular), we can consider these graphs as approximations to the case when the full two-body t matrix occurs n times. In Sec. II B we show that the general nth-order graph has no additional singularities.

The skeleton graphs have propagators ($\hbar = 2m = 1$)

$$P_i = 2(q_i^2 + \vec{q}_i \cdot \vec{q}_{i-1} + q_{i-1}^2) - E,$$

and potential terms

$$Q_i = (\vec{q}_i - \vec{q}_{i-2})^2 + \mu^2.$$

In terms of these, the nth-order skeleton graph of Fig. 1 is, up to a constant,

$$F_n = \int \prod_{i=1}^{n} d^3q_i P_i^{-1} Q_i^{-1} \tag{2.1a}$$

$$= (2n-1)! \int \prod_{i=1}^{n} d^3q_i \int \prod_{j=1}^{n} d\alpha_j d\beta_j \times [D(\alpha, \beta, q)]^{-2n} \delta\Big(1 - \sum_{l=1}^{n} (\alpha_l + \beta_l)\Big), \tag{2.1b}$$

$$D(\alpha, \beta, q) = \sum_{i=1}^{n} (\alpha_i P_i + \beta_i Q_i), \tag{2.2}$$

where we have employed Feynman parameters to collect the various denominators.

The singularities of F_n occur when D vanishes either so as to "pinch" the hypercontour or at a boundary.[2] The boundaries of integration here are $\alpha_i = 0$ and $\beta_j = 0$; the integrations over q have no boundaries. In this way we obtain the Landau equations

$$\frac{\partial}{\partial \vec{q}_i} D = 0, \quad i = 1, \ldots, n \tag{2.3}$$

$$\alpha_i \frac{\partial}{\partial \alpha_i} D = \alpha_i P_i = 0, \quad i = 1, \ldots, n \tag{2.4}$$

$$\beta_i \frac{\partial}{\partial \beta_i} D = \beta_i Q_i = 0, \quad i = 1, \ldots, n. \tag{2.5}$$

Equation (2.3) is short for $3n$ equations, one for each component of $\vec{q}_i$. Since D is homogeneous in α and β, (2.4) and (2.5) together imply that D vanishes. We know that the potential term Q_i cannot vanish for physical momenta, and so (2.5) implies that β_i vanishes for the *physical* singularities. This is precisely equivalent to saying that the potentials contribute only unphysical singularities, and hence that their detailed form is not significant here.

We will want to approximate D in the neighborhood of the points in the space of variables of integration that give rise to the singularities, i.e., these points given by the Landau equations. Since we cannot expand the δ function in (2.1b), we eliminate it by doing one α integration, i.e., eliminating one Feynman parameter; we choose to eliminate α_n. In doing this we lose the homogeneity of D and therefore must require that it vanish. Our equations become

$$\beta_i = 0, \quad i = 1, \ldots, n \tag{2.6}$$

$$\alpha_n = 1 - \sum_{i=1}^{n-1} \alpha_i, \tag{2.7}$$

$$\alpha_i \frac{\partial}{\partial \alpha_i} D = \alpha_i (P_i - P_n) = 0, \quad i = 1, \ldots, n-1 \tag{2.8}$$

$$\frac{1}{2} \frac{\partial}{\partial \vec{q}_i} D = \alpha_i (2\vec{q}_i + \vec{q}_{i-1}) + \alpha_{i+1}(2\vec{q}_i + \vec{q}_{i+1}) = 0, \quad i = 1, \ldots, n \tag{2.9}$$

where we define $\alpha_{n+1} \equiv 0$ and where $\vec{q}_0$ and $\vec{q}_{-1}$ are external momenta. We now need to consider all the sets of equations that arise for the various choices of which α's vanish, and to find the singularity structure for that choice. We start our discussion with a detailed treatment of the low-order graphs.

