PHYSICAL REVIEW D VOLUME 39, NUMBER 3 1 FEBRUARY 1989

$I_t = J_t$ rule in action

Michael P. Mattis
Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, Illinois 60637
(Received 17 August 1988)

A new large-N_c selection rule governing meson-baryon scattering is established in the context of one-boson exchange. This $I_t = J_t$ rule is equivalent to the model-independent linear relations among partial-wave amplitudes that are known to emerge from the Skyrme model.

In this Rapid Communication, we study meson-baryon quasielastic scattering via one-boson exchange (OBE). A new selection rule is shown to emerge in the large-N_c limit: The isospin of the exchanged state must equal its total angular momentum (spin+orbital). In related papers, this $I_t = J_t$ rule is also derived using the Skyrme model[1] and the large-N_c nonrelativistic quark model.[2] Ideally, the three papers should be read in tandem, as they are intended to interfere constructively.

I. πN PARTIAL-WAVE AMPLITUDES

For concreteness, we shall focus here on the case of elastic πN scattering, although our results are, in fact, much more general.[1,2] Our starting point is the effective Lagrangian of nucleons, pions, σ and ρ mesons:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{KE}} + \tfrac{1}{2} g_{\sigma\pi\pi}\sigma\vec{\pi}\cdot\vec{\pi} + g_{\rho\pi\pi}\vec{\rho}_\mu\cdot\vec{\pi}\times\partial^\mu\vec{\pi} + g_{\sigma NN}\sigma\bar{N}N + \tfrac{1}{2} g^V_{\rho NN}\vec{\rho}_\mu\cdot\bar{N}\gamma^\mu\vec{\tau}N + g^T_{\rho NN}\vec{\rho}_{\mu\nu}\cdot\bar{N}\sigma^{\mu\nu}\vec{\tau}N\,, \tag{1}$$

where N is the eight-component spinor $\binom{p}{n}$ and $\vec{\rho}_{\mu\nu} = \partial_\nu\vec{\rho}_\mu - \partial_\mu\vec{\rho}_\nu$. No explicit pion-nucleon vertex is required, as we are only interested here in the σ and ρ exchange graphs (see Fig. 1 for notation). The coupling constants $g^V_{\rho NN}$ and $g^T_{\rho NN}$ multiply terms that we shall refer to, respectively, as the vectorlike and tensorlike ρ-nucleon interactions. Although we can also couple the σ derivatively to the nucleon, such a term vanishes when the nucleons are on shell.

We remind the reader that, in large N_c, there is actually supposed to be an infinite tower of mesons in each J^{PC} channel, so that Eq. (1) might best be thought of as containing implicit summations over all higher-mass recurrences of the σ, ρ, and π; this will not affect our results.

In the center of mass, the invariant amplitude for the processes depicted in Fig. 1, with three-momentum $p = |\mathbf{p}| = |\mathbf{p}'|$, and with the external legs on shell, is given by

$$\mathcal{A} = \delta_{ab}\delta_{\alpha\beta} g_{\sigma\pi\pi} g_{\sigma NN} G_\sigma \bar{U}^{(\lambda')}_{\mathbf{p}'} U^{(\lambda)}_{\mathbf{p}} + i\epsilon_{abc}\tau^c_{\beta\alpha} g_{\rho\pi\pi} G_\rho \Big((E_{N\mathbf{p}} + E_{\pi\mathbf{p}})(g^V_{\rho NN} + 8m_N g^T_{\rho NN})\bar{U}^{(\lambda')}_{\mathbf{p}'}\gamma^0 U^{(\lambda)}_{\mathbf{p}} - \{m_N g^V_{\rho NN} + [8E_{N\mathbf{p}}(E_{N\mathbf{p}} + E_{\pi\mathbf{p}}) - 4p^2(1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')] g^T_{\rho NN}\}\bar{U}^{(\lambda')}_{\mathbf{p}'} U^{(\lambda)}_{\mathbf{p}} \Big)\,. \tag{2}$$

In this expression, $\hat{\mathbf{p}} = \mathbf{p}/p$, $\hat{\mathbf{p}}' = \mathbf{p}'/p$, $E_{N\mathbf{p}} = \sqrt{m_N^2 + \mathbf{p}^2}$, $E_{\pi\mathbf{p}} = \sqrt{m_\pi^2 + \mathbf{p}^2}$, and we have used the spinor equations of motion. The σ and ρ propagators are, respectively, $-iG_\sigma$ and $i[g_{\mu\nu} - (p_\mu - p'_\mu)(p_\nu - p'_\nu)/m_\rho^2]G_\rho$. Of course, for bare propagators, $G_{\sigma,\rho} = [2p^2(1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') + m^2_{\sigma,\rho}]^{-1}$, but in what follows we will let G_σ and G_ρ remain as arbitrary functions of p and $\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}'$, so that we can think of them, if we like, as representing the fully dressed propagators, with all vertex form factors absorbed.

