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Approximate multijet cross sections in QCD

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A method is given to approximate the matrix elements squared for all parton processes involving a quark-antiquark pair plus an arbitrary number of gluons. Detailed comparisons are made between the results of this method and the exact results which are known for the quark-antiquark plus-four-gluon-processes in four-jet production at hadron colliders. Together with Maxwell's approximate result for six-gluon processes an excellent agreement is found for the total cross section and shape of four-jet production at hadron colliders.

I. INTRODUCTION

Recently there has been a great deal of interest in multijet events at hadron colliders by both experimentalists[1] and theorists.[2] Unfortunately the theoretical situation is such that for more than four-jet production little is known of the QCD partonic cross sections. For four-jet production all QCD matrix elements exist in the literature[3] but intense computer usage is required to calculate each partonic cross section. Berends and Giele[4] have also given a recursive algorithm which allows calculation of the n-gluon amplitudes for arbitrary n but the complexity of this algorithm has so far prevented rapid and simple usage.[14]

A systematic procedure to approximate multigluon cross sections was given recently by Maxwell.[5] He also suggested the use of the effective-structure-function approximation[6] to describe processes involving quarks as well. However, this approximation for the fermionic cross sections is known to become progressively worse with an increasing number of partons.

In this paper we generalize the multigluon approximation of Maxwell and give approximate cross sections for those processes involving a quark-antiquark pair plus n gluons. We will specifically treat the n=4 case, the generalization to larger n being straightforward. Detailed comparisons are made with the exact matrix elements and with the predictions of the effective-structure-function approximation. When our results are combined with the approximate six-gluon cross section of Maxwell they provide a powerful tool for analyzing the four-jet events of hadron colliders.

II. APPROXIMATE CROSS SECTIONS

Our starting point is the exact nonzero tree-level matrix element squared, to leading order in the number of colors, for the processes involving an arbitrary number of gluons or a quark-antiquark pair plus an arbitrary number of gluons which maximally violates the conservation of helicity; e.g.,

$$g^+g^+ \to g^+g^+g^+g^+\cdots\ ,\quad q^+g^+ \to q^+g^+g^+g^+\cdots\ ,$$

and

$$g^+g^- \to q^+\bar{q}^-g^+g^+\cdots\ .$$

The matrix elements squared were given by Parke and Taylor[7,8] for the purely gluonic process and by the authors[9] for the quark-antiquark-plus-n-gluon process and can be simply given in terms of the elementary variables, $S_{ij} \equiv 2p_i\cdot p_j$. For the n-gluon process the color sum for the matrix element squared for the sum of the maximally helicity-violating amplitudes is given by

$$\sum |M_g^{\rm viol}|^2 = g_s^{2n-4}(Q^2)2N^{n-2}(N^2-1) \times\left(\sum_{i<j} S_{ij}^4\right)\left(\sum_{\rm perm'}\frac{1}{S_{12}S_{23}\cdots S_{n1}}\right)\ , \tag{2.1}$$

where $\sum_{\rm perm'}$ is the sum over all noncyclic permutations of the gluons $(1,2,\ldots,n)$. The similar expression for the quark-antiquark plus n gluons is

$$\sum |M_q^{\rm viol}|^2 = g_s^{2n}(Q^2)\,2N^{n-1}(N^2-1) \times\sum_i (S_{qi}^3 S_{\bar{q}i} + S_{qi}S_{\bar{q}i}^3) \times\left(\sum_{\rm perm}\frac{1}{S_{\bar{q}q}S_{q1}S_{12}S_{23}\cdots S_{n\bar{q}}}\right)\ , \tag{2.2}$$

where $\sum_{\rm perm}$ is now the sum over all permutations of the n gluons and $q,\bar{q}$ are the momenta of the quark and antiquark, respectively.

For the six-gluon process, Maxwell[5] has given a method for including the contribution from the more complex helicity-conserving amplitudes. His approximation is to multiply the matrix element squared for the helicity-violating processes by a factor χ_{gg}^g such that the product has the Altarelli-Parisi[10] residue for the collinear pole of the pair of gluons with the smallest $|S_{ij}|$. Maxwell refers to this procedure as infrared reduction. For the purely gluonic process, the multiplication factor is

$$\chi^g_{gg} = \frac{(1+R)\,[1+z^4+(1-z)^4]}{R+z^4+(1-z)^4}\,, \tag{2.3}$$

where R and z are determined by the pair of gluons (α,β) which have the minimum $|S_{ij}|$ in the following way:[11]

$$z = \frac{p^0_\alpha}{P^0},\quad P \equiv p_\alpha + p_\beta\ ,$$
$$R = \frac{\sum_{i<j,\neq\alpha,\beta} S^4_{ij}}{\sum_{i\neq\alpha,\beta} S^4_{iP}}. \tag{2.4}$$

Our conventions are that all momenta are outgoing, i.e., the incoming particles have negative energies.

