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Unsuppressed fermion-number violation at high temperature: An O(3) model

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The O(3) nonlinear σ model in $1+1$ dimensions, modified by an explicit symmetry-breaking term, is presented as a model for baryon- and lepton-number violation in the standard electroweak theory. Although arguments based on the Atiyah-Singer index theorem and instanton physics apply to the model, we show by explicit calculations that the rate of chiral fermion-number violation due to the axial anomaly is entirely *unsuppressed* at sufficiently high temperatures. Our results apply to unbroken gauge theories as well and may require reevaluation of the role of instantons in high-temperature QCD.

I. INTRODUCTION

It has been known for some time that the standard $SU(2)\times U(1)$ model of weak interactions does not exactly conserve baryon and lepton number. The reason is the chiral current anomaly,[1] which becomes an anomaly in the fermion-number currents as well, in a parity-violating theory. For example,

$$\partial^\mu j^B_\mu = \frac{1}{32\pi^2}\frac{N_B}{3}(-g^2 F^i_{\mu\nu}\tilde{F}^{i\mu\nu} + g'^2 F'_{\mu\nu}\tilde{F}'^{\mu\nu}) , \quad (1.1)$$

where N_B is the number of left-handed SU(2) baryon doublets, g,g' are, respectively, the SU(2) and U(1) hypercharge coupling constants and $F^i_{\mu\nu}$, $F'_{\mu\nu}$ are the corresponding field-strength tensors.

Although the current anomaly first appears in one-loop perturbation theory, the nonconservation of baryon and lepton number is essentially nonperturbative, since the relevant operator $F\tilde{F}$ is a total divergence and its contributions vanish order by order in perturbation theory. In fact, the Atiyah-Singer index theorem[2] relates the nonconservation of fermion number to the topological aspects of the SU(2) gauge theory to which the fermions couple, i.e., to the change in winding number of the gauge theory vacuum. The characteristic dependence on the coupling constant of such nonperturbative topological effects is $\exp(-\text{const}/g^2)$.

Explicit calculation of the rate of fermion-number violation may be carried out in the semiclassical approximation by expanding about the self-dual instanton solutions of the Euclidean Yang-Mills field equations.[3] These solutions interpolate between distinct topological sectors of the gauge theory vacuum and give rise to baryon- and lepton-number violation at the level of $\exp(-2S)$, where $S=8\pi^2/g^2$ is the classical action of the instanton. So, if the process can occur only via instantons, it is very strongly suppressed. The factor $\exp(-2S)$ is characteristic of a *quantum* tunneling process from one topological vacuum sector of the theory to another and for that reason vanishes exponentially as $\hbar$ approaches zero.

Although interesting in principle, these nonperturbative tunneling processes occur far too infrequently to be of any relevance to physics: the Universe would still be awaiting the very first event of this kind. A different light was cast on the issue as a result of the work of Taubes[4] and Manton,[5] who demonstrated the existence of a new kind of solution to the classical Yang-Mills-Higgs field equations. By an interesting argument related to Morse theory (which we recapitulate below) these authors argued that there should exist a time-independent, finite-energy solution to the equations that involves the symmetry-breaking Higgs field in an essential way. Because this solution is unstable, it was dubbed a sphaleron. Unlike the monopole or other finite-energy solutions it is not a soliton (which is a stable field configuration of finite extent). Instead it represents an extremum (in fact, a saddle point) of a family of static field configurations that smoothly interpolate between the vacuum sectors of the gauge theory with different winding numbers.

Although the same vacuum topology arises in the Manton-Taubes construction of the sphaleron solution as the instanton, its relevance to fermion-number nonconservation was not widely appreciated at first. That it should play an important role in baryon- and lepton-number-violating processes was suggested originally by Klinkhamer and Manton[5] and emphasized in the work of Rubakov and co-workers.[6] The most important difference between the two configurations is that whereas instanton solutions have finite (Euclidean) *action*, so that their effects are always exponentially small [i.e., of order $\exp(-2S)$, even at nonzero temperatures], the sphaleron has finite *energy*. Its effects should be suppressed instead by the Boltzmann factor, $\exp(-E_{\rm sph}/kT)$, if the system is at temperature T. Since $E_{\rm sph}\sim 4M_W/g^2$ (Ref. 5), the sphaleron could conceivably have much larger effects than those associated with instantons, at temperatures greater than M_W. The Boltzmann factor is characteristic of a *classical* thermal activation process and does not vanish as $\hbar\rightarrow 0$ (provided $E_{\rm sph}$ is held fixed in the classical limit).

