Uniqueness of quark and lepton representations in the standard model from the anomalies viewpoint

C. Q. Geng
TRIUMF, 4004 Wesbrook Mall, Vancouver, British Columbia, Canada V6T 2A3

R. E. Marshak
Physics Department, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061
(Received 27 July 1988)

The uniqueness of the Weyl representations of the standard gauge group is reexamined. We find that, prior to spontaneous breaking of the electroweak subgroup, the minimal Weyl representations and their charges are uniquely determined by insisting on all three known chiral gauge anomaly-free conditions in four dimensions: (1) cancellation of triangular anomalies; (2) absence of the global SU(2) anomaly; and (3) cancellation of the mixed-gauge-gravitational anomaly. The uniqueness question for the left-right-symmetric group and the simple (grand-unified-theory) group are discussed from the anomalies viewpoint.

The standard theory of strong and electroweak interactions has been remarkably successful experimentally[1,2] and mysteriously compliant with the anomaly-free conditions arising from the theoretical requirements of renormalizability and self-consistency. Three anomalies have thus far been identified for chiral gauge theories in four dimensions: (1) the triangular (perturbative) chiral gauge anomaly,[3] which must be canceled to avoid the breakdown of gauge invariance and, *a fortiori*, renormalizability of the theory; (2) the global (nonperturbative) SU(2) chiral gauge anomaly,[4] which must be absent in order to define the fermion integral in a gauge-invariant way; (3) the mixed (perturbative) chiral gauge-gravitational anomaly,[5,6] which must be canceled in order to ensure general covariance of the theory. The absence of all three anomalies for the observed quark and lepton representations (for each of the three generations separately) has been demonstrated with satisfaction and relief. In this paper, with an eye to "going beyond the standard model," we turn the equation around; we show that the imposition of all three anomaly-free conditions uniquely fixes the correct minimal set of massless fermion (Weyl) representations [and their $U(1)_Y$ charges] in the standard group *prior* to the spontaneous breaking of the electroweak subgroup.

Freedom from the ordinary triangular (perturbative) chiral gauge anomaly[3] was first noted for the standard model[7,8] in 1972 for each quark and lepton family. It was clear that with only the triangular anomaly-free condition,[9] one could not explain the empirically determined quark and lepton representations and their quantized hypercharges and electric charges. Moreover, the total of 15 Weyl states does not correspond to the simplest set of anomaly-free representations under $SU(3)_C \times SU(2)_L \times U(1)_Y$: e.g., the set of representations $(3,1,Q)$, $(\bar{3},1,-Q)$, $(1,2,q)$, and $(1,2,-q)$ is even simpler. One might expect that the condition that no fermion mass terms can be constructed without breaking the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetries, would help pin down the representations. In fact, this is not the case since the minimal set of triangular anomaly-free Weyl representations would then be $(3,1,Q)$, $(3,1,-Q)$, $(\bar{3},1,q)$, $(\bar{3},1,-q)$ $(q\neq 0)$, and $(1,2,0)$ (Ref. 10), a total of 14 states rather than 15. One must go beyond the constraint imposed by the triangular anomaly-free condition to establish the uniqueness of the observed quark and lepton representations (and their charges) in the standard model.[11]

Fortunately, two other types of chiral gauge-related anomalies in four dimensions have been identified whose absence is required for the self-consistency of the theory; when these anomaly-free conditions are imposed, in addition to the triangular anomaly-free condition, the Weyl representations, together with their $U(1)_Y$ charges, are determined uniquely for the standard group and are in accord with experiment.