In analogy with Maxwell's method one can show that the infrared reduction procedure applies to processes with a quark-antiquark pair plus gluons. Here the multiplication factor depends on the type of particles which make up the minimum $|S_{ij}|$. For the case where the particles with the minimum $|S_{ij}|$ are both gluons,

$$\chi^q_{gg} = \frac{(1+R)\,[1+z^4+(1-z)^4]}{R+z^4+(1-z)^4} \tag{2.5}$$

as before, but with

$$z = \frac{p^0_\alpha}{P^0},\quad P \equiv p_\alpha + p_\beta\ ,$$
$$R = \frac{\sum_{i\neq\alpha,\beta} (S^3_{qi}S_{\bar{q}i} + S_{qi}S^3_{\bar{q}i})}{S^3_{qP}S_{\bar{q}P} + S_{qP}S^3_{\bar{q}P}}. \tag{2.6}$$

If the pair with the minimum dot product is a quark and a gluon, then

$$\chi^q_{qg} = \frac{(1+R)\,(1+z_q^2)}{1+Rz_q^2}\,, \tag{2.7}$$

where

$$z_q = \frac{q^0}{Q^0},\quad Q \equiv p_\alpha + q\ ,$$
$$R = \frac{\sum_{i\neq\alpha} S^3_{Qi}S_{\bar{q}i}}{\sum_{i\neq\alpha} S_{Qi}S^3_{\bar{q}i}}. \tag{2.8}$$

The result for an antiquark-gluon pair is the same as the above quark-gluon pair but with each fermion momentum replaced by the appropriate antifermion momentum.

For the situation in which the minimum $|S_{ij}|$ pair is made up of a quark and an antiquark the multiplication factor is

$$\chi^q_{q\bar{q}} = 1 + R\ , \tag{2.9}$$

where

$$G \equiv q + \bar{q}\ ,$$
$$R = \frac{\sum_{i<j} S^4_{ij}}{\sum_i S^4_{Gi}}. \tag{2.10}$$

Thus our approximation is equal to $\chi \sum |M^{\rm viol}|^2$ times a weight factor which averages the incoming colors and helicities and also provides the appropriate statistical factor for identical particles. All of these results can be generalized to processes with more than four partons in the final state by expressing the approximate cross section as a product of the maximally helicity-violating cross section times more χ factors, two for a seven-parton process, three for an eight-parton process, and so on.

III. THE COMPARISON

To compare our results with the exact matrix elements squared we have looked at the processes $gg \to gggg$, $qg \to qggg$, $\bar{q}g \to \bar{q}ggg$, and $gg \to \bar{q}qgg$ in a proton-antiproton collider at 1.8 TeV, the Fermilab Tevatron. We omit the results for the $q\bar{q} \to gggg$ process because its rate is very small. Processes with two quark pairs can be approximated in a similar way by using the simple helicity-violating matrix elements given in Ref. 12, but their rate is totally negligible.

We have used a fixed set of structure functions throughout, Duke and Owens[13] ($\Lambda = 200$ MeV), and the cuts on the partonic jets for the transverse momentum p_t, pseudorapidity y, and separation of the jets ΔR, are

$$p_t > 25\ \text{GeV},$$
$$|y| < 3.5, \tag{3.1}$$
$$\Delta R \equiv \sqrt{\Delta\phi^2 + \Delta y^2} > 0.8.$$

We choose the Q^2 scale for the QCD evolution to be the average p_t of the event: $Q^2 = (\sum p_t/4)^2$. For both the exact and the approximate matrix elements we have plotted three differential cross sections, $d\sigma/dp_t$ versus p_t, $d\sigma/d\cos\theta_{23}$ versus $\cos\theta_{23}$, and $d\sigma/dP_{\rm out}$ versus $P_{\rm out}$ in Figs. 1–3. p_t is the transverse momentum of each

