Longitudinal modes in classical Yang-Mills plasma

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A classical nonperturbative study of the longitudinal modes in a Yang-Mills [SU(2)] plasma is carried out. A new periodic non-Abelian mode is found which alternates with the usual Abelian plasma mode. When the non-Abelian terms in the equations become large, chaotic behavior sets in.

I. INTRODUCTION

Currently, there is much interest in the theoretical and experimental investigations of the quark-gluon plasma (for recent overviews of the field see Refs. 1 and 2).

In this paper, we examine the longitudinal (plasma) oscillations of a classical Yang-Mills (YM) plasma. Our prime motivation is to find qualitatively new features that arise in the plasma due to its non-Abelian character. It ought to be mentioned that such time-dependent effects cannot be investigated with the present-day simulations of QCD on the lattice, and very likely, perturbative methods may not be adequate for fully exhibiting the non-Abelian effects. In addition, there are serious difficulties with finite-temperature perturbative QCD as emphasized by Nadkarni.[3] We believe therefore that it is useful to carry out nonperturbative studies of the YM plasma with the expectation that essential non-Abelian features that show up in such studies would survive in a full quantal treatment.

Some preliminary studies of the classical non-Abelian plasma have been carried out by Kajantie and Montonen.[4] In their work, the classical particles in the plasma are described by three variables: density n, velocity $\mathbf{V}$, and color charge I_a $(a=1,2,3)$. Consequently, the plasma particles generate a four-current density which acts as a source for the generation of Yang-Mills fields A_a^{μ}. The authors[4] write down a set of hydrodynamical equations for the plasma variables n, $\mathbf{V}$, and I_a which are consistent with the classical equations of motion for the colored particles in external Yang-Mills fields obtained by Wong.[5] A major difference from an Abelian plasma is the feature that the color charges I_a are also dynamical variables, since the particles exchange color with the non-Abelian fields. A limitation of almost all classical work (for discussion see Heinz[6]) on YM plasma is that the (thermodynamic) equilibrium background for the non-Abelian waves has been ignored. Inclusion of these "thermal gluons"[6] is difficult and remains an unresolved problem.

Kajantie and Montonen[4] examine some specific solutions for the Yang-Mills fields in the plasma. In particular, by assuming a plane-wave-type solution for the fields obtained by Coleman,[7] they determine the four-current density that would produce such a solution. Furthermore, they use Wong's[5] equations of motion for the particles to relate the current density to the non-Abelian fields. Such a self-consistent determination of the fields results in nonpropagating oscillatory solutions having frequency ω_p which is the plasma frequency. It should be noted that their solution is harmonic which can also result from an Abelian (linear) theory. In addition the solution is not purely longitudinal as the waves produce both color-electric and -magnetic fields.

Our interest is in looking for *essentially* non-Abelian (nonlinear) oscillatory modes that are purely longitudinal (no color-magnetic fields). For simplicity, we study the Yang-Mills [SU(2)] plasma, governed by the hydrodynamic equations of Kajantie and Montonen.[4] We obtain a closed set of equations for the longitudinal color fields which are derived in Sec. II. It is not possible to find analytic solutions to these equations and hence we have solved them numerically. Results of the numerical calculations are presented in Sec. III. Section IV contains a summary and some conclusions. We also discuss other questions of physical interest that ought to be examined within the classical theory.

II. EQUATIONS FOR LONGITUDINAL OSCILLATIONS

As stated in the Introduction our aim is to describe the collective longitudinal oscillations of a classical YM [SU(2)] plasma, using the simplest "hydrodynamic" description. Equations governing the behavior of such plasmas in the cold "collissionless" limit (dominated by long-range encounters) were first written by Kajantie and Montonen.[4] These authors derived a set of gauge-covariant equations based on the classical equations of motion for the gauge fields and the classical equations of motion for a colored particle in external chromofields (Wong[5]). We now briefly review these equations.