Although related, the two configurations represent quite distinct pathways between topological sectors. This is most clear from the fact that the sphaleron exists at all only because a definite scale is introduced into the equations by the symmetry-breaking Higgs field. As the

Higgs vacuum expectation value is turned off, the sphaleron becomes indistinguishable from the vacuum: its spatial extent goes to infinity and its energy goes to zero. In contrast, the instanton is a solution of the equations of the symmetric gauge theory; its scale is arbitrary and must eventually be integrated over. In addition, instantons are self-dual so that the electric and magnetic components of the field strength are equal in magnitude, whereas the sphaleron has zero electric field strength (but nonzero magnetic field strength). Therefore, it is not inconceivable that the sphaleron and instanton can make quite different contributions to fermion-number-violating amplitudes. The relative importance of the two configurations is a function of temperature. The transition from the self-dual instanton to the static sphaleron has been discussed recently by several authors.[7]

If the Rubakov suggestion is correct, and fermion-number violation at unsuppressed rates occurs in the standard model, it could have important consequences for experiments at TeV collider energies, as well as for the early Universe when such high temperatures were attained. In particular, it could require a revision of the scenarios for generating the observed baryon- and lepton-number asymmetries based on grand unified theories such as SU(5), where $B-L$ is exactly conserved.[8] This point of view has been taken up in several recent papers.[9]

An analogy to a far simpler physical system is suggested in Ref. 5: namely, a one-dimensional quantum pendulum. The Euclidean action for the pendulum is

$$S=\int d\tau\left[\tfrac{1}{2}\dot{\theta}^2+\omega^2(1-\cos\theta)\right] . \tag{1.2}$$

The analogue of the topologically distinct vacuum sectors are the minima of the potential at $\theta=2n\pi$. The instanton solutions are those solutions of the Euclidean classical equations of motion with finite action that interpolate between two such minima. At temperatures low compared to the barrier height, $V_0=2\omega^2$, these solutions dominate the amplitude for processes involving a nonzero winding number. However, as the temperature of the system is raised, it is clear that the pendulum no longer quantum-mechanically tunnels under the barrier separating the periodic minima. Instead, the thermal fluctuations allow it to swing around by 2π with a rate controlled only by $\exp(-V_0/kT)$. The winding-number-changing amplitudes are dominated by classical thermal fluctuations at high temperatures as quantum tunneling becomes irrelevant.

The barrier height V_0 is also the energy of an unstable static solution that sits atop the potential barrier: namely, $\theta=\pi$. It is a configuration that lies halfway between two minima of different winding number and is one of a member of static configurations that smoothly interpolate between these minima. This trivial static solution is the analogue of the sphaleron in the toy model. At temperatures approaching V_0 it dominates the rate of winding-number change, which becomes more and more unsuppressed. The rate remains unsuppressed as the temperature is raised above V_0, although any expansion of the functional integral around the "sphaleron" must certainly break down at these very high temperatures.

In the pendulum example it is very clear that the instanton suppression does not persist at sufficiently high temperatures, because thermal activation comes to dominate over quantum tunneling. However, the pendulum differs from the gauge theory in several important respects.

First and foremost, it is a model with only 1 degree of freedom. In such a model it is evident that heating the system must imply greater kinetic energy available to leap the potential barrier: there is nowhere else for the energy to go. In a field theory there are infinitely many degrees of freedom and the class of configurations that interpolate between vacuum states of different winding number may be very special and very few. Heating this system also increases the available energy, but it is by no means clear that the incoherent thermal energy can organize itself into the special configuration(s) necessary to leap the barrier. In other words, a significant entropy suppression is possible. We need to consider the free energy, not just the classical energy of the sphaleron.

Second, the sphaleron solution in the Higgs gauge theory makes use of the spontaneous symmetry breaking in the theory in an essential way. As the temperature is raised above the critical temperature, the symmetry is restored. Do we expect then to recover the results of the ordinary instanton configurations, together with their exponential suppression? A naive argument based on the Boltzmann factor indicates that the sphaleron-induced transitions continue to *increase* in rate as the temperature approaches the critical temperature, just as in the pendulum example for T greater than V_0. Yet we know that above T_c, $M_W=0$, and the sphaleron no longer exists at all. This is just another way of saying that the semiclassical, dilute-gas approximation must break down and entropy effects must become dominant near the critical temperature. So this second criticism is related to the first.

It should be obvious but may be worth emphasizing, nonetheless, that breakdown of the semiclassical expansion around the sphaleron is a technical difficulty that does not by itself imply that there is again suppression of topology change and fermion-number violation at temperatures above the critical temperature. It implies only that the sphaleron dilute-gas expansion is unjustified, and some other calculational technique must be employed in the symmetry-restored phase.