The second anomaly that arises in chiral gauge theories is the global (nonperturbative) anomaly, known as the Witten SU(2) anomaly. Witten[4] showed in 1982 that any SU(2) gauge theory with an odd number of (left-handed) Weyl doublets is mathematically inconsistent. There was no problem with an odd number of Dirac doublets, since each Dirac doublet is equivalent to two (an even number of) Weyl doublets. Mathematically, Witten showed that the fermion path integral (taken over the Weyl fermions) for an SU(2) gauge theory with an odd number of Weyl doublets changes sign (the change of sign is due to the properties of the chirality operator γ_5) under a topologically nontrivial SU(2) gauge transformation. This property introduces ambiguities in the evaluation of expectation values of the quantum field operators and leads to a mathematically inconsistent theory. The only remedy is to insist on an even number of SU(2) Weyl doublets.[4]

A third type of chiral gauge-related anomaly has been discovered in four dimensions that supplies the additional relation that enables one to completely fix the $U(1)_Y$ charges of the standard group. This anomaly is similar to the perturbative triangular anomaly but with the three chiral current vertices replaced by a mixture of one chiral current vertex plus two energy-momentum-tensor (gravitational) vertices. This anomaly was first pointed out by Delbourgo and Salam[5] in 1972 and its consequences discussed by Alvarez-Gaumé and Witten[6] in 1983, who concluded that a necessary condition for consistency of the standard group coupled to gravity is that the sum of the $U(1)_Y$ charges of the Weyl fermions, must vanish; i.e., $\mathrm{Tr}\,Y=0$. We shall refer to this anomaly as the mixed anomaly.

We now show that the imposition of all three anomaly-free conditions on the standard chiral gauge group $SU(3)_C\times SU(2)_L\times U(1)_Y$ leads uniquely to the correct result for the minimal number of Weyl representations of $SU(3)_C\times SU(2)_L$ and their $U(1)_Y$ charges. We start by allowing an arbitrary number of (left-handed) Weyl representations under the standard group: i.e.,

$$\begin{array}{lll} SU(3)_C & \times SU(2)_L & \times U(1)_Y \\ 3 & 2 & Q_i\ (i=1,2,\ldots,j) \\ 3 & 1 & Q_i'\ (i=1,2,\ldots,k) \\ \bar{3} & 1 & \bar{Q}_i\ (i=1,2,\ldots,l) \\ \bar{3} & 2 & \bar{Q}_i'\ (i=1,2,\ldots,m) \\ 1 & 2 & q_i\ (i=1,2,\ldots,n) \\ 1 & 1 & \bar{q}_i\ (i=1,2,\ldots,p) \end{array} \tag{1}$$

where the integers j, k, l, m, n, and p and the $U(1)_Y$ charges are all arbitrary.[12] Freedom from the triangular anomalies then leads to the following equations:

$$[SU(3)]^3:\ \sum_{i=1}^{j}2+\sum_{i=1}^{k}1-\sum_{i=1}^{l}1-\sum_{i=1}^{m}2=0\ , \tag{2a}$$

$$[SU(3)]^2U(1):\ 2\sum_{i=1}^{j}Q_i+\sum_{i=1}^{k}Q_i'+\sum_{i=1}^{l}\bar{Q}_i+2\sum_{i=1}^{m}\bar{Q}_i'=0\ , \tag{2b}$$

$$[SU(2)]^2U(1):\ 3\sum_{i=1}^{j}Q_i+3\sum_{i=1}^{m}\bar{Q}_i'+\sum_{i=1}^{n}q_i=0\ , \tag{2c}$$

$$U(1)^3:\ 6\sum_{i=1}^{j}Q_i^3+3\sum_{i=1}^{k}Q_i'^3+3\sum_{i=1}^{l}\bar{Q}_i^3+6\sum_{i=1}^{m}\bar{Q}_i'^3+2\sum_{i=1}^{n}q_i^3+\sum_{i=1}^{p}\bar{q}_i^3=0\ . \tag{2d}$$

The global SU(2) anomaly condition is

$$3j+3m+n=N\ , \tag{2e}$$

where N is an even integer.

