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Quantum stabilization of the Skyrme soliton

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We point out that the putative Skyrme soliton in a simple chiral model of pions (without the Skyrme term) can be stabilized against collapse by quantum fluctuations. This leads to a reasonable "zeroth order" description of the nucleon in terms of the single (pion decay) constant F_π.

I. INTRODUCTION

It is presently believed that QCD at very low energies can be described by an effective chiral Lagrangian constructed out of the low-lying meson fields. The leading term is

$$L = -\frac{F_\pi^2}{8}\int \mathrm{Tr}(\partial_\mu U \partial_\mu U^\dagger)d^3x\ , \tag{1}$$

where $F_\pi \simeq 132$ MeV is the pion decay constant and U is a 2×2 unitary unimodular matrix describing the pion fields. Of course, the full effective Lagrangian is expected to have many relevant terms but in the present paper we shall limit our attention to the "zeroth order" model given simply by (1). A well known[1] but remarkable feature of the pure mesonic chiral Lagrangians based on (1) is that they admit classical field configurations that describe the nucleon. Skyrme noticed many years ago that such a classical configuration in (1) is unstable with respect to collapse. To solve this difficulty he introduced a complicated term involving four derivatives of U which leads to a classically stable finite-energy soliton. However, this approach has some disadvantages when one considers more realistic and complicated chiral Lagrangians (e.g., including vector mesons or other higher-derivative pion terms) which give a better description of the low-energy mesonic sector. For one thing it is very hard to reliably extract the coefficients of the various possible Skyrme-type terms from experiment. A six-derivative term is likely, for example, to play a big role in classical stabilization, but a negligible role in low-energy mesonic scattering. Furthermore, if additional mesonic degrees of freedom are included the very task of checking classical stability becomes an onerous one. For these reasons and for its own elegance we shall investigate the quantum stabilization of the nucleonlike field configuration in (1).

Actually, quantum stability is really what we might expect from our experience with other physical systems. The s-wave state of hydrogen is a well-known example. Classically, a static s-wave solution in the potential $V=-e^2/r$ should collapse to the origin. However, in the quantum theory such a collapse would energize the kinetic term $P_r^2/2m$ (P_r is a suitable radial momentum) and quantum fluctuations would lead to an effective term of order of magnitude $1/2mr^2$. The effective energy $(1/2mr^2-e^2/r)$ then is minimized for $E=-me^4/2$, fortuitously the same result as the proper Schrödinger equation treatment.

II. COLLECTIVE-COORDINATE LAGRANGIAN

In order to demonstrate a stabilization mechanism similar to the hydrogen-atom example we clearly must introduce a collective variable which measures the size of the pion cloud. For this purpose it is convenient to adopt a particular shape for the profile function $F(r)$ which appears in the Skyrme ansatz $U=\exp[i\hat{\mathbf{x}}\cdot\boldsymbol{\tau}F(r)]$. We will consider two reasonable[2] choices,

$$F(r;R)=\begin{cases}\pi(1-r/R)\ , & r\le R\ ,\\ 0, & r>R\ ,\end{cases} \tag{2a}$$

$$F(r;R)=\pi\exp(-r/R) \tag{2b}$$

and regard R as a dynamical variable. As we will discuss later, the key aspect of these choices is their short-range nature. Granted this, one might expect the "correct" choice to be the one which minimizes the nucleon's mass. In addition, we must take account of the usual[3] "angular" collective coordinates $A(t)$ describing the spin and isospin degrees of freedom. Thus, our final ansatz for U is

$$U=A(t)\exp[i\hat{\mathbf{x}}\cdot\boldsymbol{\tau}F(r;R(t))]A^\dagger(t)\ , \tag{3}$$

where $A^\dagger(t)=A^{-1}(t)$. Substituting (3) into (1) yields the collective-coordinate Lagrangian

$$L_{\mathrm{col}}=a\dot{x}^2-bx^{2/3}+\tilde{\lambda}x^2\mathrm{Tr}(\dot{A}\dot{A}^\dagger)\ , \tag{4}$$

wherein we have set $x=R^{3/2}$ in order to put the first term in standard form. The coefficients a, b, and $\tilde{\lambda}$ are given for each profile choice in Table I. Notice that the usual[3] "moment of inertia" $\lambda=x^2\tilde{\lambda}$. The Lagrangian (4) will have the same form for any profile choice, only the values of a, b, and $\tilde{\lambda}$ will differ.

