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Deuteron as a toroidal Skyrmion: Electromagnetic form factors

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The minimal-energy solution in the baryon-number-two sector of the Skyrme model is a toroidally shaped soliton, and its ground state can be identified with the deuteron. A stringent test of this identification is provided by the electromagnetic form factors, since they probe the internal structure of the soliton. These form factors are calculated in the semiclassical approximation and are found to be in qualitative, although not quantitative, agreement with the form factors of the deuteron.

I. INTRODUCTION

Most attempts to apply the Skyrme model to nuclear physics have focused on extracting from the model a nucleon-nucleon potential.[1] A nucleus with atomic number B would then arise as a bound state of B nucleons interacting through this potential, just as in conventional approaches to nuclear physics. An alternative possibility is that nuclei in this model arise in the same way as the nucleon itself: namely, as quantum states of solitons.[2,3] A nucleus of atomic number B would be identified with a static soliton solution with topological charge B. The structure of this soliton would be determined by the nonlinear interactions between the meson fields which appear in the effective Lagrangian, and it might bear very little resemblance to a collection of B individual $B=1$ solitons.

There are many opportunities for this soliton approach to fail. First, there might not be a stable localized soliton solution with topological charge B. It is possible that the only stable solution could consist of B individual solitons at infinite separation. Second, even if there is a stable localized soliton, it might not have a quantum state with the correct spin and isospin quantum numbers of the nucleus. The quantum numbers of the soliton can be determined by semiclassical quantization and depend sensitively on the symmetries of the solution and on its structure. Third, the electromagnetic properties of the soliton might bear little resemblance to those of the nucleus. Finally, the dynamical behavior of the nucleus as measured by scattering with pions, nucleons, etc., may not be reproduced by the soliton. Unless this soliton approach to nuclear physics is fundamentally correct, we should expect it to fail miserably under most of these tests.

The simplest case in which this approach can be tested is the two-nucleon system, in which there is a single stable nucleus, the deuteron, and also several unbound but identifiable states. In Ref. 3 we studied the minimal-energy $B=2$ soliton of the Skyrme model and showed that it passes many of the tests described above. That a stable $B=2$ solution exists was first pointed out in Ref. 4, and the solution was first calculated by Verbaarschot, Walhout, Wambach, and Wyld[5] and by Kopeliovich and Shtern.[6] The symmetries of this solution are such that its ground state is the unique state with the spin, isospin, and parity quantum numbers of the deuteron. In Ref. 3 we pointed out that its first excited state has the quantum numbers of the almost bound $^1S_0$ state of the deuteron, and some of its other states can be interpreted as unbound $N\Delta$ and $\Delta\Delta$ states. (Because of the limitations of our calculational method, we were unable to predict which of these states should be bound.) We also calculated the static electromagnetic properties of the deuteron, including its charge radius, its magnetic moment, its quadrupole moment, and the transition moment for photodisintegration of the deuteron via its excitation into the $^1S_0$ state. The sign and order of magnitude for each of these quantities was consistent with the known properties of the deuteron. We believe that these qualitative successes provide striking evidence in favor of the soliton approach to nuclear physics in the Skyrme model. It is reasonable to expect that by using a more accurate effective Lagrangian and a more accurate approximation of the quantum field theory, one could obtain a quantitative description of the deuteron.

One can argue, however, that none of the properties considered in Ref. 3 really probed the internal structure of the $B=2$ soliton. In particular, the shape of the soliton in the Skyrme model, as measured either by the baryon density or the energy density, is roughly toroidal. The deuteron is identified with a quantum superposition of the classical toroidal solutions, with a wave function that corresponds to isospin $i=0$ and total angular momentum $j=1$. This novel and counterintuitive model of the deuteron bears very little resemblance to the conventional model of the deuteron as a loosely bound state of a proton and neutron. The toroidal structure would not reveal itself in any static property; it could only be probed by scattering some particle off the soliton with momentum transfer comparable to the inverse radius of the toroid.

The simplest probe to use is an electron, in which case the momentum is transferred by a photon. We will therefore calculate the electromagnetic form factors of the deuteron in the Skyrme model using the semiclassical approximation. As was the case for the static properties, we do not expect to reproduce the form factors of the deuteron at the quantitative level. We are primarily interested in their qualitative behavior. In particular, we wish to determine whether the toroidal structure of the soliton is evidenced by any unusual behavior of the form factors which would be incompatible with those of the deuteron. We will follow the notation of Ref. 3, repeating only as many formulas as are necessary to make this paper self-contained.

