Lower-bound renormalization group for gauge-Higgs systems

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The highly accurate Kadanoff lower-bound renormalization group for spin systems is generalized to models with local gauge symmetry. As an example it is applied to the Z_2 gauge-Higgs theory. The two critical exponents of the model are, respectively, predicted exactly and with 0.1% accuracy by a simple analytic calculation. The application of the generalized method to more complicated gauge groups in arbitrary spacetime dimension is described.

The Kadanoff lower-bound renormalization group (LBRG) predicts[1] critical exponents of spin systems with a precision not achievable by any other simple RG method (see Ref. 2 for a review). The LBRG gives $d\nu=2-\alpha$ to 0.1% accuracy in the $d=2$ and 3 dimensional Ising models, and to 1% accuracy in four dimensions. For ϕ^4 field theory it yields a one-dimensional integral equation whose solution gives ν correctly to first order in the ϵ expansion. It is also a direct approximation to an infinite system, so true finite-size effects are absent.

This amazing precision, coupled with the present level of interest in the renormalization-group structure of gauge-Higgs systems (see, e.g., Ref. 3), provides motivation to generalize this simple technique to systems with local gauge symmetry. The generalization presented here is applicable to all gauge groups in arbitrary spacetime dimension, and reduces to the Kadanoff method for spin systems in the limit where the gauge coupling vanishes.

The crux of the LBRG is the observation that [since $\langle \exp(\Delta S)\rangle \geq \exp(\langle \Delta S\rangle)$] the addition to the action of an operator with vanishing expectation value *lowers* the free energy f. Interaction-moving operations satisfy this criterion by translational invariance. Variational parameters are introduced and are (easily) optimized at a fixed point to give a best lower bound for f. Critical exponents then follow directly from the recursion relations.

The application of the original LBRG to gauge theories is complicated by the fact that its block lattice points lie in the centers of hypercubes, while the gauge links $\{U\}$ are only defined along its edges. It is therefore first necessary to use a prefacing transformation to map the original "plaquette" system to a "subsumed" model which allows parallel transport to the center of a hypercube. The blocking then yields a plaquette model with larger lattice spacing, ready for the next iteration.

An example facilitates explanation. Consider (Fig. 1) a single plaquette in a two-dimensional Z_2 gauge theory with vertices labeled (1,2,3,4) and action $S_{G,1234} = -\beta U_{12}U_{23}U_{34}U_{14}$ where $U_{ji}=U_{ij}=\pm 1$. Place a point c in the center of the plaquette. The corresponding subsumed model action is

$$\tilde{S}_{G,1234} = -\tilde{\beta}(V_{1c}U_{12}V_{2c}+V_{2c}U_{23}V_{3c}+V_{3c}U_{34}V_{4c} + V_{1c}U_{14}V_{4c}) - C_\beta \ , \quad (1)$$

where each of the $V_{ic}=\pm 1$ is an element of Z_2 which runs between points c and $i=1-4$. Since $\mathrm{Tr}_{\{V\}}[\exp(-\tilde{S})]=\exp(-S)$ it follows that $(\tilde{\beta})''=\beta$ and $C_\beta=-2(\tilde{\beta})'-\beta-\ln 16$, where $x''\equiv(x')'$ and $\tanh x' \equiv \tanh^2 x$. In higher spacetime dimensions, the point c is placed in the center of a hypercube. The prefacing transformation can be performed analytically for discrete groups. For larger groups more interactions [e.g., $(VUV)^2$, $(VUV)(VUV)$, etc.] must be included. Note that the prefacing is *underconstrained*—many subsumed models correspond to the one-plaquette model.

The machinery of the generalized LBRG can be illustrated using a Z_2 gauge theory in $d=2$ dimensions. Figure 2 defines the notation. The original lattice fields (e.g., U_{67}) run between the original site points, denoted by numerals 1–16. The new fields of the subsumed model (e.g., V_{6e}) go between the original site points and new points labeled by letters a through i. Block points are denoted by crosses, and block fields (e.g., U'_{ag}) connect these. Thus (a,c,i,g) is the boundary of a typical block of side length $b=2$, which is the RG scaling factor.