In any case, it seems clear that something quite essential is missing from the pendulum analogy. There is no analogue of symmetry breaking or restoration in a one-dimensional quantum-mechanical model. Closely related is the point that entropy never plays an important role in the system. In addition, the pendulum model simply does not have any nontrivial instanton solutions with finite action at sufficiently high temperatures. None of these statements is true of the Yang-Mills-Higgs system. Hence, the pendulum analogy must be viewed as highly suspect as a model for topology change in four-dimensional gauge theory. These criticisms would be blunted if we could find a model which is a bona fide field theory, but which is nevertheless simple enough to permit exact calculations.[10]

Our purpose in this paper is to present just such a model: the O(3) nonlinear σ model in $1+1$ dimensions. The completely symmetric model has been studied before for its remarkable similarities to non-Abelian gauge theories. We need to modify the symmetric model by introducing a term in the action that breaks the O(3) symmetry down to O(2), in order that a sphaleron solution exist.

Although this is an explicitly broken global symmetry, unlike the spontaneously broken local symmetry of the Weinberg-Salam theory, it shares many properties with the latter. Its main virtue is the fact that we will be able to attain closed form results for fermion-number-violating processes in this theory at temperatures larger than the symmetry-breaking scale. In the σ model the one-loop approximation also breaks down at high temperatures but it is possible to calculate the rate of fermion-number violation nonetheless. While interesting in their own right, we believe that these results are very useful in guiding one's intuition of how the Weinberg-Salam theory should behave at temperatures above M_W and above the symmetry-restoration point at $T=T_c$. In the σ model a complete picture emerges which meets the criticisms raised above with explicit calculations.

The paper is organized as follows. In the next section we review the salient features of the O(3) nonlinear σ model in relation to non-Abelian gauge theories, at first without any symmetry breaking. Coupling the theory to fermions, we show how the chiral anomaly induces fermion-number violation through instanton configurations at zero and at finite temperatures. The very same formal arguments for instanton suppression of fermion-number-violating amplitudes can be made in the O(3) σ model as in the four-dimensional gauge theory.

In Sec. III we introduce the symmetry-breaking term and argue that a sphaleron solution must exist in the modified O(3) σ model by arguments exactly paralleling those of Manton for the gauge theory. In this case the sphaleron can be found analytically and is none other than the instanton solution of the simple pendulum, with the spatial variable x playing the role of Euclidean "time."

In Sec. IV we review the general path-integral formalism for the calculation of decay rates at finite temperature by expanding about a static sphaleron background. The single negative eigenvalue of the Gaussian fluctuation operator is interpreted as giving rise to an imaginary part of the free energy function. The imaginary part of the free energy is related in turn to the classical decay rate of the metastable perturbative vacuum, i.e., to the rate of topology changing events in a theory such as the one we consider. This method automatically accounts for entropy factors since what is calculated is the free energy not just the classical energy of the background.

In Sec. V we calculate the free energy of the sphaleron by applying the general methods of Sec. IV. For the model considered, the free energy can be expressed in closed form. We examine the resulting expression as the temperature is raised above the symmetry-breaking scale, and calculate the rate of fermion-number violation it induces.

At very high temperatures the loop expansion fails and the vacuum becomes a highly disordered state, but the resulting behavior can be found by making use of the fact that at these high temperatures the model collapses again to a quantum-mechanical one. Although the sphaleron solution *per se* becomes irrelevant at these high temperatures, fermion-number violation is *not* exponentially suppressed. The simple pendulum analogy is different in its details, but the major qualitative conclusion it suggests remains in the model we consider, and is completely calculable. These results are discussed in Sec. VI. We address the question of how these results can be reconciled with the standard instanton analysis and comment on the relevance of our model to the actual weak-interaction theory.

We believe that our results put the claim of unsuppressed baryon- and lepton-number violation in the weak interactions at high temperatures on much firmer ground, by removing the severest restrictions of the pendulum analogy in a nontrivial yet explicit model. Our results are in full agreement with those Refs. 5 and 7 and indicate that a reappraisal of baryon- and lepton-number asymmetry generation in cosmology and collider energy physics is indeed warranted. In addition, we suggest that in an *unbroken* gauge theory such as QCD, instanton-based estimates of topology change may be completely misleading and should be reexamined as well.

II. THE O(3) NONLINEAR σ MODEL

In $1+1$ dimensions, the action of the O(3) nonlinear σ model is

$$S_0=\frac{1}{2g^2}\int d^2x\,(\partial_\mu\hat{\mathbf{n}}\cdot\partial_\mu\hat{\mathbf{n}}),\quad \hat{\mathbf{n}}^2(x)=1\ . \tag{2.1}$$

This model possesses some remarkable similarities with non-Abelian gauge theories in $3+1$ dimensions, and for that reason has been much studied.[11] The most important features which concern us here are the following: (i) scale invariance of the classical action; (ii) renormalizability and asymptotic freedom in the coupling constant g; (iii) existence of a topological winding number, instantons, and a chiral anomaly when coupled to fermions; (iv) generation of a dynamical mass by nonperturbative effects at zero temperature and of a thermal mass ($\sim g^2T$) at finite temperature.