The minimal-energy solution $U_2(\mathbf{r})$ in the $B=2$ sector of the Skyrme model can be brought to the form

$$U_2(\rho,\phi,z)=e^{-i\phi\tau_3}e^{-i\Theta(\rho,z)\tau_2/2}e^{iF(\rho,z)\tau_3}\times e^{i\Theta(\rho,z)\tau_2/2}e^{i\phi\tau_3}\ , \tag{1.1}$$

where the functions $F(\rho,z)$ and $\Theta(\rho,z)$ must be determined numerically subject to the boundary conditions

$$F(\rho,z)\to\begin{cases}0 & \text{as } r\to\infty\ ,\\ \pi & \text{as } r\to 0\ ,\end{cases}\qquad \Theta(0,z)=\pi\theta(-z)\ , \tag{1.2}$$

where $\theta(x)$ is the step function. The asymptotic form of the solution as $r\to\infty$ is

$$F(\rho,z)\to C\frac{e^{-m_\pi r}}{r}\ ,\qquad \tan\frac{\Theta(\rho,z)}{2}\to\tan^2\frac{\theta}{2}\ , \tag{1.3}$$

where C is a constant. The solution has a cylindrical symmetry and an independent discrete symmetry:

$$U_2(\rho,\phi+\alpha,z)=e^{-i\alpha\tau_3}U_2(\rho,\phi,z)e^{i\alpha\tau_3}\ ,\qquad U_2(\rho,-\phi,-z)=\tau_1U_2(\rho,\phi,z)\tau_1\ . \tag{1.4}$$

Applying translations, rotations, and isospin rotations to $U_2(\mathbf{r})$ generates the complete eight-parameter set of minimal-energy solutions.

In the semiclassical limit of the field theory, these are the only degrees of freedom that need be considered. We therefore make the following ansatz for the dynamical chiral field:

$$U(\mathbf{r},t)=A(t)U_2(R(B(t))[\mathbf{r}-\mathbf{X}(t)])A(t)^\dagger\ , \tag{1.5}$$

where $\mathbf{X}$ is the center-of-mass coordinate, A is an SU(2) matrix, and R is a rotation matrix that can be conveniently parametrized in terms of a second SU(2) matrix B: $R(B)_{ij}=\mathrm{Tr}(\tau^iB\tau^jB^\dagger)/2$. A convenient basis for the quantum states consists of the eigenstates of the coordinate-fixed and body-fixed isospin operators $\mathbf{I}^2=\mathbf{K}^2$, I_3, and K_3, the coordinate-fixed and body-fixed angular momentum operators $\mathbf{J}^2=\mathbf{L}^2$, J_3, and L_3, and the momentum operator $\mathbf{P}$. We denote these states by $|ii_3k_3;jj_3l_3;\mathbf{p}\rangle$. This state can be represented by a wave function which is the product of the following three factors:

$$\langle A|ii_3k_3\rangle=\left[\frac{2i+1}{2\pi^2}\right]^{1/2}D^i(i\tau^2A^\dagger)_{k_3i_3}\ ,\qquad \langle B|jj_3l_3\rangle=\left[\frac{2j+1}{2\pi^2}\right]^{1/2}D^j(i\tau^2B^\dagger)_{j_3l_3}\ ,\qquad \langle\mathbf{X}|\mathbf{p}\rangle=e^{i\mathbf{p}\cdot\mathbf{X}}\ , \tag{1.6}$$

where $D^i(A)_{m,m'}$ is a Wigner D function.

The symmetries (1.4) imply constraints on the Hilbert space generated by the states $|ii_3k_3;jj_3l_3;\mathbf{p}\rangle$. The allowed states are of the form

$$|ii_3\kappa;jj_3-2\kappa;\mathbf{p}\rangle-(-1)^{i+j}|ii_3-\kappa;jj_32\kappa;\mathbf{p}\rangle\ , \tag{1.7}$$

where the integer κ can range from 0 to $\min(i,[j/2])$. Note that for $\kappa=0$, the state (1.7) is nonzero only if $i+j$ is odd. The Hamiltonian for the soliton is obtained by inserting the ansatz (1.5) into the Hamiltonian for the field theory. The state (1.7) is found to be an energy eigenstate with energy

$$E=M_2+\frac{i(i+1)-\kappa^2}{2U_{11}}+\frac{j(j+1)-4\kappa^2}{2V_{11}}+\frac{\kappa^2}{2U_{33}}+\frac{\mathbf{p}^2}{2M_2}\ , \tag{1.8}$$

where M_2 is the classical mass of the soliton and U_{11}, V_{11}, and U_{33} are diagonal components of its inertia tensors. Explicit formulas for these quantities are given in Ref. 3. From the form of E, it is evident that the ansatz (1.5) is inherently nonrelativistic and semiclassical.