The equations of motion for the Yang-Mills gauge fields are given by

$$\partial_\mu F_a^{\mu\nu}+g\epsilon_{abc}A_{\mu b}F_c^{\mu\nu}=j_a^{\nu} . \qquad (1)$$

Here the color indices $a,b,c=1,2,3$ and the Lorentz indices $\mu,\nu=0,1,2,3$ with the metric $(1,-1,-1,-1)$. g is the dimensionless coupling constant and ϵ_{abc} is the completely antisymmetric Levi-Civita tensor. The field tensor $F_a^{\mu\nu}$ is given by

$$F_a^{\mu\nu}=\partial^\mu A_a^\nu-\partial^\nu A_a^\mu+g\epsilon_{abc}A_b^\mu A_c^\nu \ . \tag{2}$$

The source current j_a^ν is generated by the particles and is covariantly conserved

$$\partial_\nu j_a^\nu+g\epsilon_{abc}A_{\nu b}j_c^\nu=0 \ . \tag{3}$$

Following Refs. 4 and 5, the components of this four-current are written as

$$j_a^0=g\sum_A n_A I_{Aa} \ , \tag{4a}$$

$$\mathbf{j}_a=g\sum_a n_A \mathbf{V}_A I_{Aa} \ , \tag{4b}$$

where A denotes a group or a species of colored particles, n_A is their number density, and I_{Aa} is the associated ath component of the color vector. Note that the quantities I_{Aa} play a role similar to *charge* in electrodynamics. However, unlike electrodynamics, I_{Aa} are components of a vector quantity, the color vector, and are *dynamical* quantities, i.e., evolve with time. This is so because color, unlike charge, is exchanged between waves and particles. The dynamic nature of I_{Aa} forms an important *conceptual* difference between classical electrodynamic plasmas and the non-Abelian Yang-Mills plasma being studied here.

A set of nonrelativistic hydrodynamical equations consistent with Eq. (3) were obtained in Ref. 4. These are

$$\frac{\partial n_A}{\partial t}+\nabla\cdot(n_A\mathbf{V}_A)=0 \ , \tag{5a}$$

$$\frac{\partial}{\partial t}\mathbf{V}_A+(\mathbf{V}_A\cdot\nabla)\mathbf{V}_A=\frac{g}{m_A}I_{Aa}(\mathbf{E}_a+\mathbf{V}_A\times\mathbf{B}_a) \ , \tag{5b}$$

$$\left[\frac{\partial}{\partial t}+V_A\cdot\nabla\right]I_{Aa}=-g\epsilon_{abc}(A_b^0-\mathbf{V}_A\cdot\mathbf{A}_b)I_{Ac} \ . \tag{5c}$$

In Eqs. (5a)–(5c), n_A, $\mathbf{V}_A$, and I_A denote continuum functions of (x,t) and m_A denotes the mass of quark species A. Equation (5a) is the mass continuity equation Eq. (5b) is the force equation, and (5c) describes the dynamics of color charge vector I_{Aa}. Equations (1)–(5) are a closed set of equations which may be utilized for a description of self-consistent collective nonlinear oscillations of a non-Abelian plasma. These equations are deficient in that no finite-temperature contributions, either from thermal gluons or from finite quark pressure have been retained here. The former difficulty is well known and arises because to date one has not been able to find a gauge-covariant way of writing a kinetic or hydrodynamic equation for thermal gluons (Heinz[6]). The neglect of quark pressure is a reasonable approximation for long-wavelength fluctuations where the thermal dispersion effects are negligible.[6]

In order to determine the effect of gauge transformations on Eqs. (5a)–(5c) we need to know the gauge transformation properties of the variables n_A, $\mathbf{V}_A$, and I_{Aa}. Since n_A and $\mathbf{V}_A$ do not depend on the internal color degrees of freedom they are invariant under the gauge transformations. The color charge vector I_{Aa}, however, transforms covariantly under the gauge transformations—i.e., $I_A\rightarrow UI_AU^{-1}$. This can be deduced from Eqs. (1) and (4), knowing that the left-hand side of Eq. (1) transforms covariantly, and therefore the four-current density must have the same property due to the requirement of covariance for the equation of motion [Eq. (1)]. In view of these gauge properties it follows that Eqs. (5a) and (5b) are gauge invariant whereas Eq. (5c) is gauge covariant.