The first property is obvious and the second well known.[12] The winding number will be evident if we identify the points at infinity of the Euclidean plane. Then the plane has topology S^2. Since $\hat{\mathbf{n}}$ is also constrained to lied on S^2, the $\hat{\mathbf{n}}$ field is a map from S^2 to S^2. This mapping can be characterized by an integer winding number, given explicitly by

$$Q=\frac{1}{8\pi}\int d^2x\,\epsilon_{\mu\nu}\hat{\mathbf{n}}\cdot(\partial_\mu\hat{\mathbf{n}}\times\partial_\nu\hat{\mathbf{n}})\ . \tag{2.2}$$

By forming the quantity

$$\int d^2x\,[\partial_\mu\hat{\mathbf{n}}\pm\epsilon_{\mu\nu}(\hat{\mathbf{n}}\times\partial_\nu\hat{\mathbf{n}})]^2\geq 0 \tag{2.3}$$

it is easy to see that Euclidean action for any $\hat{\mathbf{n}}$ *obeying the boundary condition at infinity* is bounded from below:

$$S_0 \geq \frac{4\pi}{g^2} \mid Q \mid \; . \tag{2.4}$$

The bound is saturated by the instanton solutions which can be given explicitly in terms of the complex function

$$w = \frac{n_1 + in_2}{1 - n_3} \tag{2.5}$$

of the complex variable $z = x_1 + ix_2$. In terms of w, S_0 and Q become proportional to

$$\int d^2x \frac{1}{(1+\mid w \mid^2)^2} \left[\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} \pm \frac{\partial w}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial z} \right] , \tag{2.6}$$

respectively. Thus, the bound (2.4) is saturated when one of the terms in large parentheses vanishes and (anti-)instanton solutions are simply meromorphic functions of the complex variable $z\,(\bar{z})$. In particular,

$$w_n = c \prod_{l=1}^{n} \frac{z - a_l}{z - b_l} \tag{2.7}$$

has $Q = n$ and $S_0 = 4\pi n / g^2$.

Yet another formulation of the model is obtained by defining a two-component field

$$\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}$$

such that

$$\hat{\mathbf{n}} = \chi^\dagger \sigma \chi \; , \tag{2.8}$$

where σ are the Pauli matrices. The condition $\hat{\mathbf{n}}^2 = 1$ now becomes

$$\chi^\dagger \chi = 1 \; . \tag{2.9}$$

In terms of χ,

$$S_0 = \frac{2}{g^2} \int d^2x [(\partial_\mu \chi^\dagger)(\partial_\mu \chi) - (\chi^\dagger \partial_\mu \chi)(\partial_\mu \chi^\dagger \chi)] \; . \tag{2.10}$$

Evidently there is now a U(1) gauge invariance in this formulation since S_0 is invariant under $\chi \to e^{i\alpha(x)}\chi$. This may be made explicit by introducing a subsidiary gauge field

$$A_\mu(x) = \frac{1}{2i}(\chi^\dagger \partial_\mu \chi - \partial_\mu \chi^\dagger \chi) \tag{2.11}$$

and defining the covariant derivative

$$D_\mu = \partial_\mu - iA_\mu \tag{2.12}$$

so that

$$S_0 = \frac{2}{g^2} \int d^2x \mid D_\mu \chi \mid^2 \; . \tag{2.13}$$

In this language,

$$Q = \frac{1}{4\pi} \int d^2x \, \epsilon^{\mu\nu} F_{\mu\nu} = \frac{1}{2\pi} \int d^2x \, \partial_\mu(\epsilon^{\mu\nu} A_\nu) \; . \tag{2.14}$$

Massless fermions may now be added to the system and coupled in the usual way to the U(1) gauge field:

$$S_{\text{fermion}} = i \int d^2x \, \bar{\psi}\gamma^\mu D_\mu \psi \; . \tag{2.15}$$

Such fermions are well known[13] to possess an anomaly in the chiral current

$$j^{\mu 5} = \bar{\psi}\gamma^\mu \gamma^5 \psi \; , \tag{2.16}$$

namely,

$$\partial_\mu j^{\mu 5} = \frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu} \; . \tag{2.17}$$