There are two states with significantly lower energy than the rest:

$$|dj_3\mathbf{p}\rangle=|000;1j_30;\mathbf{p}\rangle\ ,\qquad |{}^1S_0i_3\mathbf{p}\rangle=|1i_30;000;\mathbf{p}\rangle\ . \tag{1.9}$$

We identify these as the deuteron and the $^1S_0$ state of the two-nucleon system, respectively. The higher-energy states can be interpreted as $N\Delta$ and $\Delta\Delta$ resonances and other more exotic dibaryons.

II. COULOMB AND QUADRUPOLE FORM FACTORS

The semiclassical calculation of nucleon form factors in the $B=1$ sector of the Skyrme model was presented in Ref. 7. The same method can be applied in the $B=2$ sector. The ansatz (1.5) is inserted into the expression for the electromagnetic current and the matrix element of the resulting operator is then evaluated between the states (1.9). We first describe the calculation of the Coulomb and quadrupole form factors $G_C(q^2)$ and $G_Q(q^2)$ of the deuteron. They are defined in the Breit frame ($\mathbf{p}+\mathbf{p}'=0$) by

$$\langle dj_3'\mathbf{p}'|J^0(\mathbf{r}=0)|dj_3\mathbf{p}\rangle=G_C(q^2)\delta_{j_3'j_3}+\frac{1}{6M_d^2}G_Q(q^2)U_{j_3'a}(3q^aq^b-q^2\delta^{ab})U^\dagger_{bj_3}\ , \tag{2.1}$$

where $\mathbf{q}=\mathbf{p}'-\mathbf{p}$ is the momentum transfer, $q=|\mathbf{q}|$, M_d is the mass of the deuteron, and U_{im} is the unitary matrix that relates the Cartesian and spherical bases. The isoscalar part of the charge-density operator is one-half the baryon density operator:

$$J^0(\mathbf{r})=\tfrac{1}{2}B^0(R(B)(\mathbf{r}-\mathbf{X}))=-\frac{1}{2\pi^2}\sin^2F\frac{\sin\Theta}{\rho}(F_{,z}\Theta_{,\rho}-F_{,\rho}\Theta_{,z})\bigg|_{\mathbf{r}\to R(B)(\mathbf{r}-\mathbf{X})} . \tag{2.2}$$

Its operator character resides completely in the argument $\mathbf{r}$ which contains the operators B and $\mathbf{X}$. Their matrix elements are diagonal in i, i_3, k_3, so we will suppress these quantum numbers.

The dependence of the matrix element on the momenta is evaluated as follows:

$$\begin{aligned}\langle j'j_3'l_3';\mathbf{p}'|J^0(\mathbf{r}=0)|jj_3l_3;\mathbf{p}\rangle &=\langle j'j_3'l_3';\mathbf{p}'|\tfrac{1}{2}B^0(-R(B)\mathbf{X})|jj_3l_3;\mathbf{p}\rangle\\ &=\tfrac{1}{2}\int d^3X\, e^{-i\mathbf{q}\cdot\mathbf{X}}\langle j'j_3'l_3'|B^0(-R(B)\cdot\mathbf{X})|jj_3l_3\rangle\\ &=\tfrac{1}{2}\int d^3X\, B^0(\mathbf{X})\langle j'j_3'l_3'|\exp[i\mathbf{q}\cdot R(B)^T\mathbf{X}]|jj_3l_3\rangle .\end{aligned} \tag{2.3}$$

In the second step we used the explicit formula (1.6) for the wave function $\langle\mathbf{X}|\mathbf{p}\rangle$, and in the last step we made the change of integration variable $\mathbf{X}\to-R(B)^T\mathbf{X}$, where the superscript T represents transpose. The exponential can be expanded in terms of irreducible representations of the SU(2) matrix B:

$$\exp[i\mathbf{q}\cdot R(B)^T\mathbf{X}]=4\pi\sum_{lmm'} i^l j_l(q|\mathbf{X}|)Y_m^l(\hat{\mathbf{X}})^*D^l(\tau^2B\tau^2)_{mm'}Y_{m'}^l(\hat{\mathbf{q}}) . \tag{2.4}$$