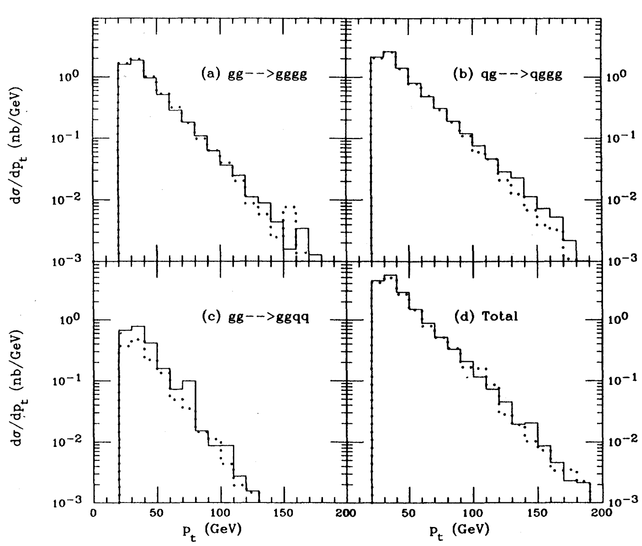

FIG. 1. The differential cross sections $d\sigma/dp_t$ versus p_t for the labeled processes of four-jet production at the Tevatron with the cuts given in Eq. (3.1). The solid line is the approximation and the dotted line the exact result.

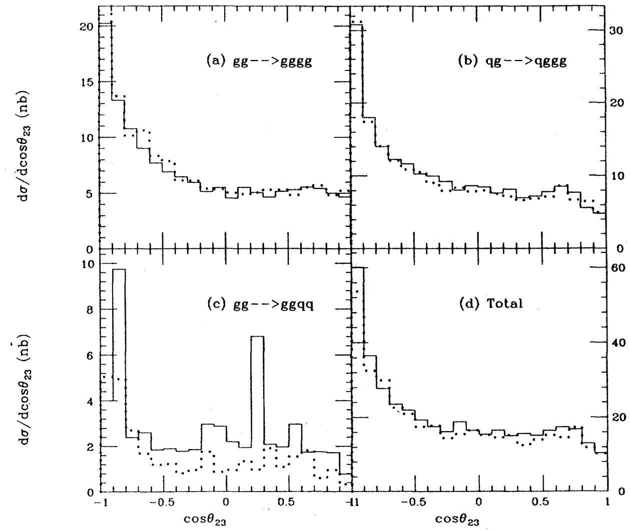

FIG. 2. The differential cross sections $d\sigma/d\cos\theta_{23}$ versus $\cos\theta_{23}$ for the labeled processes of four-jet production at the Tevatron with the cuts given in Eq. (3.1). The solid line is the approximation and the dotted line the exact result.

jet. $P_{\text{out}} \equiv \frac{1}{2}\sum |p^i_{\text{out}}|$ with p^i_{out} the momentum of the ith jet perpendicular to the plane given by the beam and the jet of largest p_t. The angle θ_{23} is the angle between the second- and third-highest-energy jets in the center of mass of the incoming partons. For each differential cross section there are four plots each appropriate for proton-antiproton collisions with five flavors of light quarks: (a) for the purely gluonic process, (b) for all processes with quark (antiquark) plus gluon to quark (antiquark) plus three gluons, (c) for the process gluon plus gluon to quark-antiquark plus two gluons, and (d) for the sum of these three. The total rates for these processes are summarized in Table I.

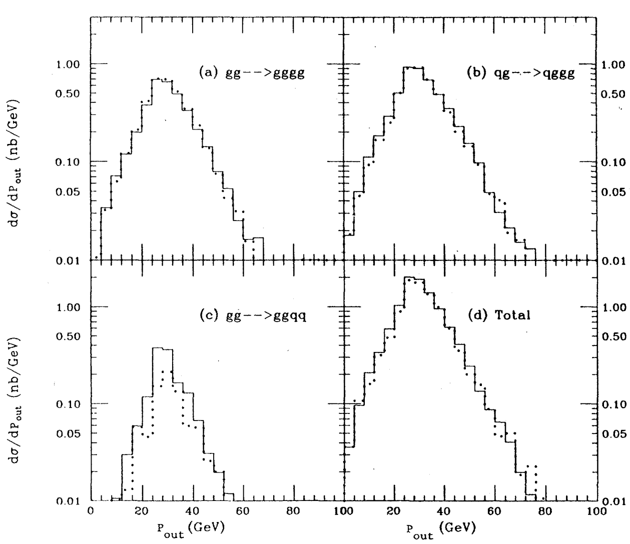

FIG. 3. The differential cross sections $d\sigma/dP_{\text{out}}$ versus P_{out} for the labeled processes of four-jet production at the Tevatron with the cuts given in Eq. (3.1). The solid line is the approximation and the dotted line the exact result.