Equations (1)–(5) are a set of nonlinear partial differential equations which are difficult to solve in their full generality. We shall look for special solutions of these equations which are nonlinear plane *stationary waves.* Thus, we assume that the potential A^μ depend only on the variables x_0 and x_3 (say) and that too only through the single variable $\xi=x_3+\beta x_0$. Mathematically, this assumption converts partial differential equations to ordinary differential equations because $\partial x_3=d/d\xi$ and $\partial x_0=\beta d/d\xi$. Physically, the crucial *assumption* here is that the nonlinear solutions are stationary in a frame moving with the phase speed β. Such nonlinear stationary plane-wave solutions are widely discussed in the electrodynamic plasma literature[8] and have also been considered for non-Abelian fields.[7,9] The phase velocity β plays the role of a parameter in the final equations. Sometimes one may find restrictions on the values that β can assume for solutions to be real; often, as in the present case, β turns out to be unrestricted.

In our subsequent discussion, we shall assume that waves we study are purely *longitudinal,* i.e., that we may ignore the coupling to color-magnetic fields. It can be directly shown from the field equations that if A^1, A^2 and their derivatives are zero at $\xi=0$, then their derivatives are zero for all values of ξ. Physically, this means that the symmetry properties of the field and plasma equations ensure that a pure longitudinal disturbance in the plasma may propagate independently, completely uncoupled to the color-magnetic perturbations. It is these nonlinear longitudinal disturbances that we shall study in the paper.

For a study of the dynamic perturbations, we now make the *gauge choice* $A^0=0$. The only nonvanishing field strength may now be written $E_a=F_a^{30}=-\partial_0 A_0^3=-\beta a_a'$ where $A_a^3\equiv a_a$ and the prime denotes differentiation with respect to ξ. One can also write Ampere's equation in this case from (1). We get

$$a_a''=\frac{1}{\beta^2}j_a^3 \ . \tag{6}$$

We next derive an expression for the source current j_a^3 in terms of the fields using the hydrodynamic description of Eqs. (4) and (5). We consider the hydrodynamic equations for two species—i.e., $A=1,2$. It should be stressed that we need at least two species to satisfactorily describe an equilibrium for the plasma. This is necessary in order that we may have overall color neutrality in the plasma in equilibrium. We thus assume that, in equilibrium, $n_{10}=n_{20}=n_0$, the velocities $v_{10}=v_{20}=0$ and the color neutrality condition $I_{1a0}+I_{2a0}=0$. Equation (5) on integration yield the following relations:

$$n_A=\beta n_0/(\beta+v_A)\ , \tag{7a}$$

$$v_A=-\beta+\beta\left[1-\frac{2g}{m_A\beta}I_{Aa}a_a\right]^{1/2}\ , \tag{7b}$$

$$I'_{Aa}=\left[\frac{g}{\beta n_0}\right]\epsilon_{abc}a_bI_{Ac}n_Av_A\ . \tag{7c}$$

Equation (7c) may be combined with Eqs. (4b) and (6):

$$\sum_A (I_{Aa})'=\frac{1}{\beta n_0}\epsilon_{abc}a_bj_c=\frac{\beta}{n_0}\epsilon_{abc}a_ba''_c\ .$$

Integration leads to the conservation law

$$I_{1a}+I_{2a}=(\beta/n_0)\epsilon_{abc}a_ba'_c\ . \tag{7d}$$

We now assume for simplicity of calculation that one species is much heavier than the other, so that (say) $m_2\gg m_1$. In this case we may set $v_2=0$, which in turn implies that $I_{2a}=\text{const}=+I_{2a0}=-I_{1a0}$. We may now write

$$j_a^3=gn_1v_1I_{1a}=g\beta n_0\left[\frac{\beta}{n_0}\epsilon_{abc}a_ba'_c+I_{1a0}\right]\times\left[1-\frac{1}{\left[1-\frac{2g}{m_1\beta}I_{1a0}a_a\right]^{1/2}}\right]\ ,$$

which may be substituted in the Ampere's law equation (6) to give the final nonlinear field equation

$$a''_a=\left[g\epsilon_{abc}a_ba'_c+\frac{gn_0}{\beta}I_{1a0}\right]\times\left[1-\left[1-\frac{2g}{m_1\beta}I_{1a0}a_a\right]^{-1/2}\right]\ . \tag{8}$$