The Feynman graph contributing to the anomaly is illustrated in Fig. 1. By integrating this equation over two-dimensional space, we obtain the index theorem

$$\Delta \int dx \, j^{05} \equiv \Delta N_5 = 2Q = 2\Delta N_{\text{CS}} \; , \tag{2.18}$$

where

$$N_{\text{CS}} = \frac{1}{2\pi} \int dx \, A_1 \tag{2.19}$$

is the Chern-Simons number corresponding to the Q of Eq. (2.14). Thus, (2.18) relates the change of chiral fermion number to the topological charge, or change of winding number, N_{CS} in going from one vacuum configuration to another. Because of the bound (2.4) on the classical action, and the interpretation of the Euclidean instanton as a tunneling event (by continuation to imaginary time), such topology changing events and concomitant fermion-number violation are strongly

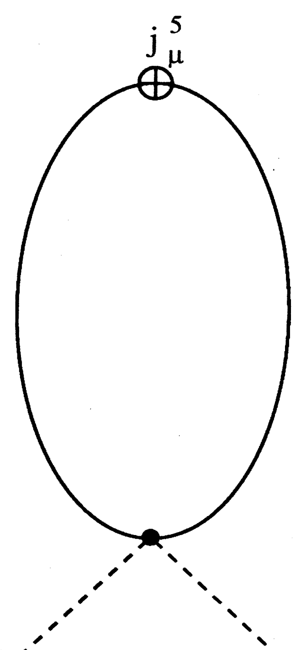

FIG. 1. The diangle graph which gives rise to the axial anomaly in the present model. Fermion propagators are denoted by solid lines and scalar χ propagators by dashed lines. It is the same graph as that in (1 + 1)-dimensional QED with the role of the U(1) gauge field played by the A_μ defined by Eq. (2.11).

suppressed at zero temperature: the rate is proportional to $\exp(-2S_0) \ll 1$.

At finite temperature, exact instanton solutions may also be constructed. We need only take the multi-instanton form Eq. (2.7) and substitute $a_l = a + il\beta$, $b_l = bil\beta$. The product over l then gives

$$w = c\frac{\sinh\pi(z-a)/\beta}{\sinh\pi(z-b)/\beta}\ , \tag{2.20}$$

which is manifestly periodic in Euclidean time with period $\beta = T^{-1}$. This corresponds to the pure gauge theory where the finite-temperature instanton construction via the 't Hooft ansatz amounts to fixing the relative orientation and scales of the general multi-instanton configuration.[14]

The action of the periodic solution (2.20) is $4\pi/g^2$ so the instanton in each Euclidean time slice $(l\beta, l\beta+\beta)$ does not interact with its neighbors, classically. The bound on the action and suppression of fermion-number-violating processes would seem to be just as valid at finite temperature as at zero temperature. If the loop expansion about the classical instanton configuration is carried out, the coupling constant must be replaced by the effective temperature-dependent coupling

$$\frac{1}{g^2(T)} = \frac{1}{g^2(T_0)} + \frac{1}{2\pi}\ln(T/T_0)\ . \tag{2.21}$$

As T increases the coupling only gets smaller which is the asymptotic freedom referred to above and which suppresses the rate even further.

Now, the other side of the coin of asymptotic freedom is infrared slavery: the coupling grows ever larger at large scales. This means that the infrared behavior of the integration over instanton scale size diverges at large scales, i.e., the instanton gas is not dilute, since large instantons begin to overlap with each other, and these overlapping configurations are the quantitatively important ones. In four-dimensional gauge theories this is a still unsolved problem at present, but in the σ model there is no need to assume diluteness of the instanton gas: one can sum over all exact multi-instanton configurations. Iwasaki[15] finds in this way that the infrared problem cures itself by the dynamical generation of a mass:

$$M_{\text{dyn}} = \frac{16\pi\mu}{eg^2}e^{-2\pi/g^2}\ . \tag{2.22}$$

The fact that a naive perturbative or semiclassical dilute-gas expansion of correlation functions breaks down in the infrared means that we must be very careful in applying arguments based on such expansions to the issue of suppression of fermion-number violation. We will show that in the σ model such arguments based on the classical bound (2.4) actually are incorrect, and that fermion-number-violating processes are not exponentially suppressed at high temperature.

The analysis of the σ model at high temperature is possible because in that limit the $(1+1)$-dimensional field theory reduces to a purely quantum-mechanical problem via the replacement[16]

$$S_0 \to \frac{1}{2g^2T}\int dx\left[\frac{d\hat{\mathbf{n}}}{dx}\right]^2\ . \tag{2.23}$$

That is, as $T\to\infty$ the time direction becomes compressed so that only static configurations are relevant and $\hat{\mathbf{n}}$ becomes a function of x only. The action (2.23) is the classical Euclidean action of a point particle constrained to move on a two-sphere, i.e., a rigid rotor with moment of inertia $1/g^2T$. Since the spatial variable x runs from $-\infty$ to $+\infty$, and this now plays the role of the Euclidean "time" for the system, the effective "temperature" of this quantum rotor is zero. Hence, the rotor lies in its ground state, which is a state of zero angular momentum with wave-function uniform and constant everywhere on the sphere.