Since $B^0(\mathbf{r})$ is a function of ρ and z only, the integral in (2.3) will be nonzero only for the $m=0$ term of (2.4). The only operator in this expression appears in the factor $D^l(\tau^2B\tau^2)_{0m'}$. Using the explicit wave function $\langle B|jj_3l_3\rangle$ in (1.6), its matrix element is evaluated as follows:

$$\begin{aligned}\langle j'j_3'l_3'|D^l(\tau^2B\tau^2)_{0m'}|jj_3l_3\rangle &=\int dB\left[\frac{2j'+1}{2\pi^2}\right]^{1/2}D^{j'}(i\tau^2B^\dagger)^*_{j_3'l_3'}D^l(\tau^2B\tau^2)_{0m'}\left[\frac{2j+1}{2\pi^2}\right]^{1/2}D^j(i\tau^2B^\dagger)_{j_3l_3}\\ &=\left[\frac{2j'+1}{2\pi^2}\right]^{1/2}\left[\frac{2j+1}{2\pi^2}\right]^{1/2}\int dB\, D^{j'}(B)_{l_3'j_3'}D^l(i\tau^2B)_{0m'}D^j(B^\dagger)_{j_3l_3}\\ &=\left[\frac{2j'+1}{2j+1}\right]^{1/2}\sum_m D^l(i\tau^2)_{0m}\langle j'll_3'm|jl_3\rangle\langle j'lj_3'm'|jj_3\rangle .\end{aligned} \tag{2.5}$$

In the second step we used the invariance of the SU(2) integration measure dB to make the change of variables $B\to B(i\tau^2)$. In the last step, we made use of the standard SU(2) integration formula

$$\int dB\, D^i(B)_{m_1m_1'}D^j(B)_{m_2m_2'}D^k(B^\dagger)_{m_3'm_3}=\frac{2\pi^2}{2k+1}\langle ijm_1m_2|km_3\rangle\langle ijm_1'm_2'|km_3'\rangle . \tag{2.6}$$

Using the identity $D^l(i\tau^2)_{0m}=(-1)^l\delta_{m,0}$ and combining (2.3), (2.4), and (2.5), we obtain the general expression for the matrix element of $J^0(\mathbf{r}=0)$:

$$\langle j'j_3'l_3';\mathbf{p}|J^0(0)|jj_3l_3;\mathbf{p}\rangle=4\pi\left[\frac{2j'+1}{2j+1}\right]^{1/2}\sum_{lm}(-i)^l\langle j'll_3'0|jl_3\rangle\langle j'lj_3'm|jj_3\rangle Y_m^l(\hat{\mathbf{q}})\tfrac{1}{2}\int d^3X\, Y_0^l(\hat{\mathbf{X}})^*j_l(q|\mathbf{X}|)B_0(\mathbf{X}) . \tag{2.7}$$

Specializing to the case of the deuteron, this becomes

$$\langle dj_3'\mathbf{p}'|J^0(0)|dj_3\mathbf{p}\rangle=\delta_{j_3'j_3}\tfrac{1}{2}\int d^3r\, j_0(qr)B^0(\rho,z)+U_{j_3'a}(3\hat{q}^a\hat{q}^b-\delta^{ab})U^\dagger_{bj_3}\tfrac{1}{4}\int d^3r\frac{\rho^2-2z^2}{r^2}j_2(qr)B^0(\rho,z) . \tag{2.8}$$

Comparing with the definition (2.1) of the Coulomb and quadrupole form factors, we find

$$\begin{aligned}G_C(q^2)&=\tfrac{1}{2}\int d^3r\, j_0(qr)B^0(\rho,z) ,\\ \frac{1}{M_d^2}G_Q(q^2)&=\frac{3}{2}\frac{1}{q^2}\int d^3r\frac{\rho^2-2z^2}{r^2}j_2(qr)B^0(\rho,z) .\end{aligned} \tag{2.9}$$

The limiting behavior of these form factors as $q^2\to0$ is $G_C(q^2)\to1-\frac{1}{6}q^2\langle r^2\rangle_d$ and $G_Q(0)=M_d^2Q$, where $\langle r^2\rangle_d$ and Q are the charge radius squared and quadrupole moment of the deuteron obtained in Ref. 3:

$$\begin{aligned}\langle r^2\rangle_d&=\tfrac{1}{2}\int d^3r\, r^2B^0(\rho,z) ,\\ Q&=\tfrac{1}{10}\int d^3r(\rho^2-2z^2)B^0(\rho,z) .\end{aligned} \tag{2.10}$$