$n = 1$

The Landau equations (2.6)–(2.9) reduce to $\beta_1 = 0$ and

$$\frac{1}{2} \frac{\partial D}{\partial \vec{q}_1} = 2\vec{q}_1 + \vec{q}_0 = 0.$$

We expand about the point $\beta_1 = 0$, $\vec{q}_1 = -\frac{1}{2}\vec{q}_0$, and obtain

$$D = \tfrac{3}{2} q_0^2 - E + 2(\vec{q}_1 + \tfrac{1}{2}\vec{q}_0)^2 + \beta_1(\mu^2 + E - \tfrac{5}{4} q_0^2 + \vec{q}_0 \cdot \vec{q}_{-1} + q_{-1}^2). \tag{2.10}$$

Notice that only the lowest-order nonvanishing terms in $\vec{q}_1 + \frac{1}{2}\vec{q}_0$, and β_1 are kept. We now extend the β_1 integration to infinity, so as to simplify the integration. Since the convergence at infinity is not sufficient, we replace D by $D + \eta$ and differentiate with respect to η; this gives a denominator $(D+\eta)^{-3}$. We now use

$$\int_{-\infty}^{\infty} \frac{d^3 q}{(A + Bq^2)^n} \propto B^{-3/2} A^{-n+3/2}, \quad n \geq 2 \tag{2.11}$$

and

$$\int_0^{\infty} \frac{d\beta}{(A + B\beta)^n} \propto B^{-1} A^{-n+1}, \quad n \geq \tfrac{3}{2} \tag{2.12}$$

to find for the first-order skeleton graph F_1

$$\frac{d}{d\eta} F_1 \propto (\tfrac{3}{2} q_0^2 - E + \eta)^{-1/2} (\mu^2 + E - \tfrac{5}{4} q_0^2 + \vec{q}_0 \cdot \vec{q}_{-1} + q_{-1}^2). \tag{2.13}$$

Thus, the singularity in F_1 is $(\frac{3}{2} q_0^2 - E)^{1/2}$. This is, of course, the usual two-body threshold singularity.

$n = 2$

Here the Landau equations are

$$\frac{1}{2} \frac{\partial}{\partial \vec{q}_1} D = \alpha_1 (2\vec{q}_1 + \vec{q}_0) + (1 - \alpha_1)(2\vec{q}_1 + \vec{q}_2) = 0,$$

$$\frac{1}{2} \frac{\partial}{\partial \vec{q}_2} D = (1 - \alpha_1)(2\vec{q}_2 + \vec{q}_1) = 0,$$

$$(1 - \alpha_1)\alpha_1 \frac{\partial D}{\partial \alpha_1} = 2(1 - \alpha_1)\alpha_1(-q_2^2 - \vec{q}_2 \cdot \vec{q}_1 + \vec{q}_1 \cdot \vec{q}_0 + q_0^2) = 0.$$

There are now three special cases, corresponding to our choice of α_1. If we take $\alpha_1 = 0$, we need $\vec{q}_1 = \vec{q}_2 = 0$. Expanding about this point, we obtain

$$D \approx -E + 2(q_1^2 + \vec{q}_1 \cdot \vec{q}_2 + q_2^2) + 2\alpha_1 q_0^2 + \beta_1(\mu^2 + E + q_{-1}^2) + \beta_2(\mu^2 + E + q_0^2). \tag{2.14}$$

Again, we extend the integrals over Feynman parameters to infinity, and now we need three more powers of the denominator for convergence, i.e., we differentiate with respect to η three times. Evaluating the integrals as before yields

$$(-E + \eta)^{-1}(2q_0^2)^{-1}(\mu^2 + E + q_{-1}^2)^{-1}(\mu^2 + E + q_0^2); \tag{2.15}$$

then integrating with respect to η three times yields

$$\frac{E^2}{q_0^2} \ln(-E). \tag{2.16}$$

We now discuss the effects of scaling the external momentum:

$$\vec{q}_i = \vec{y}_i \sqrt{E}.$$

If we do not scale, and q_0^2 is fixed (and nonzero), then the singularity in E coming from (2.16) is $E^2 \ln E$, but $E = 0$ with q_0^2 fixed is not an on-shell point for the amplitude; if we do scale, y_0^2 fixed (and nonzero), then the singularity is $E \ln E$. In the special case of $q_0^2 = y_0^2 = 0$, there is no scaling effect; instead, the coefficient of α_1 in the approximate form is zero, and therefore in integrating (2.14) one less integration is needed. The result is $E \ln E$. We see that scaled momenta are in some sense the logical variables for this problem, since with unscaled momenta the singularity is $E^2 \ln E$ for $q_0^2 \neq 0$ and $E \ln E$ for $q_0^2 = 0$; with scaled momenta no such anomaly arises. Moreover, by scaling the amplitude is kept on shell.

