Comment on Jarlskog's conditions for *CP* invariance

M. Gronau
Department of Physics, Technion, Haifa, Israel

R. Loewy
Department of Mathematics, Technion, Haifa, Israel
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In a previous work we derived for the quark mass matrices in the standard model with any number of families, n, a set of basis-independent conditions, which are necessary and sufficient for *CP* invariance. Very recently Jarlskog claimed to have reduced the number of these conditions for $n>3$ to $\frac{1}{2}(n-1)(n-2)$=the number of Kobayashi-Maskawa-type phases. We prove that in general her conditions are insufficient for *CP* conservation.

Some time ago we derived[1] necessary and sufficient conditions for *CP* invariance, which the quark mass matrices obey in the standard electroweak model with any number of fermion families n. Our basis-independent conditions consist of the reality of the traces of products of powers of the up- and down-quark mass matrices. For $n>3$ the number of these conditions is larger than $\frac{1}{2}(n-1)(n-2)$=the number of Kobayashi-Maskawa-(KM-) type phases.[2] Very recently Jarlskog[3] readdressed this question. She claimed to have found a subset of our conditions, exactly $\frac{1}{2}(n-1)(n-2)$ of them written in a somewhat different form, which are sufficient for *CP* invariance.

The purpose of this Comment is to prove that, in fact, Jarlskog's conditions are not sufficient for *CP* invariance. We will present a counterexample to her conditions for $n=4$. The weak link in her formulation will be pointed out. We will be brief, since the details of the argument as well as the complete set of sufficient conditions were already presented in Ref. 1. To avoid any possible confusion due to different notations, we will not use our own notations of Ref. 1 but rather the ones used by Jarlskog herself.

The claim made by Jarlskog [see Eq. (22b) in Ref. 3] is that the $\frac{1}{2}(n-1)(n-2)$ necessary and sufficient conditions for *CP* invariance are

$$\begin{aligned}\mathrm{Im}(\hat{K}_{1j}\hat{K}_{jk}\hat{K}_{k1}) &= \mathrm{Im}\,\mathrm{Tr}(\hat{P}_1\hat{K}\hat{P}_j\hat{K}\hat{P}_k\hat{K}) \\ &= \mathrm{Im}\,\mathrm{Tr}(P_1KP_jKP_kK) \\ &= 0 \quad (2\le j\le k\le n)\,. \end{aligned} \tag{1}$$

Here and from now on repeated indices are not summed over. $K=[S,S']$ is the commutator of the Hermitian up- and down-quark mass matrices $S=mm^\dagger$, $S'=m'm'^\dagger$, respectively. P_j are projection matrices for flavor j. P_j and K are the matrices in an arbitrary basis, while $\hat{P}_j$ and $\hat{K}$ are their forms in a basis in which S is diagonal. The down-quark matrix will be denoted in this basis by $\hat{S}'$.

A critical remark is in order. To obtain the projection matrices P_j, which appear in Eq. (1), one must first calculate all the eigenvalues of the $n\times n$ matrix S. It is well known, however, that there is no general formula in terms of radicals for the eigenvalues of matrices of dimension $n>4$. Therefore the above conditions can in general be applied only to $n=3,4$.

In the basis in which S is diagonal Eq. (1) reads [see Eqs. (20) and (22) of Ref. 3]

$$\mathrm{Im}(\hat{S}'_{1j}\hat{S}'_{jk}\hat{S}'_{k1})=0 \quad (2\le j\le k\le n)\,, \tag{2}$$

where it was assumed that the quark masses are nondegenerate. Our example will have this property.

Equation (2) states that in the basis in which S is diagonal $\frac{1}{2}(n-1)(n-2)$ cyclic products of *order 3* in $\hat{S}'$ are real. It was shown in Ref. 1 that for *CP* invariance the cyclic products of *all orders* must be real.[4] The reality of cyclic products of higher order does not follow in general from the reality of the three-cycles. As pointed out in Ref. 1, this is the result of possible zeros in the mass matrix $\hat{S}'$. There is no *a priori* physical reason to assume that $\hat{S}'$ has no off-diagonal zero elements. As in Ref. 1, we will not make such an assumption. Therefore, Eqs. (2) or (1) are insufficient for *CP* invariance (unless one explicitly assumes that $\hat{S}'$ has no off-diagonal zero element). Additional conditions, expressing the reality of cyclic products of higher order, must be implemented to guarantee the symmetry. Hence the overall number of conditions is larger than $\frac{1}{2}(n-1)(n-2)$ (Ref. 5).

Our counterexample in the case $n=4$, to which we already alluded in Ref. 1, will be based on the above argument. All the three-cycles of $\hat{S}'$ will be chosen to be real, whereas some of its four-cycles will have a nonzero imaginary part. Recall that the other Hermitian mass matrix S is diagonal in this basis. Its positive eigenvalues are arbitrary.