Our first goal is to extract the s-channel partial-wave amplitudes (PWA's) from $\mathcal{A}$. This requires projecting out both isospin and angular momentum. The projection onto total s-channel isospin $I_s = \frac{1}{2}$ or $\frac{3}{2}$ involves simple Clebsch-Gordan manipulations. The angular momentum projection is more complicated: if we let (θ,ϕ) denote the Euler angles of $\hat{\mathbf{p}}$, and represent the initial πN state by $|p\theta\phi\lambda\rangle$, then what we are after is the spin-orbit state defined by

$$|pJ_sJ_{sz}L\rangle = \frac{\sqrt{2L+1}}{4\pi}\sum_{\lambda=\pm 1/2}\langle J_s\lambda|L\,\tfrac{1}{2}\,0\lambda\rangle \times \int d\Omega\, D^{(J_s)}_{J_{sz},\lambda}{}^*(\theta\phi)\,|p\theta\phi\lambda\rangle\,, \tag{3}$$

with L the orbital, and $J_s = L \pm \frac{1}{2}$ the total, s-channel angular momentum. The identical projection must be applied to the final πN state as well.

In order to carry out these integrals over solid angle, it is first necessary to express all angular-dependent quantities in terms of D functions. Thus, $G_{\sigma,\rho}$ are expanded in Legendre polynomials $P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')$, which can, in turn, be rewritten as D functions using the addition theorem

$$G_{\sigma,\rho}(p,\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') = \sum_{l=0}^{\infty}(2l+1)G^l_{\sigma,\rho}(p)P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(2l+1)G^l_{\sigma,\rho}(p)D^{(l)}_{m0}(\theta\phi)D^{(l)}_{m0}{}^*(\theta'\phi')\,. \tag{4a}$$

Similarly, we write

$$\tfrac{1}{2}\bar{U}_{\mathbf{p}'}^{(\lambda')}(1 \pm \gamma^0)U_{\mathbf{p}}^{(\lambda)} = (E_{N\mathbf{p}} \pm m_N)(\delta_{\lambda',-\lambda} \pm \delta_{\lambda',\lambda})[D^{(1/2)\dagger}(\theta'\phi')D^{(1/2)}(\theta\phi)]_{\lambda',\lambda} \tag{4b}$$

and

$$\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}' = \sum_{k=-1,0,1} D_{k0}^{(1)}(\theta\phi)D_{k0}^{(1)*}(\theta'\phi') . \tag{4c}$$

The D functions can then be integrated, and the resulting Clebsch-Gordan coefficients summed, with the help of standard identities. Finally, $\mathcal{A}$ must be multiplied by $p/(8\pi\sqrt{s})$ to give a conventionally normalized partial-wave T-matrix element.

The result of this calculation is

$$\begin{aligned} T_{I_s J_s L} = {} & \frac{p}{8\pi\sqrt{s}} g_{\sigma\pi\pi} g_{\sigma NN}[G_\sigma^L(E_{N\mathbf{p}}+m_N) - G_\sigma^{L\pm1}(E_{N\mathbf{p}}-m_N)] \\ & + \frac{p}{2\pi\sqrt{s}} \frac{(-1)^{I_s-1/2}}{2I_s+1} g_{\rho\pi\pi} g_{\rho NN}^V \{G_\rho^L[E_{\pi\mathbf{p}}(E_{N\mathbf{p}}+m_N)+p^2] + G_\rho^{L\pm1}[E_{\pi\mathbf{p}}(E_{N\mathbf{p}}-m_N)+p^2]\} \\ & + \frac{p^3}{\pi\sqrt{s}} \frac{(-1)^{I_s-1/2}}{2I_s+1} g_{\rho\pi\pi} g_{\rho NN}^T \left[\left(\frac{1}{2L+1} \mp 1\right)(G_\rho^{L-1} - G_\rho^{L+1})(E_{N\mathbf{p}}+m_N) \right. \\ & \left. + \left(1 \pm \frac{1}{2(L\pm1)+1}\right)(G_\rho^{L\pm2} - G_\rho^L)(E_{N\mathbf{p}}-m_N) - 4E_{\pi\mathbf{p}}(G_\rho^L - G_\rho^{L\pm1}) \right] . \end{aligned} \tag{5}$$

The $\pm$ signs in (5) are determined by the choice of $J_s = L \pm \frac{1}{2}$, and $G_{\sigma,\rho}^l \equiv 0$ by definition whenever $l < 0$.