The quantum Hamiltonian of the rotor has spectrum $g^2Tl(l+1)/2$. If we now calculate a typical correlation function in the $l=0$, O(3)-symmetric ground state, we find that it decays exponentially:

$$\langle\hat{\mathbf{n}}(x)\cdot\hat{\mathbf{n}}(0)\rangle = e^{-g^2T|x|}\ . \tag{2.24}$$

Thus, the O(3) nonlinear σ model generates a mass at finite temperature equal to g^2T without breaking the O(3) symmetry. Instead, what has happened is the breakdown of perturbation theory. This breakdown may be understood from either of two points of view.

In the language of the quantum rotor the wave function is uniform on the sphere. This means that the typical state of the original field theory is completely disordered at high temperature, any direction on the $\hat{\mathbf{n}}$ field being equally likely. Since perturbation theory involves expanding about a fixed $\hat{\mathbf{n}}$ and assumes that the fluctuations from that value are small, it is clear that perturbation theory must break down. This conclusion is forced by the Mermin-Wagner-Coleman theorem,[17] which forbids the spontaneous breakdown of a continuous symmetry group in $1+1$ dimensions (at any temperature).

A second way to understand this result is to examine the infrared singularities which appear in the perturbation theory with massless modes at finite temperature. Naive power counting indicates that nth-order perturbation theory includes terms which behave like

$$g^{2n}\left[T\int_{p_{\min}}^{\infty}\frac{dp}{p^2}\right]^n \sim \left[\frac{g^2T}{p_{\min}}\right]^n\ , \tag{2.25}$$

where $p_{\min}$ is an infrared cutoff. Since the previous argument tells us a mass term is generated at finite temperature, $p_{\min}$ must be of order g^2T. But then g cancels from the above expression and we conclude that all orders of perturbation theory are equally important in generating the mass; i.e., perturbation theory has broken down in the infrared. Conversely, it is impossible to generate the mass g^2T in naive perturbation theory; at the very least, some sort of resummation of the perturbative series is necessary to remove the infrared divergences and evaluate the mass in the original field theory.

The situation is analogous to QCD_4 which is strongly believed to generate a magnetic screening mass of order g^2T, which cannot be reliably calculated in perturbation theory for the same reason as above.[18] The σ model allows us to calculate the mass easily in one dimension whereas no such technique is available in the gauge theory.

In the completely O(3)-symmetric σ model the following picture emerges. At temperatures high compared to the dynamically generated mass scale M_{dyn} of Eq. (2.22), a finite correlation length and temperature-dependent mass is generated by the thermal disordering of the system. At lower temperatures the O(3) symmetry must remain unbroken and the vacuum disordered, although the mechanism is completely different, depending as it does on the multi-instanton configurations and dimensional transmutation. Quantum fluctuations, rather than thermal ones are responsible for disordering the ground state and giving rise to a finite coherence length for O(3)-invariant operators at lower temperatures. We return to these considerations in Sec. VI in order to draw conclusions about the relevance of the present model to the physics of a theory with an *unbroken* gauge symmetry such as QCD.

III. THE SPHALERON SOLUTION

Having reviewed the symmetric O(3) model we are now ready to add to it a symmetry-breaking term. This is necessary if we want the model to possess nontrivial static finite-energy solutions such as the sphaleron, since scaling arguments tell us that no such solutions exist in the symmetric model. A simple symmetry-breaking term, inspired by the pendulum analogy is

$$S_1 = \frac{\omega^2}{g^2} \int d^2x\,(1+\hat{\mathbf{n}}_3)\ . \tag{3.1}$$

The classical energy functional of the model now reads

$$E = \frac{1}{g^2} \int dx \left[\frac{1}{2} \left[\frac{d\hat{\mathbf{n}}}{dx} \right]^2 + \omega^2(1+\hat{\mathbf{n}}_3) \right] . \tag{3.2}$$

Now, by paralleling Manton's original argument for the Yang-Mills-Higgs theory, we shall argue that an unstable static solution to the equations of motion must exist with finite energy (3.2). First, let us parametrize the sphere in the following way:

$$\hat{\mathbf{n}} = (\,\sin\eta \sin\xi,\ \sin\eta \cos\eta(\,\cos\xi - 1)\ ,$$
$$-\sin^2\eta \cos\xi - \cos^2\eta)\ . \tag{3.3}$$

This parametrization has the following properties: (i) it satisfies the constraint $\hat{\mathbf{n}}^2=1$ and is continuous in its arguments; (ii) for fixed η, ξ is the azimuthal angle of a circle, S^1; (iii) for all η, $\hat{\mathbf{n}}(\xi=0)=\hat{\mathbf{n}}(\xi=2\pi)=(0,0-1)$; (iv) for all ξ, $\hat{\mathbf{n}}(\eta=0)=\hat{\mathbf{n}}(\eta=\pi)=(0,0,-1)$; (v) each point on S^2 occurs for at least one (η,ξ) and if $\hat{\mathbf{n}}$ is not the point $(0,0,-1)$ then $\eta(\hat{\mathbf{n}})$ is unique; (vi) as η ranges from 0 to π and ξ from 0 to 2π the map (3.3) has $Q=1$.