The second end-point singularity is $\alpha_1 = 1$. Since $\alpha_2 = 0$, the propagator P_2 "does not participate" in this singularity, and we expect it to resemble $n = 1$. Indeed, we expand about $\vec{q}_1 = -\frac{1}{2}\vec{q}_2$, perform the $\vec{q}_1$, α_1, β_1, and β_2 integrations[3] (after one differentiation and subsequent integration with respect to

η) and obtain a singularity $(\frac{3}{2}q_0^2-E)^{1/2}$. We see that the singularity of F_2 corresponding to $\alpha_2=0$ is that of F_1. In the same way all singularities of F_{n+1} corresponding to $\alpha_{n+1}=0$ are those of F_n. For this reason we eliminate α_{n+1} and investigate only singularities arising from the vanishing of the remaining α's.

We now consider the third and last case, in which $\partial D/\partial\alpha_1=0$, with $\alpha_1\neq 0$ and $\alpha_1\neq 1$. Here $\vec{q}_2=\vec{q}_1=\vec{q}_0=0$. Thus, both denominators can participate in the singularity *only* if the external momentum $\vec{q}_0$ is identically zero; it does not suffice for it to be scaled and therefore to vanish as E vanishes. Differentiating and integrating twice with respect to η gives

$$E\ln(-E)\left(\frac{1}{\mu^2+E+q_{-1}^2}\right)\left(\frac{1}{\mu^2+E}\right)\int\frac{d\alpha_1}{(1-\alpha_1)^{3/2}(3+\alpha_1)}. \tag{2.17}$$

Here α_1 is not determined by the Landau equations, and so appears only in coefficients of q's.[4] The singularity is $E\ln(-E)$.

n = 3

Here we simply sketch the process of finding the nature of the singularities, except when the methods differ from those above. First, if no α vanishes, $\vec{q}_0=\vec{q}_1=\vec{q}_2=\vec{q}_3=0$, and we have a singularity

$$(-E)^{3/2}\frac{1}{(\mu^2+E)(\mu^2+E+q_{-1}^2)}\int\frac{d\alpha_1 d\alpha_2}{\Delta^{3/2}}, \tag{2.18}$$

where Δ is the determinant of the matrix of coefficients of q_iq_j. Δ does not vanish so long as no α is allowed to vanish.

With $\alpha_1=0$ the Landau equations yield $\vec{q}_1=\vec{q}_2=\vec{q}_3=0$; the singularity is therefore

$$\frac{(-E)^{5/2}}{q_0^2(\mu^2+E)(\mu^2+E+q_0^2)(\mu^2+E+q_{-1}^2)}\int\frac{d\alpha_2}{\Delta^{3/2}}. \tag{2.19}$$

Here again we must distinguish between scaled and unscaled external momenta: For $\vec{q}_0$ scaled or zero, the singularity is $E^{3/2}$; for $\vec{q}_0$ unscaled and nonzero it is $E^{5/2}$.

The case $\alpha_2=0$ is the only "new" case, inasmuch as it corresponds to allowing two separate propagators, P_1 and P_3, to contribute to the singularity. This will give rise to what we call "product" singularities, i.e., singularities arising from the product of two (or more) integrals, each of which may independently be singular. We look at the Landau equations for $\alpha_2=0$ and allow both $\alpha_1=0$ and $\partial D/\partial\alpha_1=0$. In both cases $\vec{q}_2=\vec{q}_3=0$; no other condition is common to both cases. Thus we expand D about $\vec{q}_2=\vec{q}_3=0$ and $\alpha_2=\beta_j=0$. We can integrate over α_2 and β_j from zero to infinity, and then over α_1 from zero to unity; we obtain[5]

$$\frac{1}{\mu^2+E+q_0^2}\int d^3q_2d^3q_3\frac{1}{-E+2(q_2^2+\vec{q}_2\cdot\vec{q}_3+q_3^2)}\int\frac{d^3q_1}{2q_1^2[-E+2(q_1^2+\vec{q}_1\cdot\vec{q}_0+q_0^2)][\mu^2+E+(\vec{q}_1-\vec{q}_{-1})^2][\mu^2+E+q_1^2]}. \tag{2.20}$$