The neglect of the motion of heavy species m_2 is very similar to description of high-frequency plasma oscillations in a classical electron-ion plasma where the ions (heavy quarks) only provide a (color) neutralizing background of charges.

We observe that Eq. (8) contains two types of nonlinear terms: those arising from the non-Abelian nature of the theory (first term in the first set of large parentheses) and those arising from the hydrodynamic framework used to describe the plasma (second term in the square brackets). It is well known that the latter type of terms are also present for an Abelian (electromagnetic) plasma. Since our main interest is in the study of non-Abelian effects, we expand the square root in Eq. (8) and retain only the terms linear in the field amplitudes a_a. We then obtain the equation

$$a''_a=-\frac{g^2n_0}{m\beta^2}I_{1a0}(I_{1b0}a_b)-\frac{g^2}{m\beta}(\epsilon_{abc}a_ba'_c)(I_{1d0}a_d)\ . \tag{9}$$

The assumption of weak plasma nonlinearity corresponds to the condition $(gI_{1a0}/m_1\beta)a_a\ll 1$ which is equivalent to $|v_A|\beta\ll 1$, i.e., the directed particle velocity in the wave fields is much less than the wave phase velocity. At the same time, the retention of non-Abelian terms means that we must have $ga/|k|>1$ where $|k|$ measures the magnitude of derivative term $|a'/a|$. A more detailed discussion of the range of validity of this assumption is presented in the discussion section at the end. Equation (9), which treats the plasma in an approximate and linear manner, may be given a simple derivation (see Appendix A). The oscillations in this case are interpreted as nonlinear temporal oscillations and have no spatial dependence ($d/dx_3\to 0$). For finite wavelength perturbations ($d/dx_3\neq 0$), the correct interpretation is still in terms of the variable $\xi=x_3+\beta x_0$ or its temporal analogue $\tau\equiv\xi/\beta=x_0+(x_3/\beta)$.

To proceed further we take, for simplicity, $I_{110}=I_{120}=I_{130}\equiv I_0$ and also define $\omega_p^2=g^2n_0(I_0)^2/m_1$. We further rewrite Eq. (9) in a neat symmetrical form by introducing scaled normal-mode variables, which remove the coupling between a_1,a_2,a_3 arising through the linearized first term. We thus introduce the quantities $a_a^*=a_0a_a$ (where a_0 is a normalizing scale factor for the vector potential), $x_1=a_1^*+a_2^*+a_3^*$, $x_2=\sqrt{3/2}(a_1^*-a_3^*)$, $x_3=\sqrt{1/2}(a_1^*-2a_2^*+a_3^*)$, $\omega_p(\xi/\beta)=t$. The resulting equations are

$$\ddot{x}_1=-3x_1+(\epsilon/\sqrt{3})(x_2\dot{x}_3-x_3\dot{x}_2)x_1\ , \tag{10a}$$

$$\ddot{x}_2=(\epsilon/\sqrt{3})(x_3\dot{x}_1-x_1\dot{x}_3)x_1\ , \tag{10b}$$

$$\ddot{x}_3=(\epsilon/\sqrt{3})(x_1\dot{x}_2-x_2\dot{x}_1)x_1\ . \tag{10c}$$

In Eqs. (10a)–(10c), the overdots denote a differentiation with respect to the dimensionless parameter t and $\epsilon=g^2I_0a_0^2/m_1\omega_p$ is a parameter characterizing the strength of the non-Abelian terms.