TABLE I. Cross sections for four-jet production at the Tevatron with the cuts of Eq. (3.1).

Process	Exact cross section (nb)	Approximate cross section (nb)
(a) $gg \to gggg$	14	15
(b) $qg \to qggg$	21	20
(c) $gg \to q\bar{q}gg$	3	6
(d) Total: Four jets	38	41

As is clear from Table I and from Figs. 1–3, the approximation to the purely gluonic process and to the processes with one quark in the initial state are extremely good, while the agreement between exact and approximated results for the process with a quark-antiquark pair in the final state is rather poor. Fortunately this last process has a small cross section, and the induced error on the full cross section is marginal.

The main reason underlying the accuracy of these approximations is the dominance of the helicity-violating amplitudes over the helicity-conserving ones. This in fact guarantees the stability of the infrared reduction when extrapolated from the collinear limit $s_{ij} \to 0$ to the observable kinematical configurations in which $s_{ij} \neq 0$. This dominance holds for the $gg \to gggg$ and $qg \to qggg$ processes. When integrating over phase space the helicity-conserving amplitudes contribute in average to 20–30 % of the full amplitude. Even a 30% uncertainty in estimating them (uncertainty coming from the extrapolation of the infrared reduction) would give rise to an error no larger than 10% on the full amplitude.

One way to understand why the helicity-violating amplitudes are so important is the following. Any six-parton amplitude squared can be expressed in terms of the kinematical invariants $s_{ij} = (p_i + p_j)^2$ and $t_{ijk} = (p_i + p_j + p_k)^2$. If 1 and 2 are the incoming partons, it is easy to verify that $s_{12} = s > |s_{ij}|, |t_{ijk}|$ for any choice of i, j, and k. The cuts that are imposed in the calculation of the rates, in particular the rapidity cuts and the transverse momentum cuts, make the inequality even stronger. The cross sections are then dominated by the terms with the largest power of s_{12} in the numerator. For the $gg \to gggg$ process these terms appear in the helicity-violating amplitudes, which behave as $(s_{12})^4$ as opposed to the $(s_{12})^2$ behavior of the helicity-conserving ones. Also, for the $qg \to qggg$ process the terms containing the leading power of s_{12} again appear in the helicity-violating amplitudes, which behave as $(s_{12})^3$.

In the process $gg \to ggq\bar{q}$, whose amplitude is just obtained by crossing from the amplitude for $qg \to qggg$, the situation is, however, different. Here, in fact, no power of s_{12} can appear in the numerator of the helicity-violating amplitudes, as is clear from equation (2.2). In this case the full amplitude is dominated by the helicity-conserving component. As an example consider the situation in which the smallest s_{ij} corresponds to the $q\bar{q}$ pair: in this case $\chi_{qq} = 1 + R$, with R arbitrarily large. Even if the infrared-reduction procedure is valid in the exactly collinear limit $s_{ij} \to 0$, its extrapolation to the kinematical domain of a generic collision is rather unstable. As a consequence our approximation tends to systematically overestimate the exact result. As we already noticed, nevertheless, since the contribution of this process to the total rate is quite small the total error introduced is also small.

In the perspective of applying this approximation scheme to other reactions, however, we believe that the criterion of dominance of the helicity-violating amplitudes is relevant to establish *a priori* (or in the absence of the exact result to compare with) the reliability of the approximation. The above qualitative considerations, for example, will still hold for QCD processes with more than four hard partons in the final state. The numerical analysis of total rates for seven-gluon processes, carried out in Ref. 14, shows a good agreement between the exact result and the Maxwell multigluon approximation.

IV. THE EFFECTIVE-STRUCTURE-FUNCTION APPROXIMATION

We now compare our approximation of the quark cross section to the effective-structure-function approximation. This approximation amounts to assuming that in most of the relevant phase space the differential cross sections for processes initiated by gg, by qg, and by qq or $q\bar{q}$ stand in a constant ratio:

$$d\sigma_{gg} : d\sigma_{gq} : d\sigma_{qq} \;=\; 1 : \tfrac{4}{9} : (\tfrac{4}{9})^2. \tag{4.1}$$

In this way the total cross section, weighted by the appropriate structure functions, reads

$$d\sigma_{\rm tot} = F(x_1)F(x_2)d\sigma_{gg}, \tag{4.2}$$

$$F(x) = g(x) + \tfrac{4}{9}[q(x) + \bar{q}(x)], \tag{4.3}$$

$g(x)$ and $q(x)$ being the gluon and quark structure functions.