A gauge-invariant projection operator P can be defined for the subsumed model. The projection operator determines the renormalized action $S_{G,R}$ via

$$\exp(-S_{G,R}) = \mathrm{Tr}_{\{U,V\}} P \exp(-\tilde{S}_G) \ .$$

Here P is a taken to be a product over all block links $\{U'\}$ of terms such as

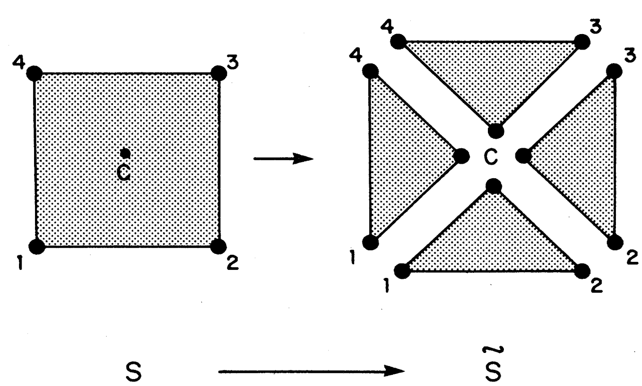

FIG. 1. Schematic diagram of prefacing transformation.

$$P_{ag}=\exp[p_1 U'_{ag}(V_{6a}V_{6d}V_{10d}V_{10g} + V_{5a}V_{5d}V_{9d}V_{9g})-N_{ag}]\ , \qquad (2)$$

where p_1 is a variational parameter. The requirement that the free energy remain invariant under the RG transformation implies that the trace of P_{ag} over U'_{ag} is unity, and thus

$$N_{ag}=(p_1)'(V_{6a}V_{6d}V_{5d}V_{5a})(V_{10d}V_{10g}V_{9g}V_{9d})+C_1$$

with $C_1\equiv(p_1)'+\ln 2$.

So far the RG transformation is *exact* (and intractable). The LB approximation consists of moving all interactions from the VUV terms and from the normalization terms (e.g., N_{ag}) in the product $P\exp(-\tilde{S}_G)$ into the shaded region (b,f,h,d), and equally to its counterparts in other blocks. The result for the contribution from block (a,c,i,g) is

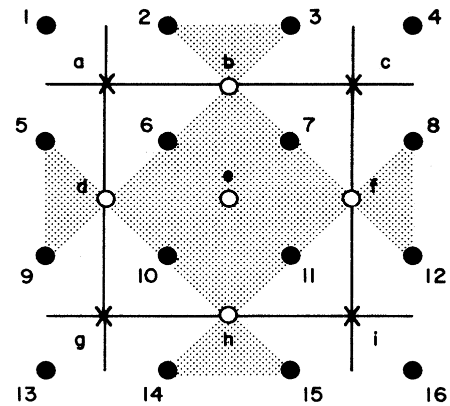

FIG. 2. Definition of notation for LBRG.

$$\begin{aligned}[P\exp(-\tilde{S}_G)]_{\mathrm{LB},acig}=\exp\{2\tilde{\beta}[&U_{6,7}(V_{6b}V_{7b}+V_{6e}V_{7e})+U_{7,11}(V_{7f}V_{11f}+V_{7e}V_{11e})\\ &+U_{10,11}(V_{10e}V_{11e}+V_{10h}V_{11h})+U_{6,10}(V_{6e}V_{10e}+V_{6d}V_{10d})]-4C_\beta\\ &+p_1[U'_{ag}(V_{6a}V_{6d}V_{10d}V_{10g})+U'_{gi}(V_{10g}V_{10h}V_{11h}V_{11i})\\ &+U'_{ci}(V_{7c}V_{7f}V_{11f}V_{11i})+U'_{ac}(V_{6a}V_{6b}V_{7b}V_{7c})]\\ &-p'_1[(V_{6b}V_{7b}V_{6e}V_{7e})(V_{10e}V_{11e}V_{10h}V_{11h})\\ &+(V_{6d}V_{6e}V_{10d}V_{10e})(V_{7e}V_{7f}V_{11e}V_{11f})]-2C_1\}\ . \qquad (3)\end{aligned}$$

The interaction-moving separates the original system into blocks within which the summations over $\{U,V\}$ can be performed independently. The renormalized couplings β_R and C_R are found from

$$\begin{aligned}\exp[\beta_R(U'_{ag}U'_{gi}U'_{ci}U'_{ac})+C_R]&=[\exp(-S_{G,R})]_{acig}\\ &=\mathrm{Tr}_{\{U,V\}}[P\exp(-\tilde{S}_G)]_{\mathrm{LB},acig}\ . \qquad (4)\end{aligned}$$