III. MAGNETIC AND TRANSITION MAGNETIC FORM FACTORS

The magnetic form factor of the deuteron is defined in the Breit frame ($\mathbf{p}+\mathbf{p}'=0$) by

$$\langle dj_3'\mathbf{p}'|J^i(\mathbf{r}=0)|dj_3\mathbf{p}\rangle = \frac{1}{2M_d}G_M(q^2)U_{j_3'a}(q^a\delta^{ib}-\delta^{ai}q^b)U^\dagger_{bj_3} \; . \quad (3.1)$$

The calculation of this form factor is much more complicated than that of $G_C(q^2)$ and $G_Q(q^2)$, so we will only quote a few intermediate results. Only the isoscalar part $\frac{1}{2}B^i$ of the electromagnetic current contributes to the matrix element. Inserting the ansatz (1.5) into the expression for J^i, the current becomes

$$\langle dj_3'\mathbf{p}'|J^i(0)|dj_3\mathbf{p}\rangle = \frac{1}{4V_{11}}\langle dj_3'\mathbf{p}'|(\{L_1,R(B)_{2i}[zB^0(\mathbf{r})]-R(B)_{3i}[\rho\sin\phi\, B^0(\mathbf{r})]\} - \{L_2,R(B)_{1i}[zB^0(\mathbf{r})]-R(B)_{3i}[\rho\cos\phi\, B^0(\mathbf{r})]\})|dj_3p\rangle \; , \quad (3.2)$$

where V_{11} is a diagonal component of one of the inertia tensors of the soliton and the functions of $\mathbf{r}$ inside the square brackets are to be evaluated at $\mathbf{r}=-R(B)\mathbf{X}$. We have dropped operators of the form $\{\mathbf{K},O\}$, where O is some operator, since $\mathbf{K}$ annihilates the deuteron. Operators of the form $\{\mathbf{P},O\}$ have also been dropped since they vanish in the Breit frame. Following steps that are similar to those leading up to (2.8), we arrive at the final result

$$\langle dj_3'\mathbf{p}'|J^i(0)|dj_3\mathbf{p}\rangle = U_{j_3'a}(\hat{q}^a\delta^{ib}-\delta^{ai}\hat{q}^b)U^\dagger_{bj_3}\frac{3}{8V_{11}}\int d^3r\frac{\rho^2+2z^2}{r}j_1(qr)B^0(\rho,z) \; . \quad (3.3)$$

Comparing with (3.1), we find that the desired form factor is

$$\frac{1}{2M_d}G_M(q^2) = \frac{3}{8V_{11}}\frac{1}{q}\int d^3r\frac{\rho^2+2z^2}{r}j_1(qr)B^0(\rho,z) \; . \quad (3.4)$$

The expression for V_{11} is given in Ref. 3:

$$V_{11} = \tfrac{1}{8}f_\pi^2\int d^3r\left[(zF_{,\rho}-\rho F_{,z})^2+\sin^2F\left[(z\Theta_{,\rho}-\rho\Theta_{,z})^2+4z^2\frac{\sin^2\Theta}{\rho^2}\right]\right] + \frac{2}{e^2}\int d^3r\left\{\sin^2F\frac{\sin^2\Theta}{\rho^2}[(zF_{,\rho}-\rho F_{,z})^2+z^2(F_{,\rho}^2+F_{,z}^2)] + \sin^4F\frac{\sin^2\Theta}{\rho^2}[(z\Theta_{,\rho}-\rho\Theta_{,z})^2+z^2(\Theta_{,\rho}^2+\Theta_{,z}^2)]+\tfrac{1}{4}r^2\sin^2F(F_{,z}\Theta_{,\rho}-F_{,\rho}\Theta_{,z})^2\right\} \; . \quad (3.5)$$

The limit of this form factor as $q^2\to 0$ is $G_M(0)=2M_d\mu_d$, where μ_d is the magnetic moment of the deuteron calculated in Ref. 3:

$$\mu_d = \frac{1}{8V_{11}}\int d^3r(\rho^2+2z^2)B^0(\rho,z) \; . \quad (3.6)$$

These methods can also be used to calculate the form factors for transitions from the deuteron to its excited states. Since these are difficult to measure experimentally, we will consider only the form factor for the transition to the ${}^1S_0$ state. We define the form factor $G_T(q^2)$ by the following matrix element evaluated in the Breit frame:

$$\langle {}^1S_0 i_3\mathbf{p}'|J^i(\mathbf{r}=0)|dj_3\mathbf{p}\rangle = -i\epsilon^{ijk}q^jU^\dagger_{kj_3}\delta_{i_30}\frac{1}{2M_d}G_T(\mathbf{q}^2) \; . \quad (3.7)$$