II. THE LARGE-N_c LIMIT AND THE SKYRME-MODEL RELATIONS

We are interested in the large-N_c limit of Eq. (5). This means, first of all, treating $E_{N\mathbf{p}}$, m_N, and $\sqrt{s}$, which are $O(N_c)$, as much larger than the other energies in the problem, which are $O(N_c^0)$, and noting that the difference $E_{N\mathbf{p}} - m_N \sim N_c^{-1}$. Second, we must understand the N_c dependence of the various coupling constants in (5). In large N_c, n-meson vertices are known to scale like $N_c^{1-n/2}$ so that mesons become stable and noninteracting.[3] Thus, $g_{\sigma\pi\pi}$ and $g_{\rho\pi\pi} \sim N^{-1/2}$.

Meson-baryon vertices are somewhat trickier. Witten's rule of thumb[4] that the meson-baryon scattering cross sections should have a smooth $O(N_c^0)$ limit as $N_c \to \infty$ would imply that the three meson-baryon couplings in (5) should scale like $N_c^{1/2}$, but this is not the case:

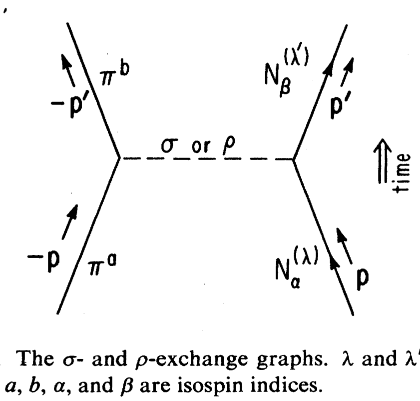

FIG. 1. The σ- and ρ-exchange graphs. λ and λ' are nucleon helicities; a, b, α, and β are isospin indices.

Specifically, whereas $g_{\sigma NN}$ and $g_{\rho NN}^T$ behave as expected, $g_{\rho NN}^V \sim N_c^{-1/2}$. There are several ways to see this, the simplest being the universality condition $g_{\rho NN}^V = g_{\rho\pi\pi}$.

All in all, in the large-N_c limit, Eq. (5) can be recast schematically as

$$\begin{aligned} T_{1/2,L-1/2,L} &= x + \frac{L+1}{2L+1} y + O(N_c^{-1}) , \\ T_{1/2,L+1/2,L} &= x - \frac{L}{2L+1} y + O(N_c^{-1}) , \\ T_{3/2,L-1/2,L} &= x - \frac{L+1}{4L+2} y + O(N_c^{-1}) , \\ T_{3/2,L+1/2,L} &= x + \frac{L}{4L+2} y + O(N_c^{-1}) , \end{aligned} \tag{6}$$

with x and y representing the N_c^0 contributions of σ exchange and tensor-coupled ρ exchange, respectively. *The contribution from vector-coupled ρ exchange is $O(N_c^{-1})$ and therefore drops out altogether as $N_c \to \infty$.*[5]

It follows immediately from Eq. (6) that to leading order in $1/N_c$, for each value of $L > 0$, our model yields two nontrivial linear relations among the four PWA's. In particular, we can solve for the two isospin-$\frac{3}{2}$ amplitudes as linear combinations of the two isospin-$\frac{1}{2}$ amplitudes:

$$T_{3/2,L-1/2,L} = \frac{L-1}{4L+2} T_{1/2,L-1/2,L} + \frac{3L+3}{4L+2} T_{1/2,L+1/2,L} , \tag{7a}$$

and

$$T_{3/2,L+1/2,L} = \frac{3L}{4L+2} T_{1/2,L-1/2,L} + \frac{L+2}{4L+2} T_{1/2,L+1/2,L} . \tag{7b}$$

These are precisely the Skyrme-model relations of Refs. 8 and 9.