Equations (10a)–(10c) may be interpreted as the equation of motion of an "effective particle" with three degrees of freedom in a nonlinear potential field. It can be shown by direct calculation that these equations have the following conservation laws:

$$\tfrac{1}{2}(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2)+\tfrac{3}{2}x_1^2=E\ , \tag{11}$$

$$\frac{\epsilon}{2}(M_1^2+M_2^2+M_3^2)-3M_1=M\ , \tag{12}$$

where

$$M_1=x_2\dot{x}_3-x_3\dot{x}_2\ ,\quad M_2=x_3\dot{x}_1-x_1\dot{x}_3\ , \tag{13}$$

and

$$M_3=x_1\dot{x}_2-x_2\dot{x}_1\ .$$

Equation (11) describes an energy conservation law. Note that the first three terms on left-hand side (corresponding to the "kinetic energy" of the effective particle) actually describe the energy in the longitudinal color-electric fields. The fourth term, i.e., the "potential energy" term describes the mean kinetic energy of plasma particles in the color-electric fields. Equation (11) thus has a clear physical interpretation in terms of exchange of energy between the color-electric fields and the kinetic energy of plasma particles. Equation (12) is related to the conservation of an angular-momentum-like vector in color space. Note from Eq. (7c) that $M_{1,2,3}$ are related to color charge fluctuations being carried by the Yang-Mills fields. In the absence of matter we have the conservation law $M_1^2+M_2^2+M_3^2=\text{const}$. The last term in Eq. (12) is a consequence of color charge being exchanged between the chromofield and the material particles. It should be emphasized that the conserved quantities E and M are gauge invariant. This is explicitly demonstrated in Appendix B by making a gauge choice $A_a^0=\delta A_a^3$ and demonstrating that E and M are independent of δ. Finally, it is worth pointing out that for the exact equations (Eq. 8) we have a different material particle kinetic energy term in the energy-conservation law, Eq. (11), but have precisely the same form for the angular-momentum-conservation law, Eq. (12).

III. NUMERICAL CALCULATIONS AND RESULTS

We have obtained numerical solutions of Eqs. (10a)–(10c) for different initial conditions and different values of the parameter ϵ. The procedure used for integrating the equations was the Runge-Kutta method with variable step size. The results are presented in Figs. 1–4.

We have chosen to present the results in terms of the scaled normal-mode variables x_1,x_2,x_3. As emphasized in Sec. II, the choice of these variables is such that only genuinely non-Abelian terms couple them to each other. These variables are therefore just right for displaying the characteristic features of non-Abelian physics. Physically, they are related to algebraic combinations of the color vector potentials A_a^3. These in turn are simply related to color-electric fields ($E_a=-\beta a_a'$) and color density fluctuations [through Eqs. (7a)–(7d)]. The choice of the initial conditions for the numerical computations is such that it displays most of the general classes of solutions observed by us. A given set of initial conditions is only to be treated as representative of a typical class of solutions.

It is obvious that when $\epsilon=0$, we have the usual plasma oscillations for x_1, with the frequency $\sqrt{3}\omega_p$ whereas x_2,x_3 increase linearly with t. For $\epsilon\neq0$ and not too large, the x_1 solutions exhibit two periodic modes. These are shown in Figs. 1 and 3. In both the cases, we first observe for small values of ϵ the plasma mode, which is followed by a new non-Abelian mode. The non-Abelian mode is different from the Abelian one both with respect to amplitude and frequency. This is seen more clearly in Fig. 2 where the oscillations between $t_i=250$ and $t_f=500$ of Fig. 1 are plotted on an expanded scale. It is easy to estimate from this figure that the frequency of the non-Abelian mode is nearly four times the plasma frequency ($\sqrt{3}\omega_p$). Although these two types of motion occur periodically it is not completely clear when the crossover from the plasma mode to the non-Abelian mode occurs. It seems to depend upon the phases of x_2 and x_3. The x_2 and x_3 motions are for $\epsilon\neq0$ bounded but show no special

