Lattice heavy-meson decay constants and fermion universality

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The mass dependence of decay constants for vector and pseudoscalar mesons is calculated using both Wilson and staggered-fermion formulations in quenched lattice QCD. For low meson masses, the agreement of the two methods is good and the determination of the vector decay constant agrees with experiment. For pseudoscalar heavy-quark–light-quark systems, we find reasonable agreement between the two fermion formulations as long as $Ma \lesssim 1$, where M is the meson mass.

The decay constants of B and D mesons provide a good test for models of hadron structure and are also necessary ingredients in determining other quantities such as weak quark mixing matrix elements. Experimentally they are not well known. Theoretically, model calculations give vastly different predictions which disagree even qualitatively,[1–7] for example, as to whether f_B and f_D are greater or less than f_π. Given the large discrepancies between different models, all of which claim to be motivated by quantum chromodynamics (QCD) in some way, it is important to investigate what lattice QCD can say about heavy-meson decay constants. Since there are now a number of calculations along these lines,[8–11] it is also important to examine the question of fermion scheme dependence as an indicator of *systematic* errors in such calculations. This is the purpose of the present paper.

In lattice QCD simulations one is most often interested in light-quark systems and a meaningful calculation requires that the hadron size be less than the total spatial extent of the lattice. At present this precludes calculations for very light quarks, although extrapolation methods can be used to go to the chiral limit. One also encounters a limitation at large quark mass that the hadron size be much greater than the lattice spacing. Although one knows that systematic errors arise as the lattice ultraviolet cutoff is approached, one does not know *a priori* how close to the cutoff meaningful calculations can be done. One way to get some information about the domain of validity is to compare calculations done with Wilson and staggered-fermion schemes. When it can be checked that physical observables are unchanged by the choice of fermion regularization, we can have more confidence that the mass is in a reasonable range. For large dimensionless masses, where the two formulations give different answers due to lattice artifacts, one would have to decide in some way which (if either) of the fermion schemes can be used reliably. We will see that the agreement of the two methods with each other, and with experiment in the vector case, is good for low meson masses. Indeed, for dimensionless meson masses up to $Ma \approx 1$, there is reasonable agreement between the Wilson and staggered-fermion calculations for the phenomenologically interesting case of pseudoscalar decay constants of mesons consisting of one heavy and one light quark. This upper limit is understandable since terms of order Ma arise between different lattice-decay-constant prescriptions. We will also see that very similar patterns of systematic differences arise between the two schemes in the pseudoscalar and vector cases at higher meson masses.

We will describe only briefly the methods of extracting the decay constants from Monte Carlo simulations since most of the techniques are, by now, standard. The vector-meson decay constant (in the continuum) is determined by the matrix element of the vector current between the vector meson $|v(k))$ and the vacuum $(0|$ states:

$$(0|V_\mu(x)|v(k))=\frac{M_v^2\epsilon_\mu e^{-ik\cdot x}}{f_v}\ , \tag{1}$$

where ϵ_μ is the polarization vector, M_v is the vector-meson mass, and f_v is the decay constant.

In the Wilson case we use the continuum-to-lattice replacement prescriptions

$$k_i \to 2a^{-1}\sin(k_i a/2)\ , \tag{2a}$$

$$k_0 \to -2ia^{-1}\sinh(k_4 a/2)\ , \tag{2b}$$

$$J_\mu^{\rm cont} \to \sqrt{2\kappa_1 2\kappa_2}\,a^{-3}J_\mu^{\rm latt}\ , \tag{2c}$$

$$|n(k)) \to (Na^3 2E_n)^{1/2}|n(k)\rangle\ , \tag{2d}$$

for momenta k_i, energies k_0, vector or axial-vector fields $J_\mu^{\rm cont}$, and particle states $|n(k))$. Similarly we use

$$k_i \to a^{-1}\sin(k_i a)\ , \tag{3a}$$

$$k_0 \to -ia^{-1}\sinh(k_4 a)\ , \tag{3b}$$

$$J_\mu^{\rm cont} \to (2a)^{-3}J_\mu^{\rm latt}\ , \tag{3c}$$

$$|n(k)) \to [N_s(2a)^3 2E_n]^{1/2}|n(k)\rangle\ , \tag{3d}$$

TABLE I. Hadron masses and decay constants for Wilson fermions.