The summations over the $\{U\}$ and $\{V\}$ are then performed. The block renormalized couplings β_R and C_R are found from the partial partition functions Z_+ and Z_-:

$$e^{2\beta_R}=\frac{Z_+}{Z_-}\ , \qquad (5a)$$

$$e^{2C_R}=Z_+Z_-\ , \qquad (5b)$$

where

$$Z_+=2e^{(2\tilde{\beta})'}(A^2+B^2)\ , \qquad (6a)$$

$$Z_-=4e^{(2\tilde{\beta})'}(AB)\ , \qquad (6b)$$

and

$$A\equiv e^{2p'_1}+\gamma e^{-2p'_1}\ , \qquad (7a)$$

$$B\equiv 1+\gamma\ , \qquad (7b)$$

$$\gamma\equiv\cosh[2(2\tilde{\beta})']=\exp[2(2\tilde{\beta})'']\ , \qquad (7c)$$

so that

$$\tanh\beta_R\doteq\tanh^2[p'_1-(2\tilde{\beta})'']\tanh^2 p'_1\ . \qquad (7d)$$

The variational parameter p_1 is determined from the extremum condition

$$0=\frac{\partial f}{\partial p_i}=\frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial p_i}\ , \qquad (8)$$

where $\alpha=1,\ldots,n$; $K_{R,n}\equiv C_R$. This can be solved easily at a fixed point,[1] for there $\partial f/\partial K_\alpha\equiv e_\alpha$ is a left eigenvector of the matrix $D_{\alpha\beta}\equiv\partial K_{R,\alpha}/\partial K_\beta$ with eigenvalue b^d $(=2^2)$. In fact, Eq. (8) is then a determinant, since e_α is proportional to $\mathrm{cof}(\Delta_{\alpha n}),\Delta_{\beta\alpha}\equiv D_{\alpha\beta}-b^d\delta_{\alpha\beta}$.

For a system with two coupling constants (β and C), the extremum condition at a fixed point is

$$\frac{\partial C_R}{\partial\beta}\frac{\partial\beta_R}{\partial p_1}+\frac{\partial C_R}{\partial p_1}\left[\frac{\partial C_R}{\partial C}-\frac{\partial\beta_R}{\partial\beta}\right]=0\ , \qquad (9a)$$

where here

$$\frac{\partial C_R}{\partial C}\equiv b^d=4\ . \qquad (9b)$$

The solution of Eqs. (9) is

$$p'_1=\tfrac{1}{4}\ln\gamma \qquad (10a)$$

$$=\tfrac{1}{2}(2\tilde{\beta})''\ . \qquad (10b)$$

When Eqs. (10) are substituted into Eq. (7d), the result is the RG flow equation

$$\beta_R = [\tfrac{1}{2}(2\tilde{\beta})'']'' \tag{11}$$

which can be compared (cf. Fig. 3) with the exact result $\beta_R = [(\tilde{\beta})'']'' = \beta''$. [Note however that the extremum equation Eq. (9a) is strictly valid only at a fixed point.]

The critical exponent for the fixed point at infinite β can be easily calculated:

$$\lim_{\beta \to \infty} \frac{\partial \beta_R}{\partial \beta} = 1 = b^y \,. \tag{12}$$

The fixed point is therefore predicted to be *marginal* ($y=0$) as per the known result. The critical exponent for the pure Z_2 gauge theory is thus given *exactly* by the LBRG.

Scalar "Higgs" fields are easily included in the formalism. Consider for simplicity fixed-length Z_2 fields $\sigma_n = \pm 1$ (variable-length scalars are treated in Ref. 1). The Higgs terms in the original action S as well as in the renormalized action S_R can be written entirely in terms of gauge-invariant objects such as $h_{6,7} \equiv \sigma_6 U_{6,7} \sigma_7$ and $h'_{ac} \equiv \sigma'_a U'_{ac} \sigma'_c$, respectively. For scaling factor $b=2$ in two dimensions nothing larger than a plaquette can be included. Thus the allowable Higgs terms for plaquette (6,7,10,11) are

$$\begin{aligned} O_{\text{plaq}} &\equiv h_{6,7} h_{7,11} h_{10,11} h_{6,10} \,, \\ O_2 &\equiv \tfrac{1}{2}(h_{6,7} + h_{7,11} + h_{10,11} + h_{6,10}) \,, \\ O_3 &\equiv \tfrac{1}{2} h_{6,7}(h_{7,11} + h_{6,10})(1 + O_{\text{plaq}}) \,, \\ O_4 &\equiv 2 O_{\text{plaq}} O_2 \,, \\ O_5 &\equiv \tfrac{1}{2} h_{6,7} h_{10,11}(1 + O_{\text{plaq}}) \,, \end{aligned} \tag{13}$$