Consider the Hermitian form

$$\hat{S}'=\begin{bmatrix} a & x & 0 & u \\ x^* & b & y & 0 \\ 0 & y^* & c & z \\ u^* & 0 & z^* & d \end{bmatrix}\,. \tag{3}$$

The diagonal elements are real and arbitrary and some of the off-diagonal elements are complex such that

$$\mathrm{Im}(\hat{S}'_{12}\hat{S}'_{23}\hat{S}'_{34}\hat{S}'_{41})=\mathrm{Im}(xyzu^*)\neq 0 \ . \tag{4}$$

The matrix of Eq. (3) obeys all of Eq. (2) (Ref. 6). Yet in this example *CP* is violated since some of the four-cycles have a nonzero imaginary part. This proves that Jarlskog's conditions (2) are not sufficient for *CP* conservation.

In conclusion, the number of basis-independent necessary and sufficient conditions of *CP* invariance on quark mass matrices in the standard model with $n>3$ families is, in fact, larger than the corresponding number of KM-type phases. There is no one-to-one correspondence between the *CP* invariance "measures" in the quark mass matrix and the *CP* phases of the quark mixing matrix. This is not surprising, since in terms of the quark mixing matrix *CP* may be conserved not only when all the *CP* phases vanish but also, for instance, when some of the mixing angles are zero.[7]

In Ref. 1 we obtained the basis-independent conditions of *CP* invariance for $n=4$ (for which we found six conditions) and $n=5$. The commutator K of the up- and down-quark mass matrices plays no special role in these conditions. The determinant of this commutator, first introduced by Jarlskog[8] for $n=3$, was shown to be replaced in the general case by traces of products of powers of the quark mass matrices. For the special case $n=3$ a simple identity relates $\mathrm{Det}K$ to $\mathrm{Im}\,\mathrm{Tr}(S^2S'SS'^2)$. Also the projection matrices P_j, for which the eigenvalues would have to be calculated first, do not appear in our conditions of *CP* invariance.

When making an *ad hoc* assumption (for no sound physical reason, except that zeros may seem "accidental") that $\hat{S}'$ has only nonzero elements, the number of conditions may in general be reduced to $\frac{1}{6}n(n-1)(n-2)$ (Ref. 1). Then in the special case $n=4$ the corresponding four conditions may be further reduced by Jarlskog's three conditions. This requires solving first the fourth-order mass-eigenvalue equation. A similar reduction cannot be obtained in general for a larger number of families.

Finally, we wish to make a critical and constructive comment on the reason for the difference between Jarlskog's results and ours. We have already stressed that whereas our conditions are equivalent to requiring that all the cyclic products of $\hat{S}'$ are real in the basis in which S is diagonal, Jarlskog's reality conditions apply only to some of the three-cycles. Our results were derived from the general definition of *CP* invariance in the multifamily standard model. Namely, *CP* symmetry is defined by the existence of three unitary matrices U_L, U_R^u, U_R^d such that the Lagrangian is invariant under the following transformations of the quark fields:[9]

$$\begin{aligned} u_L &\to U_L C u_L^*, \quad d_L \to U_L C d_L^* \ , \\ u_R &\to U_R^u C u_R^*, \quad d_R \to U_R^d C d_R^* \ . \end{aligned} \tag{5}$$

This means that, among the different equivalent weak-eigenstate bases related to each other by the above three kinds of unitary matrices, there exists a basis for which the *CP* transformations of the fermion fields have the usual textbook form (namely, without the above unitary matrices). No equivalent definition was given in Ref. 3 and hence no adequate derivation of the *CP* invariance conditions was presented. Although some remarks related to particular algebraic consequences of *CP* invariance were made in Secs. IV and V of this paper, a more precise definition of the symmetry is required for obtaining the actual conditions for *CP* conservation.

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[1]M. Gronau, A. Kfir, and R. Loewy, Phys. Rev. Lett. **56**, 1538 (1986).

[2]K. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[3]C. Jarlskog, Phys. Rev. D **36**, 2128 (1987).

[4]See Eq. (5) of Ref. 1. These relations for cycles of any length were then "analytically continued" into a basis-independent form. This was apparently overlooked by Jarlskog. See, in particular, the comments following Eq. (22a) of Ref. 3.

[5]Furthermore, Eqs. (2) are not even sufficient for all the three-cycles to become real. They only refer to the cycles involving elements of the first row. The number of distinct three-cycle products in a Hermitian mass matrix is $\frac{1}{6}n(n-1)(n-2)$. For the reality of these three-cycle products one needs this number of conditions. See Ref. 1.

[6]All the four H's of Eq. (33) in Ref. 3 vanish, which by Jarlskog's criterion would imply *CP* invariance for $n=4$.

[7]For a convenient description of the quark mixing matrix for any n in terms of mixing angles and *CP* phases see M. Gronau, R. Johnson, and J. Schechter, Phys. Rev. D **32**, 3062 (1985).

[8]C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).

[9]C is the Dirac charge-conjugation matrix. We use the notations of Ref. 1. See also G. Ecker, W. Grimus, and H. Neufeld, J. Phys. A **20**, L807 (1987).