The angles η and ξ are easily visualized geometrically by the diagram in Fig. 2: for given η between 0 and π, $\hat{\mathbf{n}}$

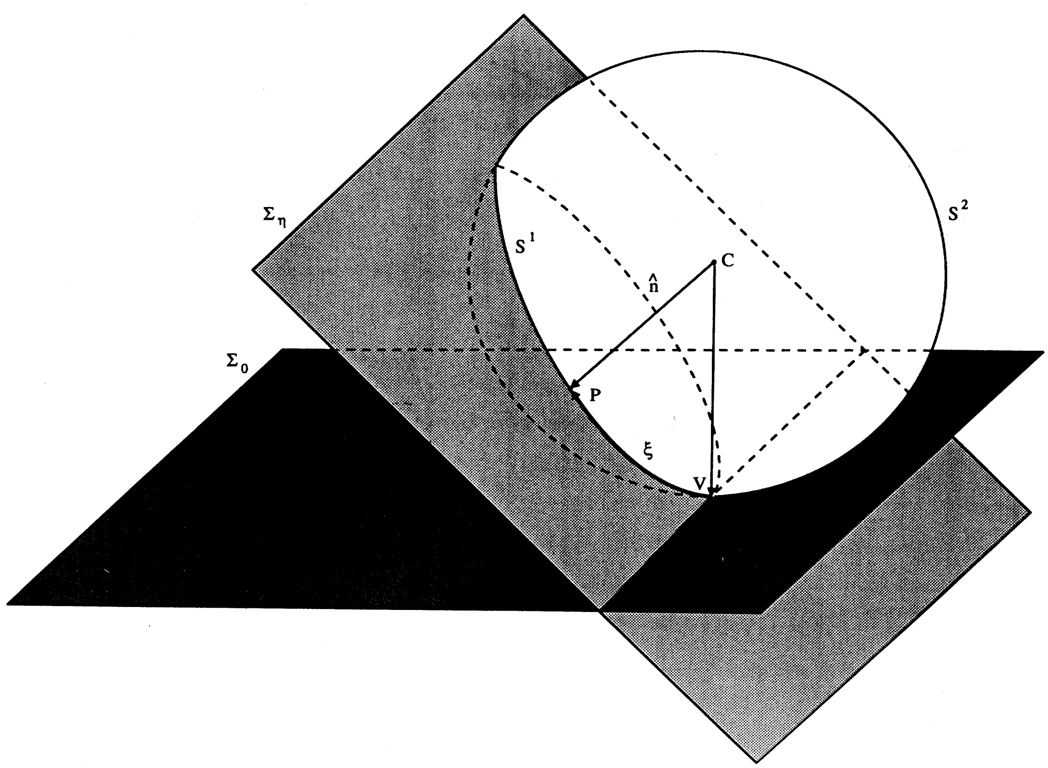

FIG. 2. Geometrical representation of the parametrization of the sphere S^2 with C at the origin, as defined by Eq. (3.3). The circle S^1 is the intersection of the sphere with the plane $x_2\sin\eta + x_3\cos\eta = -\cos\eta$, labeled by Σ_η and making dihedral angle η with the plane $x_3=-1, \Sigma_0$. ξ is the azimuthal angle along this circle measured from $V=(0,0,-1)$ to the generic point P.

lies on the circle S^1 which is the intersection of the unit two-sphere with the plane:

$$x_2 \sin\eta + x_3 \cos\eta = -\cos\eta \ .$$

As in Ref. 4, we are interested in noncontractible loops in configuration space which begin and end at the vacuum. Because of the symmetry-breaking term S_1, this is the point $\hat{\mathbf{n}}_V = (0,0,-1)$. We may now consider static configurations, $\hat{\mathbf{n}}(x)$ at fixed η, with $\xi(x)$ ranging from 0 to 2π as x ranges from $-\infty$ to $+\infty$. Because of (iii) this satisfies the boundary condition for finite energy. Because of (iv) this set of configurations reduces identically to the vacuum at $\eta=0$ and π. Because of (vi) this one-parameter (i.e., η) family of loops which begins and ends at the vacuum is noncontractible: that is, the whole sequence cannot be simultaneously continuously deformed to the vacuum. The energy functional (3.2) for fixed η and $\xi = \xi(x)$ is

$$E = \frac{\sin^2\eta}{g^2} \int dx \left[\frac{1}{2} \left[\frac{d\xi}{dx} \right]^2 + \omega^2 (1-\cos\xi) \right] . \tag{3.4}$$