and their contribution to the action is $-\sum_{i=2}^{5} K_i O_i$. The Higgs contribution to the full action is the sum over all plaquettes of this quantity (note that $\beta \equiv K_1$ and likewise $\beta_R \equiv K_{R,1}$; $C_R \equiv K_{R,6}$, etc.). The projection operator Q for the scalars is a product over all block fields of terms such as this for σ'_a:

$$\begin{aligned} Q_a &= \exp[p_2 \sigma'_a \Sigma - \tfrac{1}{2} L_2(\Sigma^2 - 4) \\ &\quad - M_2 \sigma_1 \sigma_2 \sigma_5 \sigma_6 V_{1a} V_{2a} V_{5a} V_{6a} - C_2] \,, \\ \Sigma &\equiv V_{1a}\sigma_1 + V_{2a}\sigma_2 + V_{5a}\sigma_5 + V_{6a}\sigma_6 \,, \end{aligned} \tag{14}$$

where the normalization $\mathrm{Tr}_{\sigma'} Q = 1$ implies that

$$4L_2 = (2p_2)', \quad 2M_2 = (2L_2)' \,,$$

$$-C_2 = -2L_2 + M_2 - \ln 2 \,.$$

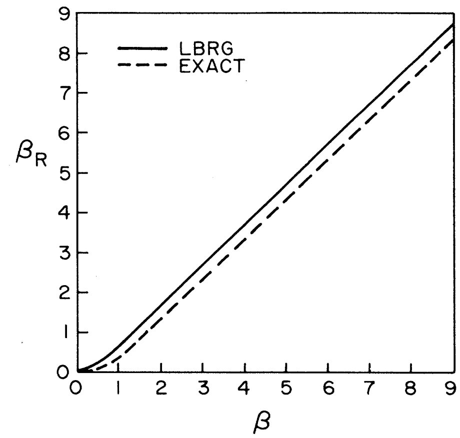

FIG. 3. Plot of LBRG recursion relation (solid line) and exact recursion relation (dashed line) for a pure Z_2 gauge theory in two dimensions.

The LB approximation of moving all interactions equally into the region (b,f,h,d) and its counterparts in other blocks is made as before. The LB contribution to $\exp(-S_R)$ from block (a,c,i,g) is the sum over all contained $\{U,V\}$ of

$$\begin{aligned} [PQ\exp(-\tilde{S})]_{\text{LB},acig} = \exp\{ & 4(2K_2 O_2 + K_3 O_3 + K_4 O_4 + K_5 O_5) - \tfrac{1}{2} L_2[(\sigma_6 V_{6e} + \sigma_7 V_{7e} + \sigma_{10} V_{10e} + \sigma_{11} V_{11e})^2 - 4] \\ & - M_2(\sigma_6 \sigma_7 \sigma_{10} \sigma_{11} V_{6e} V_{7e} V_{10e} V_{11e}) - C_2 \\ & + p_2(\sigma'_a V_{6a}\sigma_6 + \sigma'_c V_{7c}\sigma_7 + \sigma'_i V_{11i}\sigma_{11} + \sigma'_g V_{10g}\sigma_{10})\}[P\exp(-\tilde{S}_G)]_{\text{LB},acig} \,. \end{aligned} \tag{15}$$

The six block couplings $\{K_R\}$ are evaluated analytically in terms of the $\{K\}$, and the p_i are determined by Eq. (8). The familiar[1] Ising fixed point is recovered at $p_2 = 0.76$ and infinite β and p_i, yielding $d\nu = 2 - \alpha = 1.998$ to 0.1% accuracy. No distinct new fixed points appear at finite β [β always decreases when Eqs. (8) are applied].

Thus it is seen that by the use of a prefacing transformation the accurate Kadanoff LBRG can in fact be applied to systems with local gauge symmetry. The method can be applied to gauge-Higgs systems with arbitrary gauge group in any spacetime dimension, though in general numerical techniques[4] may be needed. It should be noted, however, that invariant subspaces of coupling constants often exist and can vastly simplify the calculation.[2] The (marginal) critical exponent for two-dimensional Z_2 gauge theory was predicted exactly, and good qualitative agreement with the flow equation was found. When Z_2 scalars were also included, the good results (0.1% accuracy) for the Ising limit were recovered as expected. No unexpected additional spurious fixed points appeared, and again a good flow diagram resulted.

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