III. THE $I_t = J_t$ RULE

In order to derive the $I_t = J_t$ rule, we must cross from an s-channel ($\pi N \to \pi N$) to a t-channel ($\pi\pi \to N\bar{N}$) description of the collision. Reference 1 explains how to do this, sparing us the need to go into great detail. For πN elastic scattering, the allowed values of t-channel isospin are obviously $I_t = 0$ or 1. Furthermore, in large N_c, the nucleon, being much more massive than the pion, remains essentially at rest during the collision, so that the total (spin+orbital) t-channel angular momentum is likewise restricted to $J_t = 0$ or 1. The transformation linking the s- and t-channel PWA's for a given value of L can be shown to be

$$T_{I_t J_t L} = \sum_{I_s = 1/2, 3/2} \sum_{J_s = L \pm 1/2} (-1)^{I_t + I_s + J_t + J_s + L} (2I_s + 1)(2J_s + 1) \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & I_t \\ 1 & 1 & I_s \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & J_t \\ L & L & J_s \end{Bmatrix} T_{I_s J_s L} . \quad (8)$$

This is Eq. (5) or Ref. 1, applied to the case at hand.

Evaluating the $6j$ symbols, one can rewrite Eq. (8) explicitly as

$$T_{0,0,L} = \frac{2(L+1)}{\sqrt{3(2L+1)}} \left[\frac{L}{L+1} T_{1/2,L-1/2,L} + T_{1/2,L+1/2,L} + \frac{2L}{L+1} T_{3/2,L-1/2,L} + 2T_{3/2,L+1/2,L} \right]$$
$$= \left[\frac{12}{2L+1} \right]^{1/2} [L T_{1/2,L-1/2,L} + (L+1) T_{1/2,L+1/2,L}] , \quad (9a)$$

$$T_{0,1,L} = \frac{2}{3} \left[\frac{L(L+1)}{2L+1} \right]^{1/2} (T_{1/2,L-1/2,L} - T_{1/2,L+1/2,L} + 2T_{3/2,L-1/2,L} - 2T_{3/2,L+1/2,L})$$
$$= 0 , \quad (9b)$$

$$T_{1,0,L} = \frac{4}{3} \frac{(L+1)}{\sqrt{2(2L+1)}} \left[\frac{L}{L+1} T_{1/2,L-1/2,L} + T_{1/2,L+1/2,L} - \frac{L}{L+1} T_{3/2,L-1/2,L} - T_{3/2,L+1/2,L} \right]$$
$$= 0 , \quad (9c)$$

and

$$T_{1,1,L} = \frac{4}{3} \left[\frac{L(L+1)}{6(2L+1)} \right]^{1/2} (T_{1/2,L-1/2,L} - T_{1/2,L+1/2,L} - T_{3/2,L-1/2,L} + T_{3/2,L+1/2,L})$$
$$= \left[\frac{2L(L+1)}{3(2L+1)} \right]^{1/2} (T_{1/2,L-1/2,L} - T_{1/2,L+1/2,L}) , \quad (9d)$$

where, in the final equalities, we have used the large-N_c relations (7). As promised, only $T_{0,0,L}$ and $T_{1,1,L}$, which have $I_t = J_t$, survive in the large-N_c limit, whereas $T_{0,1,L}$ and $T_{1,0,L}$ vanish like $1/N_c$. This illustrates what is meant by the $I_t = J_t$ rule. In fact, the rule holds, not just for $\pi N \to \pi N$, but for two-flavor meson-baryon quasielastic scattering in general.[1,2]

We should mention an important caveat, also noted in Ref. 1 in the context of Skyrmion physics. The large-N_c analysis carried out above, which culminated in the $I_t = J_t$ rule, explicitly assumed that the meson energies are $O(N_c^0)$. This is the natural kinematic region when one considers meson-baryon scattering. However, a t-channel process such as $\pi\pi \to N\bar{N}$ can only proceed if the meson energies are $O(N_c)$, in which case the above analysis is totally inapplicable. In other words, although we have obtained our selection rule by recasting the scattering matrix in terms of t-channel quantities, the rule only applies in the s-physical, and not the t-physical, region.

We should also dispel the possible misapprehension that the $I_t = J_t$ rule is somehow built into the Lagrangian (1), since both the σ and the ρ happen to have equal spin and isospin. In our usage, J_t is not just meson spin, but includes orbital angular momentum, so that, in the present case, the suppression of vector-coupled ρ exchange (corresponding to $J_t = 0$) was crucial. In addition, it can be shown that in the more complicated process $\pi N \to \rho N$, the $I_t = J_t$ rule follows from the exchange of *(inter alia)* π's and vector-coupled ω's, neither of which has equal spin and isospin.[10]

IV. DISCUSSION

Taken together, the present work and Ref. 1 contain two principal findings.

(i) The model-independent linear relations among meson-baryon PWA's that emerge as $N_c \to \infty$ from Skyrmion physics[8,9,11] also emerge, in the same limit, from OBE. This can be turned around: The relations can be used as a guideline for determining the N_c dependence of the various meson-nucleon (and meson-Δ) interaction terms in the low-energy effective Lagrangian of QCD.