This approximation is extremely good in the case of two partons in the final state, but becomes less and less accurate when increasing the complexity of the final state. Phenomenological applications of this approximation for multijet physics were given by Kunszt and Stirling in Ref. 15. These authors, however, used a simplified version of the multigluon approximation. Namely, they chose for χ^g_{gg} a constant value given by the ratio of the total number of helicity configurations with the number of helicity-violating helicity configurations contributing to a multigluon process. For n=6 we have $\chi^g_{\rm KS} = \frac{5}{3}$.

We have compared the prediction for the $qg \to qggg$ process obtained through the Kunszt and Stirling (KS) approximation and through the Maxwell (M) approximation with the exact calculation, which in turn agrees with our approximation (MP) within numerical (Monte Carlo) errors:

$$d\sigma^q_{\rm MP} = \chi^q d\sigma^{\rm viol}_q \;, \tag{4.4}$$

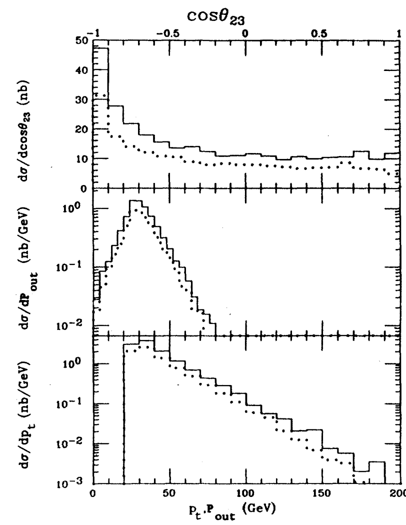

FIG. 4. Comparison of the differential cross sections for the subprocess $qg \to qggg$ of our approximation (dots) versus the approximation of Kunszt and Stirling together with the use of the effective-structure-function approximation (solid line).

$$d\sigma^q_{\rm KS} = \tfrac{4}{9}\chi^g_{\rm KS} d\sigma^{\rm viol}_g \;, \tag{4.5}$$

$$d\sigma^q_{\rm M} = \tfrac{4}{9}\chi^g_{gg} d\sigma^{\rm viol}_g . \tag{4.6}$$

The resulting distributions are shown in Figs. 4 and 5. The total rates are (in nanobarns)

$$\sigma^q_{\rm exact} = 21 \;, \quad \sigma^q_{\rm MP} = 20 \;, \quad \sigma^q_{\rm KS} = 30 \;, \quad \sigma^q_{\rm M} = 24. \tag{4.7}$$

From these results we conclude that the effective-

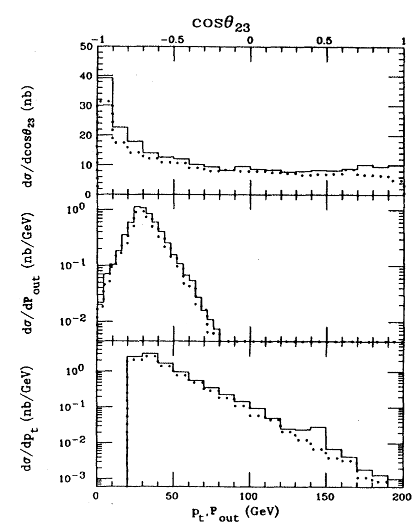

FIG. 5. Comparison of the differential cross sections for the subprocess $qg \to qggg$ of our approximation (dots) versus the approximation of Maxwell together with the use of the effective-structure-function approximation (solid line).

structure-function approximation tends to overestimate the contribution of quark-initiated processes. This suggests that for a large number of partons the purely gluonic matrix elements dominate over the matrix elements with quarks (this is not necessarily true of the rates, because of the effect of the structure functions). However the mismatch between the exact result (or our approximation) and the result of the effective-structure-function approximation is certainly compatible with the intrinsic uncertainty associated with these calculations, due to the absence of higher order corrections, uncertainty in the choice of α_s, of Q^2, and of structure functions.

V. CONCLUSION

In conclusion we have presented an approximation procedure to describe multijet QCD processes. Our prescription completes Maxwell's work on multigluon processes by generalizing it to processes involving quarks as well. The calculation of four-jet production in $p\bar{p}$ collisions at 1.8 TeV shows excellent agreement between the exact results and our approximation. The agreement holds for both total rates and differential distributions. This is a net improvement over calculations based on the effective-structure-function approximation, with which we have compared our results. Qualitative arguments suggest that this agreement should persist for higher order processes.

On completion of this manuscript we received a paper by Maxwell[16] which contains similar results to this paper.

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