The form of (2.20) shows the promised factorization. The first integral yields an $E^2\ln(-E)$ singularity. The second gives a singularity

$$(\tfrac{3}{2}q_0^2-E)^{1/2}(q_0^2-E)^{-1}; \tag{2.21}$$

this is obtained by analysis very similar to that above. The factor $(q_1^2)^{-1}$ in (2.20) accounts for the denominator in (2.21). Naturally, the F_3 has no pole at $q_0^2=E$ [just as F_2 has none at $q_0^2=0$, in spite of Eq. (2.16)]; the term is significant only for scaled momenta, for which the singularity is $E^{-1/2}$. Thus, in addition to the previously found singularities $E^{3/2}$ (with scaled momenta), we have singularities $E^2\ln(-E)$ and $E^{3/2}\ln(-E)$ arising from the product of integrals.[6]

General n

We sketch here an inductive argument to show that we can find the nature of all physical singularities of F_n by the use of Feynman parameters. Since we restrict ourselves to scaled momenta, these singularities are all at $E=0$. If all α's are nonvanishing, then $\vec{q}_0=\vec{q}_1=\cdots=\vec{q}_n=0$ and we have $E^{n/2}\ln(-E)$ for n even, and $E^{n/2}$ for n odd. If $\alpha_1=0$, then $\vec{q}_1=\cdots=\vec{q}_n=0$; for scaled momenta the same singularities arise. (In the case of unscaled momenta there is an additional power of E.) If $\alpha_n=0$, the singularities are those of F_{n-1}, and therefore known.

Products of singularities are found when $\alpha_i=0$, $2\leq i\leq n-1$. First, take $\alpha_n\neq 0$ and $\alpha_{n-1}=0$, with the other parameters unspecified. This implies $\vec{q}_n=\vec{q}_{n-1}=0$; we expand about that point and likewise about $\alpha_{n-1}=0$. The coefficient of α_{n-1} is $2q_{n-2}^2$. We then do all the integrations over Feynman parameters; we obtain the product of two integrals. The first is

$$\int d^3q_nd^3q_{n-1}[-E+2(q_n^2+\vec{q}_n\cdot\vec{q}_{n-1}+q_{n-1}^2)]^{-1}=\int d^3q_nd^3q_{n-1}P_n^{-1}. \tag{2.22}$$

This is understood as only over some neighborhood of $\vec{q}_n = \vec{q}_{n-1} = 0$, and has singularity $E^2\ln(-E)$. The second is

$$\int \left(\prod_{i=1}^{n-2} d^3q_i P_i^{-1}\right) Q(2q_{n-2}^2)^{-1} \tag{2.23}$$

in which Q contains potential terms that serve only to yield convergence at infinity. This is of the form for F_{n-2}, except for the $(q_{n-2}^2)^{-1}$ factor, which introduces a factor E^{-1} in any singularity of F_{n-2} (only for scaled momenta) and adds no other singularity. The result is a singularity $E\ln(-E)$ times any singularity in F_{n-2}. Next, we take $\alpha_n \neq 0$, $\alpha_{n-1} \neq 0$, and $\alpha_{n-2} = 0$; this implies $\vec{q}_n = \vec{q}_{n-1} = \vec{q}_{n-2} = 0$. Here α_{n-1} has a zero coefficient, and α_{n-2} has coefficient q_{n-3}^2. (Recall that α_n is eliminated and thus has no coefficient in our expansions.) Again we have two integrals. The first is

$$\int d^3q_n d^3q_{n-1} d^3q_{n-2} P_n^{-1} P_{n-1}^{-1}, \tag{2.24}$$

with singularity $E^{5/2}$. The other is like (2.23), with $n-3$ substituted for $n-2$; it has the singularities of F_{n-2} times E^{-1}. Clearly, this process can be continued to find all singularities in F_n, knowing all those in F_m with $m < n$.