For orientation, let us make a very rough estimate using the uncertainty principle in analogy to the hydrogen atom discussion above. Neglecting the third (angular variable) term in (4) we are led to ask for the mean value $\bar{x}$ for which the effective energy

$$\frac{1}{4a\bar{x}^2}+b\bar{x}^{2/3}$$

is minimized. This yields a ground-state energy $E_{\mathrm{gnd}}=(b^{3/4}/\sqrt{2}a^{1/4})\,[3^{-3/4}+3^{1/4}]=0.89$ GeV [0.71 GeV] for the profile (2a) [(2b)] and a mean "size" $\bar{R}=(3/4ab)^{1/4}=0.56$ fm [0.40 fm].

It is clear that both the mass and size of the nucleon ground state seem to be in the right ballpark.

TABLE I. Collective Lagrangian parameters.

Profile choice	a (GeV^2)	b (GeV^2)	$\tilde{\lambda}$ (GeV^2)
(2a)	$\frac{4\pi^3 F_\pi^2}{45}=0.048$	$(\pi+\pi^3/3)F_\pi^2=0.235$	$\frac{2F_\pi^2}{3}\left(\frac{\pi}{3}-\frac{1}{2\pi}\right)=0.010$
(2b)	$\frac{\pi^3 F_\pi^2}{3}=0.180$	0.269	0.128

III. SCHRÖDINGER EQUATION FOR THE BARYONS

The construction of an appropriate Hamiltonian operator from (4) involves the slight subtlety that the last term contains x as well as the angular variables. The Lagrangian is of the general form $\dot{\xi}_i g_{ij}(\xi)\dot{\xi}_j$, where the ξ_i comprise the dynamical variables. The standard procedure[4] for quantizing this object is to replace it by

$$(\sqrt{g})^{-1}\frac{\partial}{\partial \xi_i}\left[g_{ij}^{-1}\sqrt{g}\frac{\partial}{\partial \xi_j}\right],$$

where $g=\det(g_{ij})$. In (4) the nonangular factor in $\sqrt{g}$ is proportional to x^3 so we have the Hamiltonian operator

$$H=-\frac{1}{4ax^3}\frac{\partial}{\partial x}\left[x^3\frac{\partial}{\partial x}\right]+bx^{2/3}+\frac{I(I+1)}{2x^2\tilde{\lambda}}, \quad (5)$$

wherein the "angular coordinates" have been quantized in the standard[3] way leading to the last ("centrifugal") term involving the isospin eigenvalue I. The integration measure in x space should be $\sqrt{g}dx=x^3dx$. We remark that one can gain some intuition about the significance of the first term in (5) by introducing a new wave function $\phi=x^{3/2}\psi$. Then ϕ obeys a differential equation with just the ordinary second-derivative term but containing an extra repulsive potential term $3/(16ax^2)$.

Now we shall describe the results of a numerical solution of the Schrödinger equation $H\phi=E\phi$. First let us consider $I=0$. This is really not an allowed solution (since[1] $I=J$ must be half integral to represent the nucleon states) but should roughly correspond to the hedgehog mass in the usual approach to the Skyrme model. We find that the ground-state solution which vanishes at the origin[5] has an energy of 0.96 GeV for the exponential profile (2b) and 1.21 GeV for the linear profile (2a). It is reassuring that the two values are similar to each other. To proceed we shall restrict our attention to the exponential profile (2b). That choice, as can be seen from Table I, gives a much larger (and more realistic) value of the reduced moment of inertia $\tilde{\lambda}$ because it does not completely cut off the "pion tail" which is very important for rotational properties.