κ_1	κ_2	$M_p a$	$f_p a$	$M_v a$	f_v
0.130	0.130	1.47(7)	0.108(12)	1.47(8)	11(2)
0.148	0.148	0.71(2)	0.085(9)	0.73(2)	5.3(6)
0.154	0.154	0.37(2)	0.067(9)	0.47(2)	3.5(5)
0.154	0.148	0.55(2)	0.076(9)		
0.154	0.130	0.99(3)	0.083(8)		

in the staggered case. N is the total number of points in the lattice and N_s is the total number of staggered hypercubes ($N_s=N/16$). Equations (2d) and (3d) are obtained by comparing lattice and continuum (covariant) completeness.[12,13] In the staggered case the momenta and energies are defined on the doubled lattice. These prescriptions are certainly not unique. We claim only that they are reasonable and can be used to examine the influence of the systematic errors which arise in these types of calculations.

In the Wilson case we use the correlation functions (only the equal-mass case is considered for the vector decay constant)

$$G_{\tilde V\tilde V}(t)=\sum_{\mathbf{x}}\langle \tilde V_3(\mathbf{x},t)\tilde V_3^\dagger(0)\rangle \tag{4a}$$

$$\underset{1\ll t\ll N_t}{\Longrightarrow}\frac{C_{\tilde V\tilde V}}{(2\kappa)^2}(e^{-M_v at}+e^{-M_v a(N_t-t)}) \tag{4b}$$

and

$$G_{V\tilde V}(t)=\sum_{\mathbf{x}}\langle V_3(\mathbf{x},t)\tilde V_3^\dagger(0)\rangle \tag{5a}$$

$$\underset{1\ll t\ll N_t}{\Longrightarrow}\frac{C_{V\tilde V}}{(2\kappa)^2}(e^{-M_v at}+e^{-M_v a(N_t-t)})\ . \tag{5b}$$

Here $\tilde V_\mu(\mathbf{x},t)$ is the local lattice vector field and $V_\mu(\mathbf{x},t)$ is the conserved Wilson vector current. Using these with the lattice transcription of Eq. (1) gives

$$\frac{C_{V\tilde V}}{\sqrt{C_{\tilde V\tilde V}}}=\frac{(M_v a)^{3/2}}{\sqrt{2}f_v}\ . \tag{6}$$

In the staggered case we measure the correlation function

$$g_v(2t_x)=\sum_{\mathbf{x}}\langle V_3(2\mathbf{z},2t_z)V_3^\dagger(0)\rangle \tag{7a}$$

$$\underset{1\ll 2t_x\ll N_t}{\Longrightarrow} c_v(e^{-M_v a(2t_z)}+e^{-M_v a(N_t-2t_z)})\ , \tag{7b}$$

where $V_\mu(2\mathbf{z},2t_z)$ is the conserved nonlocal staggered vector current. Using the somewhat complicated expression that results, one can then measure directly the vector decay constant from

$$\left[\frac{c_v}{N_f}\right]^{1/2}=\frac{2(M_v a)^{3/2}}{f_v}\ . \tag{8}$$

The factor of $\sqrt{N_f}$ with $N_f=4$ in this expression converts the four-flavored staggered lattice matrix element to a single flavor.[14] We are not aware of previous measurements of the vector decay constant in the staggered scheme.

The pseudoscalar-meson decay constant f_p is determined in the continuum by the matrix element of the PCAC (partially conserved axial-vector current):

$$(0|A_\mu(x)|p(k))=ik_\mu\sqrt{2}f_p e^{-ik\cdot x}\ . \tag{9}$$

An alternative way[15] of getting f_p is to use the PCAC relation with the divergence of Eq. (9). Then f_p can be obtained from

$$(m_1+m_2)(0\,|\,P(x)\,|\,p(k))=k_\mu k^\mu\sqrt{2}f_p e^{-ik\cdot x}\ , \tag{10}$$

where $P(x)$ is the local pseudoscalar field.

