# Brief Reports

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## Three-derivatives in three dimensions

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A Hamiltonian formulation of linearized topologically massive three-dimensional gravity is presented.

There are very few pathology-free wave equations which are of order higher than two in derivatives. Only two examples are known to the author: the singleton dipole,[1] which has four, and topologically massive three-dimensional gravity,[2] which has three. Presumably, there are further examples, but it is clear that they are unusual, and hence they may deserve some attention.

Here we will give a Hamiltonian analysis of the linearized limit of the latter example, which will enable us to see at a glance how the number of degrees of freedom rises from zero (in the massless case) to one, due to the extra time derivative.

A Hamiltonian treatment of this system was actually given recently by Evens and Kunstatter,[3] starting from a reduced form of the action given in Ref. 2. This is excellent if one's primary interest is to reach the reduced Hamiltonian quickly, but it has the drawback that the locality properties of the theory are obscured, and the overall view of the initial-value problem is lost (the same goes, *mutatis mutandis,* for the light-front approach of Refs. 4 and 3).

Anyway, the local action is[2]

$$S=S_H+S_{\rm CS}\ ,$$

where

$$S_H=\int d^3x\,(\Gamma^{\sigma\nu}{}_{\lambda}\Gamma^{\lambda}{}_{\nu\sigma}-\Gamma^{\lambda\nu}{}_{\nu}\Gamma^{\sigma}{}_{\sigma\lambda})\ ,$$
$$S_{\rm CS}=\frac{1}{2\mu}\int d^3x\,\epsilon^{\alpha\beta\gamma}\Gamma^{\sigma}{}_{\nu\gamma}\Gamma^{\nu}{}_{\sigma\beta,\alpha}\ , \tag{1}$$

where the linearized Christoffel symbol is defined as

$$\Gamma_{\sigma\mu\nu}=\tfrac{1}{2}(h_{\sigma\mu,\nu}+h_{\sigma\nu,\mu}-h_{\mu\nu,\sigma})\ . \tag{2}$$

(Greek indices run from 0 to 2, latin indices from 1 to 2, and we use a spacelike metric. Derivatives are denoted by commas.)

From now on, we will set the mass term $\mu=1$. After a modest amount of manipulation, the action can be brought to the form

$$S=\int\{K_{ij}K_{ij}-K^2+\Gamma_{ijk}\Gamma_{kji}-\Gamma_{ikk}\Gamma_{jji}+\tfrac{1}{2}n(h_{kk,ii}-h_{ik,ik})+\epsilon_{ij}[K_{ik}K_{jk,0}+\tfrac{1}{2}N_k\Gamma_{lkj,il}+\tfrac{1}{2}K_{ik}(h_{kj,ll}-h_{jl,kl}-2n_{,kj})]\}\ , \tag{3}$$

where $n,N_i$ are the linearized version of the lapse and shifts, i.e.,

$$h=h_{00},\quad N_i=h_{i0}\ , \tag{4}$$

and $K_{ij}$ is the linearized version of the extrinsic curvature, which is defined in terms of the dynamical variables by

$$K_{ij}=\tfrac{1}{2}(N_{i,j}+N_{j,i}-h_{ij,0})\ . \tag{5}$$

$K$ is the trace of $K_{ij}$.

We are now ready to perform the canonical analysis. The action involves time derivatives of order higher than one, but it is well, although perhaps not widely, known how to deal with this. We will proceed without too much comment, and refer the reader to Ref. 5 for clarification of the procedure. It is instructive to check it out on something simple, such as the two-dimensional point-particle action

$$S=\int d\tau(\tfrac{1}{2}\dot{x}_\mu\dot{x}^\mu+\epsilon_{\mu\nu}\dot{x}^\mu\ddot{x}^\nu)\ . \tag{6}$$

The following "momenta" are defined:

 

$$p_{mn} \equiv \frac{\partial \mathcal{L}}{\partial h_{mn,00}} = -\tfrac{1}{2}\epsilon_{i(m}K_{n)i} ,$$
$$\pi_{mn} \equiv \frac{\partial \mathcal{L}}{\partial h_{mn,0}} - p_{mn,0}$$
$$= \delta_{mn}K - K_{mn} + \epsilon_{i(m}K_{n)i,0} + \tfrac{1}{4}\epsilon_{j(m}(h_{n)j,ii} - h_{jl,ln)} - 2n_{,n)j}) , \quad (7)$$
$$p_m \equiv \frac{\partial \mathcal{L}}{\partial N_{m,0}} = \tfrac{1}{2}\epsilon_{mi}K_{ik,k} + \tfrac{1}{2}\epsilon_{ji}K_{mi,j} ,$$
$$p \equiv \frac{\partial \mathcal{L}}{\partial n_{,0}} = 0 .$$

Note that $p_{mn}$ is automatically traceless. The canonical variables in the naive phase space obey the naive brackets [here and elsewhere, $\{A,B\}$ is shorthand for $\{A(x),B(Y)\}$]

$$\{q_{ij},p_{kc}\} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})\delta(x-y) ,$$
$$\{h_{ij},\pi_{kl}\} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(x-y) , \quad (8)$$
$$\{N_i,p_j\} = \delta_{ij}\delta(x-y) ,$$
$$\{n,p\} = \delta(x-y) .$$

Here, $q_{ij}$ = the traceless part of $h_{ij,0}$ (and is hence a canonical variable).