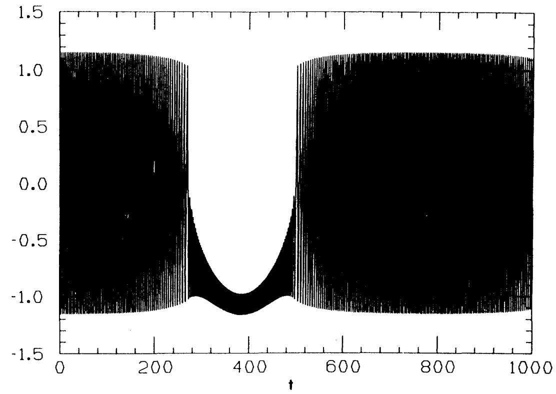

FIG. 1. Oscillations of the field variable x_1. $\epsilon/\sqrt{3}=0.05$ and the initial conditions are $x_1=x_2=x_3=0$ and $x_1'=2$, $x_2'=1$, and $x_3'=3$.

features. For a large value of ϵ (Fig. 4), we find intermittency or chaos in x_1 motion. The x_2 and x_3 motions are also quite irregular.

In order to better understand the non-Abelian mode seen in Figs. 1–3, we write Eq. (10) in a form which is similar to Euler's equations for rigid body rotation. With the color vector M defined in Eq. (13) and $\mathbf{r}=(x_1,x_2,x_3)$ we can write Eq. (10) as

$$\frac{d\mathbf{M}}{dt}-\epsilon x_1(\mathbf{r}\times\mathbf{M})=\boldsymbol{\tau}\ , \tag{14}$$

where the torque $\boldsymbol{\tau}=3\omega_p x_1(0,-x_3,x_2)$. Euler's equation for a rigid body rotation is

$$\frac{d\mathbf{L}}{dt}+\boldsymbol{\omega}\times\mathbf{L}=\mathbf{N}\ ,$$

where $\boldsymbol{\omega}$ is the angular velocity. Very qualitatively, we therefore think that the non-Abelian mode corresponds to some kind of precession in color space.

It is worth recalling here that the numerical results shown in this section do not include the nonlinear plasma terms [see discussion after Eq. (8) in Sec. II]. We have found after extensive calculations, however, that the in-

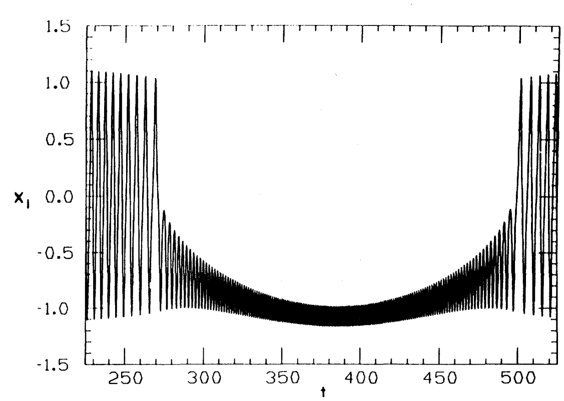

FIG. 2. Oscillations of the field variable x_1. The values of the parameter ϵ and the initial conditions are the same as those in Fig. 1. The scale for the variable t is expanded.

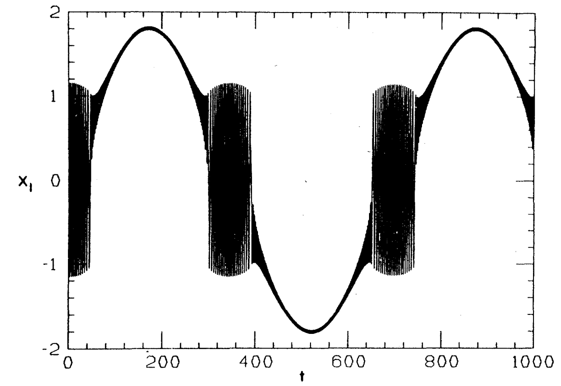

FIG. 3. Oscillations of the field variable x_1. $\epsilon/\sqrt{3}=0.05$ and the initial conditions are $x_1=x_2=x_3=0$ and $x_1'=2$, $x_2'=0.1$, and $x_3'=0.3$.

clusion of these terms does not affect in any significant way the basic features seen in the four figures.