In summary, we can derive all the singularities of the skeleton graphs in every order. They are found to be of the form

$$E^{n/2}[\ln(-E)]^m, \tag{2.25}$$

$$m = 0, 1, 2, \ldots; \quad n = 2m, 2m+1, \ldots .$$

For a given order of perturbation theory there is a maximum power of $\ln(-E)$ that occurs, but all lower powers appear, so that the leading singularity is $E\ln(-E)$ for all graphs beyond $n=1$. Thus when the perturbation expansion has nonzero radius of convergence near $E=0$, the entire amplitude also has $E\ln(-E)$ as its leading three-body singularity.

B. General Graphs

We now want to find the nature of the singularities of graphs other than the skeleton graphs previously treated. The additional features are multiple interactions between the same two particles and interactions between all three pairs of particles. If we consider identical particles, these have an additional contribution that appears first in third order; both possibilities are indicated in Fig. 2.

The graphs in Fig. 2 differ from that yielding F_3 only in the momenta that participate in the propagator P_3 (and, of course, in the potentials). For Figs. 2(a) and 2(b) we have, respectively,

$$P_3 = 2(q_1^2 + \vec{q}_1 \cdot \vec{q}_3 + q_3^2) - E$$

and

$$P_3 = 2[(\vec{q}_1 + \vec{q}_2)^2 + (\vec{q}_1 + \vec{q}_2)\cdot\vec{q}_3 + q_3^2] - E.$$

The graph in Fig. 2(a) need only be considered in the case where P_2 and P_3 both contribute (i.e., with α_2 and α_3 both nonvanishing). Every other case is identical with a singularity of F_3 (a relabeling of momenta may be needed to show this). We use the Feynman methods of the previous sections for the two cases $\alpha_1 = 0$ and $\alpha_1 \neq 0$. In both cases $\vec{q}_1 = \vec{q}_2 = \vec{q}_3 = 0$ is the point of singularity given by the Landau equations, and so D is expanded about that point and the integrations over momentum performed. For $\alpha_1 \neq 0$, we obtain $\vec{q}_2 = 0$, and the singularity is $E^{3/2}$; for $\alpha_1 = 0$, the singularity is $E^{5/2}/q_0^2$, which is $E^{3/2}$ for scaled momenta. The graph of Fig. 2(b) has a different structure only when all three propagators participate (i.e., α_1, α_2, and α_3 all nonvanishing). Here $\vec{q}_0 = \vec{q}_1 = \vec{q}_2 = \vec{q}_3 = 0$ is the point of singularity, and we find $E^{3/2}$ behavior. Clearly all higher-order graphs can be handled similarly.

We saw in section A that the basic singularities were those associated with the vanishing of a single propagator (i.e., one nonvanishing α); these gave us $E^{1/2}$ and $E\ln(-E)$ singularities. The vanishing of several adjacent propagators in skeleton graphs (i.e., several sequential α's nonvanishing) was seen to yield only these same singularities; thus the most general singularity, arising from products of these "basic" ones, was as given

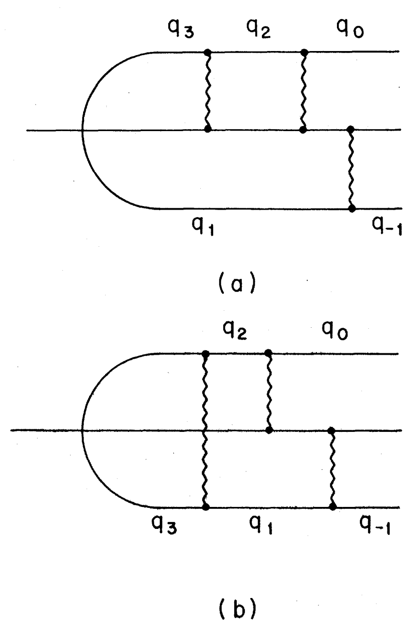

FIG. 2. Third-order graphs not included in the skeleton graphs: (a) successive interaction between the same pair and (b) all three pairs interacting.

above. We have just shown that the vanishing of several adjacent propagators in the most general graph also yields no new type of singularity. Since this is the only new feature of such graphs, as argued above, we see that they introduce no new singularities into the total amplitude. In fact, the most that can happen is that some general nth-order graph, due to its particular configuration, will lack some of the singularities present in the skeleton graph of that order.