If we assume that the masses of the Weyl fermions come only from the spontaneous breaking of the $SU(3)_C\times SU(2)_L\times U(1)_Y$ symmetries, it is straightforward to show, from Eqs. (2a)–(2c) and (2e), that the minimal values for j, k, l, m, and n are, respectively, 1, 0, 2, 0, and 1 *or* 0, 2, 0, 1, and 1. These two sets of values become equivalent when chirality is redefined. We use the first set of values for the remainder of the discussion. We rewrite Eqs. (2) as

$$2Q_1+\bar{Q}_1+\bar{Q}_2=0\ , \tag{3a}$$

$$3Q_1+q_1=0\ , \tag{3b}$$

and

$$6Q_1^3+3\bar{Q}_1^3+3\bar{Q}_2^3+2q_1^3+\sum_{i=1}^{p}\bar{q}_1^3=0\ . \tag{3c}$$

To determine the minimal value of p, we first assume that $p=0$. For this value, one finds an unphysical equation from Eq. (3c),

$$\bar{Q}_1^2+2Q_1\bar{Q}_1+4Q_1^2=0\ , \tag{4}$$

unless one accepts the trivial results that the $U(1)_Y$ charges of all the Weyl fermions are zero. Hence, p cannot be zero. It turns out that the minimal value is $p=1$. We therefore arrive at the well-known result that the minimal Weyl representations under $SU(3)_C$ and $SU(2)_L$ are the standard quark and lepton representations.

However, the triangular and global SU(2) anomaly-free conditions are not sufficient to completely fix the $U(1)_Y$ charges of the Weyl representations for the standard group. These $U(1)_Y$ charges cannot be uniquely determined [since there are four unknown parameters for only three equations—the fifth parameter is the scale of the $U(1)_Y$ charge and can be fixed in a variety of ways]. Fortunately, there is still one more anomaly-free condition that has not be used: namely, the mixed anomaly-free condition

$$\mathrm{Tr}\,Y=0 \tag{5}$$

that requires

$$2q_1+\bar{q}_1=0\ , \tag{6}$$

Eq. (6) receives no contribution from SU(3)-color triplets (quarks) because of the triangular anomaly-free condition (3a), so that combining Eq. (6) with Eqs. (3), one gets

$$Q_1=-\tfrac{1}{3}q_1,\ \ \bar{Q}_1=\tfrac{4}{3}q_1,\ \ \bar{Q}_2=-\tfrac{2}{3}q_1,\ \ \bar{q}_1=-2q_1\ . \tag{7}$$

It is seen from Eq. (7) that all the $U(1)_Y$ charges are determined in terms of a single $U(1)_Y$ charge, say q_1, choosing the normalization $q_1=-1$—consistent with zero electric charge for the neutrino—the resulting Weyl representations of $SU(3)_C$ and $SU(2)_L$ and their $U(1)_Y$ charges are shown in Table I, in agreement with the standard model.

We thus find that minimality and freedom from both the triangular and global SU(2) anomalies yield a unique set of Weyl representations of the standard group that correspond to the observed quarks and leptons of one family. Furthermore, the $U(1)_Y$ charges of these quarks and leptons are quantized and correctly determined by adding the mixed anomaly-free condition.[13] Clearly, if one lifts the minimality requirement, it is possible to obtain as many copies of a quark-lepton family with the proper quantum numbers as one wishes. However, no limit is placed on the number of replications and so the imposition of all three anomaly-free conditions does not shed any immediate light on the "generation problem." It is conceivable that freedom from some as-yet-unidentified anomaly in four dimensions, possibly for a larger (but not simple) group in which the standard group is embedded, would place a constraint on the number of families.

Apart from the generation problem, it should be noted that our demonstration, that all three anomaly-free conditions are needed to determine the correct quantum numbers of one family of Weyl fermions, is based on the acceptance of the standard group as a starting point. But the standard group only allows for left-handed neutrinos and the situation changes if, for example, the standard gauge is enlarged to the left-right-symmetric (LRS) gauge group $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ (Ref. 14). With this group (and invoking minimality, as we did with the standard group), it is easily shown that the (left-handed) Weyl representations are those shown in Table II. In deriving Table II, it is only necessary to impose the first two anomaly-free conditions: the triangular and global SU(2) anomaly-free conditions; the mixed anomaly-free condition is automatically satisfied in a manifestly left-right-symmetric theory such as the LRS model. As in the case of the standard group, the anomaly-free conditions do not, by themselves, help with the generation problem.