IV. SUMMARY AND CONCLUSIONS

We have studied the effects of non-Abelian terms on the longitudinal oscillations of a classical Yang-Mills plasma. In contrast with earlier work, we have retained the non-Abelian terms entering through the dynamical equations for the color charge vector. We find that for certain ranges of the parameter ϵ, which measures the strength of the non-Abelian term, there exists a new periodic mode which alternates with the usual (Abelian) plasma mode. It is very difficult to see how this novel behavior could have been obtained from perturbative calculations. We have also shown that for large values of ϵ, these classical modes lead to chaotic behavior.

In carrying out the calculations, we have assumed the hydrodynamic plasma nonlinearity to be small and have only retained the non-Abelian terms entering through dynamics of the color charge vector. In terms of natural units, this assumption is justified if $\beta g a_a/\omega_p > 1 > g I_0 a_a / m\beta$. In order to get a feeling for these inequalities in terms of physical quantities we rewrite the longitudinal electric field $E_a \sim \partial_0 a_a \sim \omega_p a_a$ in terms of the wave energy density $\epsilon_w \sim E_a^2$ and use a normalizing energy density $\epsilon_c \sim n_c mc^2 \sim 2-5$ GeV/fm^3 (the typical energy density in the plasma needed for a deconfining transition). We may then write the above inequalities as

$$\frac{10^{-2}}{\bar{g}(\beta/c)} < \left[\frac{\epsilon_w}{\epsilon_c}\right]^{1/2} < \frac{\beta}{c}\left[\frac{n}{n_c}\right]^{1/2},$$

where we have assumed that the phase velocity $\beta/c \sim 1$ and the typical wavelength $k^{-1} \sim c/\omega_p \sim$ few femtometers. From the above inequality we may see that our treatment is valid for ϵ_w/ϵ_c as low as 10^{-4}, i.e., wave energy density of order hundreth of a percent of the typical plasma energy density.

Our results show that the non-Abelian plasma is stable at the classical level. It would be worth investigating whether this stability persists in the presence of quantum fluctuations. This is particularly of interest because perturbative QCD (bare one loop) results[10] indicate an instability for the plasma mode at high temperature.

It ought to be clear from our study that even in the relatively simple case examined by us, there is a great deal of richness in the observed phenomena. In view of this, it seems worthwhile to systematically investigate, for the classical Yang-Mills plasma, other interesting questions, such as color-electric and -magnetic screening, existence of different types of collective modes of solitonic solutions, etc.

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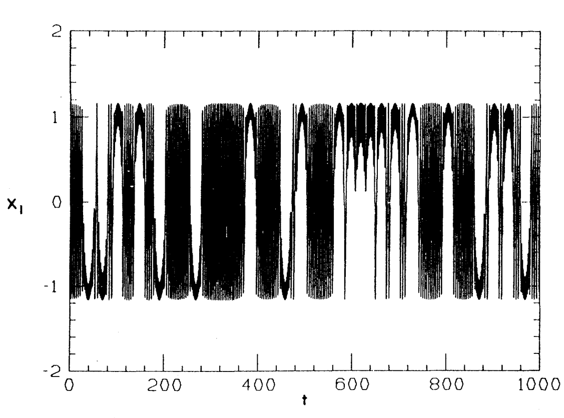

FIG. 4. Oscillations of the field variable x_1. $\epsilon/\sqrt{3}=0.5$ and the initial conditions are $x_1=x_2=x_3=0$ and $x_1'=2$, $x_2'=0.1$, and $x_3'=0.3$.