III. UNITARITY ARGUMENTS

In Sec. II we derived the singularity structure of three-body decay amplitudes in perturbation theory. We would now like to show that the singularities obtained there can all be accounted for, at least heuristically, by a judicious use of unitarity arguments.

Unitarity gives the imaginary part of an amplitude in terms of other amplitudes, and by the usual analyticity arguments the threshold point for that imaginary part should be a singular point of the amplitude in the variable appropriate to the threshold. The nature of the singularity is obtained by putting the threshold behavior of the imaginary part into a dispersion relation and integrating near the threshold. In most well-known cases the threshold behavior of the imaginary part is given by phase space by assuming the amplitudes themselves are finite at threshold. For example, the argument for establishing the nature of the singularity in a two-body amplitude T_{22} near an n-body threshold gives[7]

$$T_{22;n} \sim E^{(3n-5)/2}\ln(-E), \tag{3.1a}$$

for n odd, and

$$T_{22;n} \sim E^{(3n-5)/2}, \tag{3.1b}$$

for n even. For $n=2$ we get the well-known $\sqrt{E}$ singularity and for $n=3$ the well-known $E^2\ln(-E)$ singularity.

Let us apply similar unitarity arguments to the three-body weak-decay amplitude. Unitarity gives

$$\mathrm{Im}A_{13} = A_{13}\rho_3 T_{33}^{*}, \tag{3.2}$$

where A_{13} is the decay amplitude and T_{33} the 3 to 3 scattering amplitude. Equation (3.2) assumes only three-body intermediate states (no two-body bound states etc.) and is linear in A_{13} from the assumption of weak decay. The 3 to 3 amplitude is the sum of three disconnected terms (the three two-body terms with fly-bys) and a completely connected amplitude as follows:

$$T_{33} = \sum_i T_{22}^{(i)}\delta_i + T_{33}^{(C)}. \tag{3.3}$$

The threshold of a disconnected term's contribution to unitarity is at $E=\frac{3}{2}q_i^2$. The singularity it produces in A is of the square-root type $(E-\frac{3}{2}q_i^2)^{1/2}$, since it is just a two-body threshold in the three-body space. These unitarity arguments are sketched graphically in Fig. 3. The treatment of pair subenergy thresholds is further facilitated by the fact that each term in the decay amplitude carries such a threshold in only one of its pair energies, and thus the decay amplitude can be written as the sum of terms each with its own subenergy cut as in the Khuri-Treiman representation.[8] The three pair subenergies are not independent, but are connected by the total energy E. The amplitude has a singularity in E coming from the connected 3 to 3 amplitude to unitarity. This connected amplitude has its threshold at $E=0$. If A_{13} and T_{33} were finite at $E=0$, the contribution to the singularity structure of A_{13} from this threshold would be $E^2\ln(-E)$, as is the case for $T_{22;3}$. A_{13} is easily seen to be finite as $E\to 0$ in any reasonable model of the decay mechanism. The situation for $T_{33}^{(C)}$ is entirely different. If we keep all the pair subenergies fixed and let $E\to 0$, T_{33} will remain finite, but such a procedure goes off the energy shell. We can ensure that the amplitude stays on shell as $E=0$ by scaling the external momenta as in (1.2). In that case we can study the limit $E\to 0$ for fixed y. We find

$$\langle\vec{y}|T_{33}^{(C)}(E)|\vec{y}'\rangle = \frac{A(\vec{y}_1\vec{y}')}{E} + \frac{B(\vec{y}_1\vec{y}')}{E^{1/2}} + C(\vec{y}_1\vec{y}')\ln(-E) + O(1) \tag{3.4}$$