(ii) These relations, which appear somewhat awkward when formulated in terms of s-channel quantities, can be expressed concisely and elegantly as the $I_t = J_t$ rule when crossed to the t channel.

In the end, the picture that suggests itself is one of *complementarity*. The Skyrme-model literature abounds with comparisons to more traditional approaches to hadron physics: lattice and bag models, the nonrelativistic quark model, and now, the OBE models that have proved useful in describing hadron collisions at low momentum transfer. None of these methodologies has a monopoly on the truth; each has its own domain of validity, and each sheds light on the others. Thus, the overall experimental success[8] of the Skyrme-model linear relations (7) can be viewed from the OBE perspective as reflecting a lopsided ratio in the strength of the $O(N_c^0)$ tensor-coupled versus the $O(N_c^{-1})$ vector-coupled ρ-exchange terms in (5). Is this a legitimate conclusion in a world such as ours, with $N_c = 3$? Yes, it is: If one trusts the phenomenological estimates for $g^T_{\rho NN}/g^V_{\rho NN}$ (Ref. 12) and uses the bare propagator for G_ρ, one can calculate from Eq. (5) that the tensor term gives a contribution to the F-wave N and Δ resonances (for example) that is three to four times larger than the vector term.

A noteworthy feature of the higher πN partial waves —and a particularly nice illustration of complementarity—is the "big-small-small-big" pattern discussed in Ref. 13. This describes the fact that, by and large, the "outer" amplitudes in Eq. (6), $T_{1/2,L-1/2,L}$ and $T_{3/2,L+1/2,L}$, are much more strongly resonant than the "inner" amplitudes $T_{1/2,L+1/2,L}$ and $T_{3/2,L-1/2,L}$. The pattern finds an appealing explanation in the Skyrme model, based on the relative sizes of the group-theoretic factors involved.[13] An equally satisfactory explanation can be offered in the OBE framework: witness the signs of the coefficients multiplying y, the tensor-coupled ρ-exchange term in Eq. (6); the positive signs denote a resonant, and the negative signs a repulsive, contribution to the total amplitude.

I have profited from discussions with Eric Braaten, Marek Karliner, Michael Peskin, and Jon Rosner, and am indebted to Harry Lee for pointing out Ref. 12. This work was supported in part by the National Science Foundation Grant No. PHY-85-21588.

---

[1] M. P. Mattis and M. Murkerjee, Phys. Rev. Lett. **61**, 1344 (1988).

[2] M. P. Mattis and E. Braaten, Enrico Fermi Institute, University of Chicago Report No. EFI 88-66, 1988 (unpublished).

[3] G. Veneziano, Nucl. Phys. **B117**, 519 (1976).

[4] E. Witten, Nucl. Phys. **B160**, 57 (1979), Sec. 6.

[5] Another way to convince oneself of this is to recall that, as Sakurai showed long ago (Ref. 6), the contribution of vector-coupled ρ exchange is equivalent to the Weinberg-Tomozawa (WT) formula for the πN scattering lengths (Ref. 7); this can be checked from Eq. (5). But the WT scattering lengths are proportional to $[f_\pi^2(m_\pi^{-1}+m_N^{-1})]^{-1}$, which scales like $1/N_c$ in the large-N_c limit. Q.E.D.

[6] J. J. Sakurai, *Currents and Mesons* (Univ. of Chicago Press, Chicago, IL, 1969), Chap. V.

[7] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966); Y. Tomozawa, Nuovo Cimento **46A**, 707 (1966).

[8] M. P. Mattis and M. Peskin, Phys. Rev. D **32**, 58 (1985).

[9] A. Hayashi, G. Eckart, G. Holzwarth, and H. Walliser, Phys. Lett. **147B**, 5 (1984).

[10] M. P. Mattis (unpublished).

[11] M. P. Mattis, Phys. Rev. Lett. **56**, 1103 (1986).

[12] An optimal fit to the nucleon-nucleon potential requires $g^T_{\rho NN}/g^V_{\rho NN} = 0.75 \pm 0.1\ \mathrm{GeV}^{-1}$ [T.-S. H. Lee and F. Tabakin, Nucl. Phys. **A191**, 332 (1972), Table 1], while assumed ρ dominance of the isovector nucleon form factor gives $g^T_{\rho NN}/g^V_{\rho NN} \approx 1\ \mathrm{GeV}^{-1}$ [V. Barger and M. Ebel, Phys. Rev. **138**, B1148 (1965)].

[13] M. P. Mattis and M. Karliner, Phys. Rev. D **31**, 2833 (1985).