In the Wilson case, the point-split current $\hat A_\mu(x)$ and the local current $\tilde A_\mu(x)$ do not satisfy PCAC but, in the continuum limit, matrix elements of $\hat A_\mu(x)$ and $\tilde A_\mu(x)$ can be rescaled to yield the correct axial-vector current matrix elements.[16] For the lattice calculation we construct the correlation function (allowing now for different quark masses)

$$G_{\hat AP}(t)=\sum_{\mathbf{x}}\langle \hat A_4(\mathbf{x},t)P^\dagger(0)\rangle \tag{11a}$$

$$\underset{1\ll t\ll N_t}{\Longrightarrow}\frac{C_{\hat AP}}{2\kappa_1 2\kappa_2}(e^{-M_p at}-e^{-M_p a(N_t-t)}) \tag{11b}$$

and a similarly defined correlation function $G_{\tilde AP}(t)$. The pseudoscalar-meson correlation function is

TABLE II. Hadron masses and decay constants for staggered fermions.

$m_1 a$	$m_2 a$	$M_p a$	$f_p a$	$M_v a$	f_v
0.50	0.50	1.647(2)	0.177(2)	2.05(8)	4.0(9)
0.20	0.20	1.081(3)	0.151(2)	1.31(3)	4.4(4)
0.075	0.075	0.673(3)	0.108(3)	0.84(3)	4.1(3)
0.025	0.025	0.393(3)	0.070(3)	0.59(7)	3.3(3)
0.025	0.075	0.553(3)	0.088(3)		
0.025	0.20	0.821(3)	0.108(3)		
0.025	0.50	1.220(3)	0.116(3)		

$$G_{PP}(t)=\sum_{\mathbf{x}}\langle P(\mathbf{x},t)P^{\dagger}(0)\rangle \tag{12a}$$

$$\underset{1\ll t\ll N_t}{\Longrightarrow}\frac{C_{PP}}{2\kappa_1 2\kappa_2}\left(e^{-M_p a t}+e^{-M_p a(N_t-t)}\right). \tag{12b}$$

Then using the above continuum to lattice replacements, the renormalized pseudoscalar-meson decay constant $f_p=f_p^u Z_{\hat{A}}$ from Eq. (9) is given for Wilson fermions by

$$f_p a=\frac{Z_{\hat{A}}C_{\hat{A}P}\sqrt{M_p a}}{2\sqrt{C_{PP}}\sinh(M_p a/2)}\ . \tag{13}$$

From Eq. (10) we get alternatively

$$f_p=\frac{(m_1+m_2)\sqrt{C_{PP}(M_p a)}}{4\sinh^2(M_p a/2)}\ , \tag{14}$$

where m_1 and m_2 are the quark masses. Determining the quark masses can be problematic for Wilson fermions. Fortunately a combination of quark mass and axial renormalization constant can be determined from two-point functions. Following Ref. 17 we consider the ratio

$$R_{\hat{A}}(t)=\frac{\sum_{\mathbf{x}}\langle\nabla_t\hat{A}_4(\mathbf{x},t)P^{\dagger}(0)\rangle}{\sum_{\mathbf{x}}\langle P(\mathbf{x},t)P^{\dagger}(0)\rangle}\ , \tag{15}$$

where

$$\nabla_t\hat{A}_4(\mathbf{x},t)=\hat{A}_4(\mathbf{x},t)-\hat{A}_4(\mathbf{x},t-1)\ . \tag{16}$$

Then

$$R_{\hat{A}}(t)\underset{1\ll t\ll N_t}{\Longrightarrow}(m_1+m_2)/Z_{\hat{A}}\ . \tag{17}$$

Combining Eqs. (14) and (17) again gives the renormalized decay constant f_p. A nice advantage of using Eq. (17) (besides the fact that it is nonperturbative) is that it does not require knowledge of κ_{cr}. Similar expressions can be obtained by substituting $\tilde{A}_4(\mathbf{x},t)$ for $\hat{A}_4(\mathbf{x},t)$ above.