In this case, it is the isovector part I_3^i of the current which contributes. Upon inserting the ansatz (1.5) into I_3^i, one finds that as far as the isospin quantum numbers i, i_3, and k_3 are concerned, the only nontrivial operator is an overall multiplicative factor of $\mathrm{Tr}(A\tau^3A^\dagger\tau^a)/2$. Its matrix element is

$$\langle 1i_30|\tfrac{1}{2}\mathrm{Tr}(A\tau^3A^\dagger\tau^a)|000\rangle = -\frac{1}{\sqrt{3}}\delta^{a3}\delta_{i_30} \; . \quad (3.8)$$

Given this information, the matrix element reduces to

$$\langle {}^1S_0 i_3\mathbf{p}'|J^i(0)|dj_3\mathbf{p}\rangle = -\frac{1}{\sqrt{3}}\delta_{i_3,0}\langle 000;\mathbf{p}'|R(B)_{ki}[I(\rho,z)\epsilon^{3kj}r^j]|1j_30;\mathbf{p}\rangle \; , \quad (3.9)$$

where the expression in square brackets is to be evaluated at $\mathbf{r}=-R(B)\mathbf{X}$ and the function $I(\rho,z)$ is

$$I(\rho,z) = \tfrac{1}{2}f_\pi^2\sin^2F\frac{\sin^2\Theta}{\rho^2}+\frac{2}{e^2}\sin^2F\frac{\sin^2\Theta}{\rho^2}[F_{,\rho}^2+F_{,z}^2+\sin^2F(\Theta_{,\rho}^2+\Theta_{,z}^2)] \; . \quad (3.10)$$

After calculating the matrix elements, the final result is

$$\langle {}^1S_0 i_3 \mathbf{p}'|J^i(0)|d j_3 \mathbf{p}\rangle = i\epsilon^{ijk}\hat{q}^{j} U^{\dagger}_{kj_3}\delta_{i_3 0}\tfrac{1}{2}\int d^3r \frac{\rho^2}{r} j_1(qr) I(\rho,z) \; . \tag{3.11}$$

Comparing with (3.7), we read off the transition form factor:

$$\frac{1}{2M_d} G_T(q^2) = -\frac{1}{2}\frac{1}{q}\int d^3r \frac{\rho^2}{r} j_1(qr) I(\rho,z) \; . \tag{3.12}$$

Its value at $q^2=0$ is $G_T(0)=2M_d\mu_{d\to np}$, where $\mu_{d\to np}$ is the transition moment for the photodisintegration of the deuteron calculated in Ref. 3:

$$\mu_{d\to np} = -\tfrac{1}{6}\int d^3r\, \rho^2 I(\rho,z) \; . \tag{3.13}$$

IV. DISCUSSION OF RESULTS

We have calculated the form factors numerically for the same parameter set used in Ref. 3: $f_\pi=108$ MeV, $e=4.84$, $m_\pi/ef_\pi=0.263$. This parameter set optimizes the predictions for static properties of the nucleon.[8] It is not the optimal set for describing the $B=0$ sector of the Skyrme model, since the physical value of the pion decay constant is $f_\pi=186$ MeV. We should, therefore, not expect it to give quantitative predictions in the $B=2$ sector either. The static electromagnetic properties of the deuteron for these parameters are[3]

$$\begin{aligned} \langle r^2\rangle_d^{1/2} &= 0.92 \text{ fm} \; , \\ Q &= 0.082 \text{ fm}^2 \; , \\ \mu_d &= 0.74\mu_N \; , \\ \mu_{d\to np} &= -4.4\mu_N \; , \end{aligned} \tag{4.1}$$

where μ_N is the nuclear magneton. These quantities agree in sign and order of magnitude with the experimentally measured properties of the deuteron.

The absolute values of the form factors G_C, G_Q, G_M, and G_T are plotted in Figs. 1 and 2. The form factors have been all normalized to have the value 1 at $q^2=0$. The qualitative behavior of these form factors is in fact similar to that of more conventional models of the deuteron.[9] The dips in the Coulomb and magnetic form factors are due to the form factor going through 0 as it changes sign. In contrast, the quadrupole form factor falls off monotonically. We see that the toroidal structure of the soliton does not lead to any unreasonable behavior in the form factors.