The canonical Hamiltonian density is defined as

$$\mathcal{H}_c \equiv h_{ij,00}p_{ij} + h_{ij,0}\pi_{ij} + N_{i,0}p_i - \mathcal{L} \quad (9)$$

and turns out to be

$$\mathcal{H}_c = q_{ij}\pi_{ij} + N_{i,i}\pi_{jj} - \tfrac{1}{2}(\pi_{ii} + \tfrac{1}{4}\epsilon_{ji}h_{jl,li})^2 - \hat{K}_{ij}\hat{K}_{ij} + \Gamma_{ikk}\Gamma_{jji} - \Gamma_{ijk}\Gamma_{kji}$$
$$+ \tfrac{1}{2}n(h_{ij,ij} - h_{ii,jj}) + \tfrac{1}{2}\epsilon_{ij}[\hat{K}_{ik}(2n_{,kj} + h_{jl,lk} - h_{jk,ll}) - N_k\Gamma_{ikj,il}] , \quad (10)$$

where $\hat{K}_{ij}$ = the traceless part of $K_{ij}$.

By inspection, we find the primary constraints

$$\phi_{mn} = p_{mn} + \tfrac{1}{2}\epsilon_{i(m}K_{n)i} \approx 0 ,$$
$$\phi_m = p_m + \tfrac{1}{2}\epsilon_{im}K_{ik,k} + \tfrac{1}{2}\epsilon_{ij}K_{mi,j} \approx 0 , \quad (11)$$
$$\phi = p \approx 0 .$$

They obey a naive Poisson-brackets algebra whose only nonvanishing brackets are

$$\{\phi_{ij},\phi_{kc}\} = \tfrac{1}{8}\epsilon_{il}\delta_{jk}\delta(x-y) + (i\leftrightarrow j) + (k\leftrightarrow l) ,$$
$$\{\phi_i,\phi_j\} = -\epsilon_{ij}\partial_k\partial_k\delta(x-y) , \quad (12)$$
$$\{\phi_i,\phi_{kl}\} = -\tfrac{1}{4}[\epsilon_{ki}\partial_l + \epsilon_{kp}\delta_{il}\partial_p + (k\leftrightarrow l)]\delta(x-y) .$$

(Here and elsewhere, the derivatives on the right-hand side are derivatives with respect to $x$.)

So there are second-class constraints in this model. There is some freedom in the choice of which ones we regard as second class, however. $\phi$ is a possible choice, but it leads to nonlocal Dirac brackets, and also it precludes the later use of the gauge $N_i = 0$ (the linearized version of Gaussian coordinates). For this reason we prefer to regard $\phi_{ij}$—which is traceless, so this is again two constraints—as second class. Hence $p_{ij}$ is removed from phase space, which is natural since the action is linear in the second-order time derivatives. Computing the Dirac brackets of the remaining primary constraints, one finds that they are zero, so that only two second-class constraints are present:

$$\{\phi_i,\phi_j\}^* \equiv \{\phi_i,\phi_j\} - \{\phi_i,\phi_{kl}\}\{\phi_{kl},\phi_{mn}\}^{-1} \times \{\phi_{mn},\phi_j\} = 0 . \quad (13)$$

The Dirac brackets among the remaining phase-space variables turn out to be

$$\{q_{ij},q_{kl}\}^* = -\tfrac{1}{2}\epsilon_{il}\delta_{jk}\delta(x-y) + (k\leftrightarrow l) + (i\leftrightarrow j) ,$$
$$\{q_{ij},p_k\}^* = \tfrac{1}{2}(\delta_{ik}\delta_j + \delta_{jk}\partial_i - \delta_{ij}\partial_k)\delta(x-y) ,$$
$$\{h_{ij},\pi_{kl}\}^* = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{ki})\delta(x-y) , \quad (14)$$
$$\{p_i,p_j\}^* = -\tfrac{1}{8}(\epsilon_{ki}\partial_k\partial_j - \epsilon_{kj}\partial_k\partial_i + \epsilon_{ji}\partial_k\partial_k)\delta(x-y) ,$$
$$\{n,p\}^* = \delta(x-y) ,$$
$$\{N_i,p_j\}^* = \delta_{ij}\delta(x-y) .$$

Finally, some secondary constraints are needed for consistency of the time evolution:

$$\psi = h_{ii,kk} - h_{ij,ij} - 2\epsilon_{ij}\hat{K}_{ik,kj} \approx 0 ,$$
$$\psi_i = 2\pi_{ij,j} + \tfrac{1}{4}\epsilon_{kj}(h_{jl,lki} - h_{ji,kll}) \approx 0 . \quad (15)$$

These are the linearized versions of the Hamiltonian and vector constraints in the nonlinear theory. The most economical way to compute the secondary constraints is to take the naive Poisson brackets of the canonical Hamiltonian with the first-class combination

$$\phi_i \equiv \phi_i - 2\phi_{ij,j} \quad (16)$$

and to implement the second-class constraints afterwards. At this point, we have all the constraints, and one checks that their Dirac-brackets algebra is Abelian.

To sum up, the phase space of the model under study is spanned by the eight variables $h_{ij}$, $\pi_{ij}$, and $q_{ij}$ (plus the lapse and shifts), subject to the three first-class constraints in Eq. (15). The Dirac brackets are given in Eq. (14) and the canonical Hamiltonian in Eq. (10). The number of first-class constraints is the same as in the massless model without the extra term $S_{\text{CS}}$ in the action—naturally, since the gauge invariance is the same—but the presence of double time derivatives in the action has led to two new dynamical variables $q_{ij}$ = the traceless part of $h_{ij,0}$. The latter also have to be specified before one has a well-defined initial-value problem. One can check in various ways[2–4] that these extra degrees of freedom in fact describe a massive excitation.

We have not studied the interacting model, although it has some very interesting properties for a toy model. In particular, one expects the mass $\mu$ to be quantized in a Riemannian space, but not in a Lorentzian space-time.[2] Figuring out how this hangs together in the quantum theory might teach one something about the signature of our world.

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