In the $I=J=\frac{1}{2}$ case the three lowest-lying solutions are at 1.14, 1.47, and 1.74 GeV. They should be compared to the experimentally observed particles $N(940)$, $N(1440)$, and $N(1710)$. The wave functions in the size variable x are shown in Fig. 1. It should be stressed that we have no adjustable parameters in this model once a particular profile shape is assumed. All masses and other dimensional quantities are scaled by the constant F_π. If we were to artificially lower F_π to 82% of its experimental value the nucleon mass would be correctly predicted. For comparison, the usual treatment of the Skyrme model requires[3] that F_π be lowered to 69% of its value to fit the nucleon. Actually, we prefer to keep F_π at its experimental value and attribute the mass discrepancy to effects which have been neglected. One might speculate that the neglected

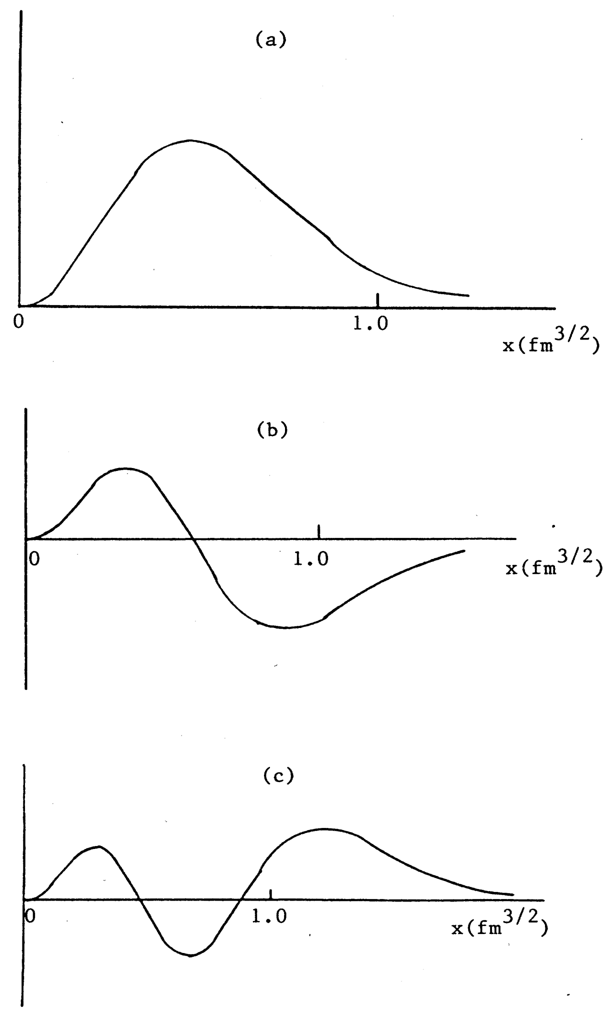

FIG. 1. Arbitrarily normalized wave functions for the $I=\frac{1}{2}$ states at (a) 1.14, (b) 1.47, and (c) 1.74 GeV plotted against the variable x.

effects are related to the "core" of the nucleon rather than to its surrounding pion cloud which is described by the soliton. From this point of view one might expect that the more extended structures (higher radial states) should be more accurately described by the present model. Indeed, our mass predictions are better for the $N(1440)$ and $N(1710)$ than for the nucleon.

The solutions for the three lowest-lying $I=\frac{3}{2}$ states have masses of 1.46, 1.72, and 1.96 GeV. These should be compared to the experimental states $\Delta(1232)$ and, presumably, to $\Delta(1600)$ and $\Delta(1920)$. It is also encouraging to note that our predicted Δ-N mass difference is 0.32 GeV, in good agreement with the experimental value 0.29 GeV.