While the qualitative behavior of the form factors is reasonable, they do not fare so well in a quantitative comparison with experimental data. The combinations of form factors that are measured directly in electron-deuteron scattering are

$$\begin{aligned} A(q^2) &= G_C(q^2)^2 + \tfrac{8}{9}\eta^2 G_Q(q^2)^2 + \tfrac{2}{3}\eta G_M(q^2)^2 \; , \\ B(q^2) &= \tfrac{4}{3}\eta(1+\eta) G_M(q^2)^2 \; , \end{aligned} \tag{4.2}$$

where $\eta=q^2/4M_d^2$ and $q^2=\mathbf{q}^2-q_0^2$. The predictions of the Skyrme model for these structure functions are shown in Fig. 3, along with some recent experimental

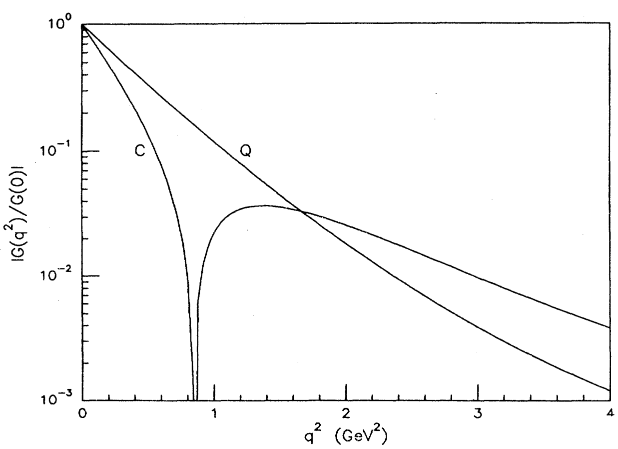

FIG. 1. Skyrme-model predictions for the absolute values of the Coulomb (C) and quadrupole (Q) form factors of the deuteron, normalized to 1 at $q^2=0$.

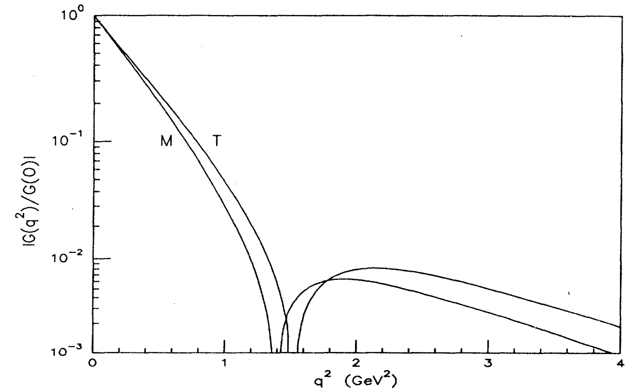

FIG. 2. Skyrme-model predictions for the absolute values of the magnetic (M) and transition magnetic (T) form factors of the deuteron, normalized to 1 at $q^2=0$.

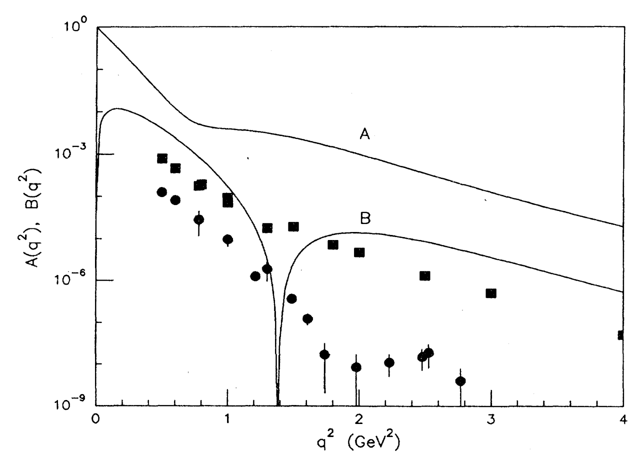

FIG. 3. Skyrme-model predictions for the structure functions $A(q^2)$ and $B(q^2)$ for elastic electron-deuteron scattering, compared with experimental data for A (solid squares) and B (solid circles).

data.[10] The most obvious discrepancy is that the Skyrme-model predictions fall off much more slowly than the data; they differ by several orders of magnitude at large q^2. The asymptotic behavior of the form factors, however, depends sensitively on the shape of the baryon density $B(\rho,z)$. It is relatively easy to find functions $F(\rho,z)$ and $\Theta(\rho,z)$ for which the form factors do fall off at the desired rate. We have not, however, succeeded in finding simple forms (i.e., simple enough so that the form factors can be calculated analytically) for $F(\rho,z)$ and $\Theta(\rho,z)$ which give good fits to both $A(q^2)$ and $B(q^2)$ simultaneously.