For the staggered case, the interpolating field for pseudoscalar mesons is constructed from the χ fields at the corners of a hypercube on the lattice.[18] The correlation function of meson fields can be expressed in terms of the function[13,18]

$$g_p(t)=\sum_{\mathbf{x}}\langle(-1)^x\bar{\chi}(\mathbf{x},t)\chi(\mathbf{x},t)\bar{\chi}(0)\chi(0)\rangle \tag{18a}$$

$$\underset{1\ll t\ll N_t}{\Longrightarrow}c_p\left(e^{-M_p a t}+e^{-M_p a(N_t-t)}\right), \tag{18b}$$

which involves local combinations of the χ fields. The pseudoscalar decay constant is calculated using the PCAC relation. The staggered fermion analogy of Eq. (14) is

$$f_p=\frac{(m_1+m_2)\sqrt{c_p(M_p a)}\cosh(M_p a/2)}{\sqrt{N_f}\sinh^2 M_p a}\ . \tag{19}$$

The factor $\cosh(M_p a/2)$ arises because the pseudoscalar meson is actually defined by an operator which is not local in χ fields.[19]

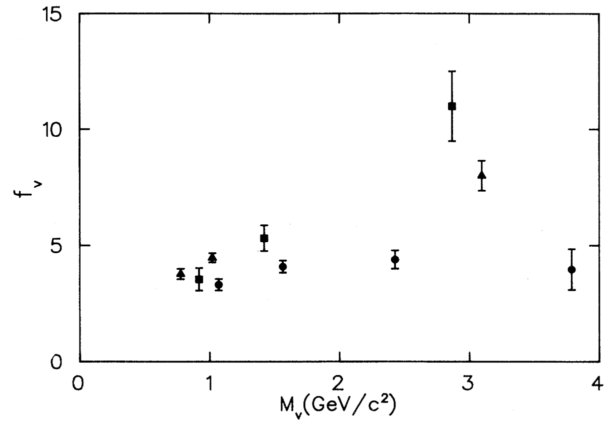

FIG. 1. Vector-meson decay constant f_v vs vector-meson mass for Wilson (■) and staggered (●) fermions. Also shown are experimental values (▲).

Gauge field configurations were prepared in quenched approximation using the Monte Carlo method with the Cabibbo-Marinari pseudo heat bath.[20] The SU(3)-color Wilson plaquette action was used with $\beta=6.0$. The lattice size was $10^3\times 20$. The gauge field was thermalized for 5000 sweeps after a cold start. Configurations were saved every 500 sweeps. A total of 18 configurations were constructed. The average plaquette $\frac{1}{3}\mathrm{Re}\langle\mathrm{Tr}U_{\square}\rangle$ equals 0.5936(2) in good agreement with previous determinations[17,21] for the same value of β.

Masses of pseudoscalar and vector mesons are presented in Table I (for Wilson fermions) and in Table II (for staggered fermions). Parameters were chosen so that the lowest mass Wilson and staggered pions had comparable dimensionful masses. The meson masses are in good agreement with previous calculations.[17,21] When needed, the lattice spacing determined by Hamber[21] is used. For Wilson fermions this is $a_w^{-1}=1950(120)$ MeV and for staggered fermions $a_s^{-1}=1850(60)$ MeV.

Tables I and II also contain our results for the decay

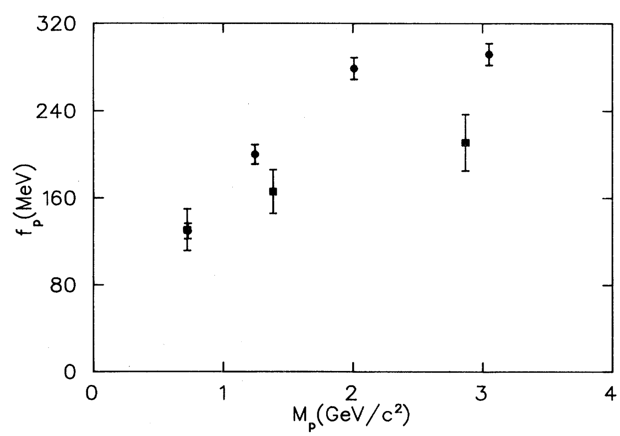

FIG. 2. Pseudoscalar decay constant vs pseudoscalar-meson mass in physical units for equal-mass quarks. Squares (■) are for Wilson fermions, and circles (●) are for staggered fermions. Experimentally, $f_\pi\approx 93$ MeV.