The situation changes dramatically if one enlarges the standard group or the LRS group to a simple (grand unification) group. In that case, only the triangular anomaly-free condition suffices to fix the Weyl representations and their $U(1)_Y$ charges. This can be seen as follows. As is well known, the only candidate grand-unified-theory (GUT) groups are[15] $SU(N)$ ($N \geq 5$), $SO(4n+2)$ ($n \geq 2$), and E_6. But is has been shown[16] that the absence of the global SU(2) anomaly is guaranteed by the triangular anomaly-free condition as long as the SU(2) group is embedded in a simple group G with the property $\pi_4(G)=0$ (where π_4 is the four-dimensional homotopy group). Since every one of the candidate GUT groups mentioned above satisfies the conditions of this theorem, the global SU(2) anomaly-free condition is redundant in determining the representations of a GUT group. The mixed anomaly is also redundant for a GUT group, albeit for another reason. The point is that freedom from the mixed anomaly requires the condition $\mathrm{Tr}\,Y=0$ to be satisfied and this follows automatically because the hyperchange Y is a generator of a GUT group and therefore must be traceless. This is why the use of the triangular anomaly-freedom condition alone has sufficed for the three most studied GUT groups: [SU(5), SO(10), and E_6] with the respective triangular anomaly-free representations: ($\bar{\mathbf{5}}+\mathbf{10}$, **16**, and **27**). It is intriguing that the quantization of hyperchange (and therefore electric charge) is such a trivial result for a GUT group but requires the cancellation of the mixed anomaly for the standard group.

We conclude that the resolution of the question of the uniqueness of the massless fermion representations and the $U(1)_Y$ charges for the standard group, when viewed from the standpoint of the three known chiral gauge anomalies in the four dimensions, argues strongly for some form of quark-lepton unification,[17] at least for one family. This is so unless the absence of the global SU(2) chiral gauge anomaly and the cancellation of the mixed chiral gauge-gravitational anomaly are sheer accidents in the standard model, which seems very unlikely.

TABLE I. The quantum numbers of the (left-handed) Weyl representations under $SU(3)_C \times SU(2)_L \times U(1)_Y$ when all three anomaly-free conditions are satisfied.

Particles ($i=1,2,3$)	$SU(3)_C \times SU(2)_L \times U(1)_Y$		
$q_L^i = \begin{bmatrix} u \\ d \end{bmatrix}_L^i$	3	2	$\frac{1}{3}$
$\bar{u}_L^i$	$\bar{3}$	1	$-\frac{4}{3}$
$\bar{d}_L^i$	$\bar{3}$	1	$\frac{2}{3}$
$l_L^i = \begin{bmatrix} \nu \\ e \end{bmatrix}_L^i$	1	2	-1
$\bar{e}_L^i$	1	1	2

TABLE II. The quantum numbers of the (left-handed) Weyl representations under $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ when the triangular and global SU(2) anomaly-free conditions are satisfied.

Particles ($i=1,2,3$)	$SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$			
$q_L^i = \begin{bmatrix} u \\ d \end{bmatrix}_L^i$	3	2	1	$\frac{1}{3}$
$\bar{q}_L^i = \begin{bmatrix} \bar{u} \\ \bar{d} \end{bmatrix}_L^i$	$\bar{3}$	1	2	$-\frac{1}{3}$
$l_L^i = \begin{bmatrix} \nu \\ e \end{bmatrix}_L^i$	1	2	1	-1
$\bar{l}_L^i = \begin{bmatrix} \bar{\nu} \\ \bar{e} \end{bmatrix}_L^i$	1	1	2	1

One of the authors (C.Q.G.) thanks Dr. J. N. Ng for useful discussions and the other (R.E.M.) thanks the University of Rochester for the opportunity to present our work at the Montreal-Rochester-Syracuse-Toronto meeting in Toronto. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (C.Q.G.) and DOE Contract No. DE-AS05-80ER10713 (R.E.M.).