as $E\to 0$.[9] The coefficients A, B, C can be calculated from the two-body scattering lengths and some kinematic integrals, but do not require a full solution of the three-body problem. The effect of these terms in (3.4) is to take the leading $E^2\ln(-E)$ term and make it $E\ln(-E)$, $E^{3/2}\ln(-E)$, and $E^2\ln^2(-E)$. The higher powers of $\ln(-E)$ we discovered in Sec. II come from $E^n\ln^m(-E)$ factors in A_{13} and T_{33}, times the $\ln(-E)$ from threshold. The $E^2\ln^2(-E)$ discussed above is one such case. This compounding of singularities in the three-body system is absent in the two-body case since powers of square-root singularities have at most a square-root singularity, whereas powers of logar-

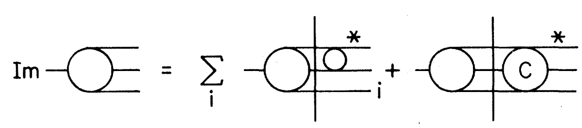

FIG. 3. Schematic representation of the unitarity equation for the imaginary part of the three-body theory amplitude. The first term on the right-hand side is the contribution from two-body scattering and the second from the connected three-body scattering. (The C stands for connected.) The vertical line means that only the δ-function part of the intermediate state is to be taken.

ithmic singularities yield new singularities. In the four-body problem there is again a $\sqrt{E}$ threshold, but there are logarithms in the three-particle subenergies to complicate matters.

IV. CONCLUSIONS

We have seen that an amplitude for weak three-body decay has only threshold singularities for physical values of its variables and that the nature of these singularities is easily understood so long as proper attention is paid to the behavior of the related amplitudes at threshold. In perturbation theory we have seen that the amplitude has square-root thresholds in the pair subenergies and that the leading three-body singularity is $E\ln(-E)$. This singularity comes essentially from the usual $E^2\ln(-E)$ from phase space and the leading $1/E$ part of the 3-3 connected amplitude, if the decay is studied as a function of the momenta scaled by the total energy rather than as a function of the momenta themselves.

The existence of the subenergy thresholds of the pairs means that the decay amplitude cannot be expanded in powers of the subenergies.[10] Expanding in the square root of subenergies would be allowed from the point of view of the subenergy singularities, since the subenergy threshold is of the square-root type. However, on shell, the subenergies are connected by the total energy and it is less clear how to write a representation that simultaneously contains the subenergy square-root cuts and the total-energy logarithms.

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[1]R. D. Amado and J. V. Noble, Phys. Rev. 185, 1993 (1969).

[2]R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966), Chap. 2.

[3]If one integrates immediately over $\vec{q}_1$ and $\vec{q}_2$, the resulting integral over α_1 diverges at $\alpha_1=1$. This corresponds to the vanishing determinant mentioned above. The present method leaves an integral over $\vec{q}_2$ as an overall coefficient which does not contribute to the singularity.

[4]Since we have explicitly excluded the point $\alpha_1=1$, the apparent singularity there does not harm us.

[5]Because we have expanded about $\vec{q}_2=\vec{q}_3=0$, the first integral must be over some neighborhood of this point, and, in fact, diverges at infinity.

[6]Because the integral over $\vec{q}_1$ was not restricted to some compact neighborhood, since $\vec{q}_1$ was not given by the Landau equations, we include its nonsingular contributions as well as its singular contribution. This is why we have the singularity $E^2\ln(-E)$, which comes wholly from the integral over $\vec{q}_2$ and $\vec{q}_3$.

[7]Cf. R. D. Amado, in *Elementary Particle Physics*, edited by M. Chrétien and S. S. Schweber (Gordon and Breach, New York, 1970), and references therein.

[8]N. N. Khuri and S. B. Treiman, Phys. Rev. 119, 115 (1960).

[9]R. D. Amado and Morton H. Rubin, Phys. Rev. Letters 25, 194 (1970).

[10]Cf. S. Weinberg, Phys. Rev. Letters 4, 87 (1960).