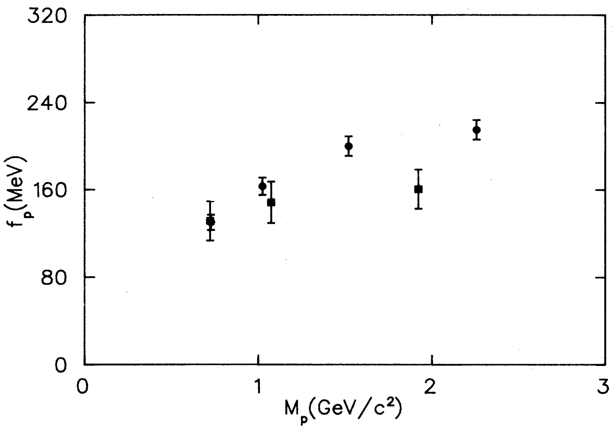

FIG. 3. Pseudoscalar decay constant vs pseudoscalar-meson mass in physical units for unequal-mass quarks. Squares (■) are for Wilson fermions with $\kappa_2=0.154$, and circles (●) are for staggered fermions with $m_2 a=0.025$.

constants of pseudoscalar and vector mesons. Since experimental values are available for vector decay constants, we discuss this case first. Figure 1 shows the vector decay constants, using Eq. (6) for Wilson fermions and Eq. (8) for staggered, as a function of vector meson mass in physical units. Also shown are the values extracted from the experimentally measured width for vector-meson decay into e^+e^- pairs, $\Gamma(V\rightarrow e^+e^-)$. The experimental values are for ρ, ϕ, and ψ mesons. There is good agreement between the lattice QCD and experimental values for small meson masses. It appears as if quenched lattice QCD explains the decay constants for ρ and ϕ mesons. At higher masses the Wilson results are systematically high and the staggered results are systematically low compared to experiment.

A similar set of results holds in the pseudoscalar case, shown in Figs. 2 and 3. Again, the Wilson matrix element comes out to be systematically small relative to the staggered one for large meson masses. [Notice the inverse methods of relating f_v and f_p to their matrix elements in (1) and (9).] Figure 2 is a comparison of Wilson and staggered decay constants for mesons with quarks of equal mass. The Wilson results use PCAC, Eqs. (14) and (17), and the staggered results are from Eq. (19). We have also calculated pseudoscalar decay constants directly from the axial-vector-current matrix element, Eq. (13), without the use of PCAC. The two Wilson results agree excellently over the entire range of κ values used. This would not have been the case if the replacement (2b) had not been made for the continuum energy k_0. We have not carried out an independent calculation of the point-split axial renormalization constant, $Z_{\hat{A}}=0.86$, determined nonperturbatively at $\beta=6.0$ in Ref. 17.

Figure 3 presents a fermion scheme comparison for pseudoscalar decay constants of mesons with quarks of unequal mass, with the light-quark mass held fixed and the heavy-quark mass allowed to vary. There is good agreement between the two fermion schemes up to $Ma\approx 1$ (which in this case means $M\approx 2$ GeV) at which point the central values differ by $\approx 20\%$. In the mass region of the D mesons, the results found for Wilson fermions are consistent with those of Ref. 10 (if the different values of lattice spacing are taken into account), and consistent with, but somewhat higher than, those of Ref. 11, although there is no attempt here to extrapolate to the chiral limit. Notice however that the staggered results are about 40 MeV higher at this point. If the same pattern holds here as was seen in the vector case, the experimental values would be expected to lie in the middle of these two extremes.

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