APPENDIX A: SIMPLE DERIVATION OF EQ. (9)

We start with the approximate linearized form of the plasma equations (5b) and (5c), viz.,

$$\frac{\partial}{\partial t}\mathbf{V}_A = \frac{g}{m_A} I_{Aa}\mathbf{E}_a , \tag{A1}$$

$$\frac{\partial}{\partial t} I_{Aa} = g\epsilon_{abc}(\mathbf{V}_A \cdot \mathbf{A}_b) I_{Ac} . \tag{A2}$$

We have used the gauge condition $A^0 \equiv 0$. The neglect of the *plasma* nonlinearity through $(\mathbf{V}_A \cdot \nabla)$ terms is fully justified when $\partial/\partial x_3 \to 0$. Equation (A2) shows that $A_a \partial I_{Aa}/\partial t = 0$. Noting that $\mathbf{E}_a = -\partial_0 A_a^3$, we may now integrate Eq. (A1) to get

$$v_A = -\frac{g}{m_A} I_{Aa} a_a . \tag{A3}$$

Again, noting from Eq. (4b), $j_a^3 \simeq g n_0 v_A I_{Aa}$ and the Ampere's law $\ddot{a}_a = j_a^3$, we may use Eq. (A2) to derive the equation

$$\sum_A I_{Aa} = \frac{g}{n_0} a_b \dot{a}_c ,$$

where an overdot denotes differentiation with respect to time variable $x_0 \equiv t$. Assuming as in the text, that species 2 is heavy such that $I_{A2} = \text{const} = I_{A20} = -I_{A10}$, we finally get the field equation

$$\ddot{a}_a = -\frac{g^2 n_0}{m_1} I_{0a}(I_{0b} a_b) - \frac{g^2}{m_1} \epsilon_{abc} a_b \dot{a}_c (I_{0d} a_d) \ . \qquad \text{(A4)}$$

Equation (A4) describes nonlinear temporal oscillations. If $a_a(t=0)$ are independent of space (variable x_3), then Eq. (A4) have similar properties to the ones discussed in the text. However, if $a_a(t=0)$ are x_3 dependent, the general initial value problem will give complicated solutions with phase mixing (similar to the ones discussed in Ref. 10). In the special case where the initial conditions are prepared specially such that $\beta(\partial/\partial x_3) \equiv (\partial/\partial x_0)$ we again recover the stationary waves discussed in the text.

APPENDIX B: GAUGE INVARIANCE OF CONSERVED QUANTITIES E AND M

The quantities E and M are defined by the conservation laws, Eqs. (11) and (12), and have been physically interpreted in the text. We expect them to be gauge invariant. To explicitly demonstrate this gauge invariance let us make a general gauge choice $A_a^0 = \delta A_a^3$. The parameter δ is a constant which was chosen to be zero in the text. We now demonstrate by explicit calculation that E and M are independent of δ.

Following the same procedure as in the text for the derivation of the field equations and defining variables x_1, x_2, x_3 as before we get the new equations

$$\ddot{x}_1 + 3x_1 = \left[\frac{\epsilon}{\sqrt{3}} \left[1 + \frac{\delta}{\beta} \right]^2 x_1 + g\frac{\delta}{\beta} \right] (x_2 \dot{x}_3 - x_3 \dot{x}_2) \ , \qquad \text{(B1)}$$

$$\ddot{x}_2 = \left[\frac{\epsilon}{\sqrt{3}} \left[1 + \frac{\delta}{\beta} \right]^2 x_1 + g\frac{\delta}{\beta} \right] (x_3 \dot{x}_1 - x_1 \dot{x}_3) \ , \qquad \text{(B2)}$$

$$\ddot{x}_3 = \left[\frac{\epsilon}{\sqrt{3}} \left[1 + \frac{\delta}{\beta} \right]^2 x_1 + g\frac{\delta}{\beta} \right] (x_1 \dot{x}_2 - x_2 \dot{x}_1) \ . \qquad \text{(B3)}$$

For $\delta = 0$, we recover Eqs. (10a)–(10c) derived in the text.

From Eqs. (B1)–(B3) it may be directly verified that the constants of motion are

$$\frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} + \frac{\dot{x}_3^2}{2} + 3\frac{x_1^2}{2} = E \ ,$$

$$\frac{\epsilon}{2}(M_1^2 + M_2^2 + M_3^2) - 3M_1 = M \ ,$$

where the $M_{1,2,3}$ are defined in the text. Thus, we explicitly note that the physically conserved quantities E and M are independent of the choice of the gauge.

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