---

[1]S. L. Glashow, Nucl. Phys. **22**, 579 (1961); S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1964); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8),* edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968).

[2]H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. **47B**, 365 (1973); S. Weinberg, Phys. Rev. Lett. **31**, 494 (1973); D. Gross and F. Wilczek, Phys. Rev. D **8**, 3633 (1973).

[3]S. Adler, Phys. Rev. **177**, 2426 (1969); J. Bell and R. Jackiw, Nuovo Cimento **51A**, 47 (1969).

[4]E. Witten, Phys. Lett. **117B**, 324 (1982).

[5]R. Delbourgo and A. Salam, Phys. Lett. **40B**, 381 (1972); see T. Eguchi and P. Freund, Phys. Rev. Lett. **37**, 1251 (1976), who also discuss "gravitational instantons."

[6]L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1983).

[7]C. Bouchiat, J. Illiopoulos, and Ph. Meyer, Phys. Lett. **38B**, 519 (1972).

[8]D. J. Gross and R. Jackiw, Phys. Rev. D **6**, 477 (1972).

[9]S. Adler, Phys. Rev. **177**, 2426 (1969); J. Bell and R. Jackiw, Nuovo Cimento **51A**, 47 (1969); W. Bardeen, Phys. Rev. **184**, 1848 (1969).

[10]One doublet fermion cannot form a mass term without breaking SU(2) since $2\times 2 \rightarrow 1_A + 3_S$ where A and S stand for antisymmetric and symmetric, respectively.

[11]For reviews, see S. L. Glashow, in *Quarks and Leptons,* proceedings of the Cargése Summer Institute, Cargése, 1979, edited by Jean-Luis Basdevant *et al.* (NATO Advanced Study Institute Series, Series B: Physics, Vol. 61) (Plenum, New York, 1980); in *Fundamental Interactions: Cargése 1981,* edited by J. S. Basdevant *et al.* (NATO Advanced Study Institute, Series B, Physics: Vol. 85) (Plenum, New York, 1982). In the second reference, Glashow remarks (p. 674): "At this pre-unification level, the curious values of quark and lepton charges remain mysterious."

[12]Adding arbitrary nonfundamental and singlet representations in (1) does not change our conclusions.

[13]It has been pointed out [H. Georgi (private communication)] that the $U(1)_Y$ charges can be fixed by considering the fermion mass generated by the breaking of $SU(2)_L \times U(1)_Y$ to $U(1)_{EM}$. That is true but we are interested in fixing the $U(1)_Y$ charges prior to breaking and we show how the mixed anomaly-free condition achieves this.

[14]J. C. Pati and A. Salam, Phys. Rev. D **10**, 275 (1974); R. N. Mohapatra and J. C. Pati, *ibid.* **11**, 566 (1975); **11**, 2559 (1975); G. Senjanović and R. N. Mohapatra, *ibid.* **12**, 1502 (1975); R. E. Marshak and R. N. Mohapatra, Phys. Lett. **91B**, 222 (1980); M. A. B. Bég and A. Sirlin, Phys. Rep. **88**, 1 (1982).

[15]Cf. R. N. Mohapatra, *Unification and Supersymmetry* (Springer, New York, 1986).

[16]C. Q. Geng, R. E. Marshak, Z. Y. Zhao, and S. Okubo, Phys. Rev. D **36**, 1953 (1987); S. Okubo, C. Q. Geng, R. E. Marshak, and Z. Y. Zhao, *ibid.* **36**, 3268 (1987). These papers give a group-theoretic proof; a "topological" proof will be found in S. Elitzur and V. P. Nair, Nucl. Phys. **B243**, 205 (1984).

[17]Cf. R. E. Marshak, Pramana (in press).