IV. STATIC PROPERTIES OF THE NUCLEON

It is well known[3] that the simplest version of the Skyrme model (including the Skyrme term) predicts fair but not excellent values for the static properties such as the isoscalar electric and magnetic charge radii $\langle r^2\rangle_{I=0}^{1/2}$ and $\langle r^2\rangle_{M,I=0}^{1/2}$ as well as the axial-vector-coupling constant in neutron decay g_A. Our zeroth-order model gives very similar predictions. We can understand this feature by noting first from Fig. 1(a) that the nucleon wave function is peaked at a size parameter $R\approx 0.67$ fm and that the corresponding moment of inertia λ is (see Table I) about 4.6 GeV^{-1}, both of which are similar to usual[3] Skyrme model values. In carrying out the computations of the static properties in the present model it is necessary to take expectation values of the appropriate operators with respect to the wave function $\psi(x)$. For example, the isoscalar electric-charge squared radius is computed from the isoscalar density $B(r;x)=-(2/\pi)F'\sin^2F$ as

$$\langle r^2\rangle_{I=0}=\int_0^\infty x^3dx\,\psi^2(x)\int_0^\infty r^2dr\,B(r;x)$$
$$=\mathcal{I}\int_0^\infty x^{13/3}\psi^2(x)dx\,, \qquad (6)$$

where $\mathcal{I}=2\int_0^\infty dr\,r^2e^{-r}\sin^2(\pi e^{-r})$. We find $\langle r^2\rangle_{I=0}^{1/2}=0.58$ fm (0.72 fm) and, similarly, $\langle r^2\rangle_{M,I=0}^{1/2}=1.00$ fm (0.81 fm), $g_A=0.65$ (1.23). The experimental values are in parentheses.

V. CHOICE OF PROFILE

Although not a rigorous statement, since we are extracting out of the infinite number of degrees of freedom in the field theory (1) only four variables, we might still expect the "correct" profile choice to be the one which minimizes the nucleon mass. Here we would like to argue that the class of profiles F under consideration should be restricted to short-range ones. In particular, we consider it reasonable that F fall off at large distances perhaps even faster than $1/r^2$, which is the expected value based[3] on the classical Yukawa theory. A pion mass term, of course, will enforce a faster falloff in the Yukawa theory too. We will discuss also the unreasonable possibility of a $r^{-3/2}$ falloff which plays a role for technical reasons.

There is an important physical reason for restricting the profile to be short range; namely, the collective quantization procedure being employed corresponds to the implicit assumption that the motion of each part of the nucleon system is instantaneously correlated with all other parts. This clearly violates relativity in a serious way if the system is too large (e.g., possesses too long a tail). A related example of this concerns the rigid rotational motion of the soliton. The rotational speed of a point near the "center of the soliton" is classically $J\bar{R}/\lambda\approx 0.75J$, where J is the angular momentum, $\bar{R}$ is the mean size, and λ the moment of inertia. Clearly the nonrelativistic approximation is highly dubious for the $J=\frac{3}{2}$ states and terrible for the $\frac{5}{2}$ states. This suggests that the $I=J=\frac{5}{2}$ states predicted by the Skyrme model should be disregarded (they should presumably fall apart before reaching the speed of light) and the predictions for the $I=J=\frac{3}{2}$ states be taken with a grain of salt. As far as the collective radial, or "breathing" modes are concerned, we would expect a serious violation of causality if the size of the profile, in natural units, were to be appreciably larger than a characteristic time scale or inverse-mass scale of the system. Taking this mass scale to be F_π suggests that profile sizes much larger than 1.5 fm are not reasonable.

To see where the mentioned $r^{-3/2}$ falloff comes from it is helpful to record explicit expressions for the Lagrangian parameters a, b, and $\tilde{\lambda}$ with an arbitrary profile:

$$a=\pi F_\pi^2\int_0^\infty \eta^4[F'(\eta)]^2d\eta\,, \qquad (7a)$$

$$b=\pi F_\pi^2\int_0^\infty \eta^2\left[(F')^2+\frac{2}{\eta^2}\sin^2F\right]d\eta\,, \qquad (7b)$$

$$\tilde{\lambda}=\frac{4\pi}{3}F_\pi^2\int_0^\infty \eta^2\sin^2F d\eta\,, \qquad (7c)$$