While the deuteron form factors in the Skyrme model have the same qualitative behavior as in more conventional models, this behavior arises in a very different way. The starting point for conventional calculations of the form factors is the impulse approximation. For example, the simplest approximation to the Coulomb form factor is

$$G_C(q^2)=[G_E^p(q^2)+G_E^n(q^2)] \times\int_0^\infty dr[u(r)^2+w(r)^2]j_0(qr/2)\ , \qquad (4.3)$$

where G_E^p and G_E^n are the electric form factors of the proton and neutron and $u(r)$ and $w(r)$ are the $^3S_1$ and $^3D_1$ components of the radial wave function of the deuteron. Note that in (4.3), the q^2 dependence arises both from the Fourier transform of the wave function of the deuteron and from the nucleon form factors. In the Skyrme model prediction (2.9) for G_C, the q^2 dependence arises only from a Fourier transform of the baryon density of the toroidal soliton. That this calculation produces a qualitatively reasonable form factor is rather remarkable, since the toroidal soliton bears no resemblance to a composite of two individual $B=1$ solitons.

There has been a previous attempt by Nyman and Riska[11] to calculate the deuteron form factors in the Skyrme model. Instead of the minimal energy solution $U_2(\mathbf{r})$, they used a product ansatz configuration of the form $A_1U_1(\mathbf{r}-\mathbf{X}_1)A_1^\dagger A_2U_1(\mathbf{r}-\mathbf{X}_2)A_2^\dagger$, where $U_1(\mathbf{r})=\exp[iF(r)\hat{\mathbf{r}}\cdot\boldsymbol{\tau}]$ and $F(r)$ was determined phenomenologically by fitting the isoscalar nucleon form factor. The electromagnetic current can then be decomposed into three terms

$$J^\mu(\mathbf{r})=J_1^\mu(\mathbf{r}-\mathbf{X}_1,A_1)+J_2^\mu(\mathbf{r}-\mathbf{X}_2,A_2) +J_{\rm ex}^\mu(\mathbf{r};\mathbf{X}_1,\mathbf{X}_2,A_1,A_2)\ . \qquad (4.4)$$

There is a corresponding decomposition of the form factors; for example, $G_C=G_C^1+G_C^2+G_C^{\rm ex}$. Nyman and Riska replaced $G_C^1+G_C^2$ by the conventional result (4.3) of the potential model in the impulse approximation. They calculated the exchange term $G_C^{\rm ex}$ by sandwiching the operator $J_{\rm ex}^\mu$ in (4.4) between deuteron potential model wave functions for the coordinate $\mathbf{X}_1$ and $\mathbf{X}_2$ and Skyrme-model nucleon wave functions for the SU(2) matrices A_1 and A_2. While they obtained very reasonable results, we regard that as rather fortuitous. We believe that the use of the product ansatz has been discredited by the discovery that it does not probe the lowest-energy configurations of the $B=2$ system.[5] Furthermore, a separation of the form factor into impulse and exchange contributions is very unnatural in a soliton model. Skyrmions are extended objects which lose their individual identities when they overlap. Finally, Nyman and Riska used the Skyrme model only as a supplement to conventional potential-model calculations, and as such their results cannot be regarded as predictions of the Skyrme model alone.

In this paper we have calculated the electromagnetic form factors for the deuteron in the Skyrme model under the assumption that the deuteron should be identified with the ground state of the toroidal $B=2$ Skyrmion. The qualitative behavior of these form factors is indeed similar to those obtained from more conventional models of the deuteron. This provides support for this unconventional and counterintuitive model of the deuteron. In spite of the fact that the constituent nucleons have completely lost their individuality, this model seems to have a remarkable ability to reproduce qualitatively the physical properties of the deuteron. At the quantitative level, this approach is far from competitive with conventional treatments of the deuteron based on a potential model.[9] However, by using a more accurate effective Lagrangian and going beyond the semiclassical limit, one could hope to develop this model into a quantitative description of the deuteron.

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