where $\eta=r/R$ and $F'=dF/d\eta$. We see that both a and $\tilde{\lambda}$, but not b, will diverge if $F(\eta)$ does not decrease faster than $\eta^{-3/2}$ for large η. If one imagines tacking on a tail which behaves for large η as $\eta^{-(3/2+\epsilon)}$, where ϵ is a small positive number, to a given profile it is possible to recapture a result reminiscent of the classical collapse. To see this, consider the $I=0$ "hedgehog" solution. From the argument in Sec. II we learn that the hedgehog energy is proportional to $(b^3/a)^{1/4}$. This result actually follows from dimensional analysis applied to the Schrödinger equation. In the present case b is finite but $a\propto 1/\epsilon$ so the energy can be reduced as low as desired. Similarly, the mean size $\bar{R}\propto(1/ab)^{1/4}$ can be reduced as much as desired. However, as we have stressed above, the relatively slow $\eta^{-(3/2+\epsilon)}$ falloff is unphysical. A similar unphysical lowering of the soliton mass can be induced by adding small wiggles to the profile at large η. Another extreme possibility corresponds to $F(\eta)$ falling very quickly at the origin. But (7a) and (7b) show that this would increase b more than a so the energy of such a configuration would be large.

Clearly it is interesting to pursue further the problem of "fine tuning" the profile function. This may help when a pion mass term, which results in an addition to $L_{\rm col}$ in (4) of $-mx^2$, where $m=4\pi F_\pi^2m_\pi^2\int_0^\infty\eta^2\sin^2Fd\eta$, is included. For the profile (2b) this term somewhat worsens the predictions for the first three $I=\frac{1}{2}$ levels to 1.22, 1.64, and 1.99 GeV, respectively.

VI. DISCUSSION

Perhaps our main result is that stabilization against collapse of hedgehog-type configurations of chiral fields is naturally provided by quantum fluctuations in a collective variable which describes the size of the pion cloud. This suggests that it may be artificial to demand classical stability or even stationarity for realistic (and complicated) chiral models. This may simplify the program of developing a detailed low-energy chiral Lagrangian.

In effect, the somewhat ad hoc Skyrme term has been traded for the reasonable physical assumption that the nucleon is a compact object (i.e., short-range profile).

It also seems quite striking that the pressure due to the quantum fluctuations is of just the right strength to explain the mass and size of the nucleon. This is highly nontrivial since the nucleon mass is about 7 times that of F_π, the only mass scale in the model, rather than being of the order of F_π. Furthermore, the model does not contain any other parameters.

There are evidently many directions for further development and understanding of this approach. As discussed in Sec. V, the fine tuning of the required short-range profile should be further studied. This aspect is in progress. It would be nice to include the effects of vector mesons.[6] Furthermore, since the relevant collective variable is related to scaling properties of the theory, it would be interesting to investigate generalizations[7] of the Skyrme model which mock up the QCD scaling behavior.

Finally, we should mention that attempts were made in the literature to stabilize the Skyrmion by rotations[8] rather than by quantum fluctuations. In addition, collective variables associated with radial modes have been previously used to discuss[9] nucleon excited states like the $N(1440)$.

Note added in proof. Some additional references related to the stability problem have recently come to our attention [J. Boguta, Phys. Lett. B **198**, 384 (1987); J. Donoghue, C. Ramirez, and G. Valencia, Phys. Rev. D **38**, 2195 (1988); J. Blaizot and G. Ripka, *ibid.* **38**, 1556 (1988)].

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[3] Adkins, Nappi, and Witten (Ref. 1).

[4] See, for example, T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, London, 1981), p. 156. Geometrically, this prescription amounts to realizing the kinetic energy as the Laplacian operator with respect to the metric g_{ij}. It is thus independent of changes of the coordinates ξ_i.

[5] Near $x=0$ the wave functions will behave as $\phi \propto x^\delta$, where $\delta = \frac{1}{2} \pm \sqrt{1+c}$ and $c = 2aI(I+1)/\tilde{\lambda}$. Only the positive-sign choice will give a normalizable wave function.

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