Consider now the extremizing of this functional. As a function of the parameter η, E clearly attains its maximum at $\eta=\pi/2$. This is physically obvious from the fact that the energy may be viewed as that of a physical pendulum in a uniform gravitational field: for given $\xi(x)$ the maximal energy is achieved by the farthest excursion from the pendulum's point of rest at $\hat{\mathbf{n}}_V = (0,0,-1)$. With η fixed at this maximal value of $\pi/2$, now consider minimizing the positive-definite energy functional with respect to $\xi(x)$. The resulting Euler-Lagrange equation for ξ is precisely that of a simple pendulum in Euclidean "time" x. Since ξ varies from 0 to 2π as x varies from $-\infty$ to ∞, the solution of this equation is none other than the instanton solution of the pendulum problem:

$$\xi_{\mathrm{sph}}(x) = 2 \arcsin(\operatorname{sech} \omega x) \ . \tag{3.5}$$

This "minimax" procedure of extremizing the energy functional by minimizing over a set of maxima of noncontractible loops is precisely the Manton-Taubes argument for the existence of an finite energy unstable classical solution, transferred to the present model. Actually this procedure does not guarantee a solution to the full field equations. However, it is straightforward to verify that the ansatz (3.3) with $\eta=\pi/2$ and $\xi=\xi_{\mathrm{sph}}(x)$ as given by Eq. (3.5) indeed does satisfy the full field equations. The energy of the sphaleron solution for the σ model is

$$E_{\mathrm{sph}} = 8\omega/g^2 \ . \tag{3.6}$$

We see that ω plays the role of M_W in the Weinberg-Salam theory. However, one might wonder how accurate the analogy is, since ω is an explicit symmetry-breaking scale in this model, unlike the spontaneous symmetry breaking in the electroweak theory. There can be no spontaneous symmetry breaking of a continuous symmetry in two dimensions because of Coleman's theorem and, as a corollary, the exact O(3) symmetry cannot be recovered no matter how much the system is heated. These issues are addressed in the Appendix, where we calculate the finite-temperature effective potential in this model. The result is that although there is no phase transition in the strict sense, there is a gradual changeover at temperatures around ω/g^2, where the system's wave functional begins to distribute itself widely over the sphere, in contrast with lower temperatures where the symmetry-breaking term keeps it sharply peaked around the classical minimum, $\hat{\mathbf{n}}_V = (0,0,-1)$. The strict lack of spontaneous symmetry breaking or restoration in two dimensions is a technical distinction from four-dimensional gauge theory, but it is a difference which does not affect the conclusions in an essential way.

IV. CALCULATION OF THE RATE: GENERAL THEORY

We review in this section the derivation by path-integral methods of the decay rate of an unstable phase at finite temperature.[19] Our aim is to provide a general framework that we can apply to the specific sphaleron solution described in the previous section.

To illustrate the general method consider a single scalar field in $d+1$ dimensions with the action

$$S[\Phi] = \int_0^\beta d\tau \int d^d x \left[\frac{1}{2} \left[\frac{\partial \Phi}{\partial \tau} \right]^2 + \tfrac{1}{2} (\nabla \Phi)^2 + \mathcal{U}(\Phi) \right] . \tag{4.1}$$

Let $\Phi = \phi(x)$ be a static solution of the equation

$$-\nabla^2 \phi + \frac{\partial \mathcal{U}}{\partial \phi} = 0 \tag{4.2}$$

and expand S to second order in $\Phi - \phi$. The Gaussian fluctuation operator is

$$\mathcal{G} = -\frac{\partial^2}{\partial \tau^2} - \nabla^2 + V(x), \quad V(x) = \left. \frac{\partial^2 \mathcal{U}}{\partial \Phi^2} \right|_{\Phi=\phi(x)} . \tag{4.3}$$

The eigenfunctions of this operator have the general form $e^{2\pi i n \tau/\beta} \psi_m(x)$ where $\beta = 1/kT$ and

$$H\psi_m = [-\nabla^2 + V(x)]\psi_m(x) = \epsilon_m^2 \psi_m(x) \ . \tag{4.4}$$

The corresponding eigenvalues are

$$\left[\frac{2\pi n}{\beta} \right]^2 + \epsilon_m^2 \ . \tag{4.5}$$

The path-integral expression for the partition function is

$$Z = e^{-\mathcal{F}/kT} = \int \mathcal{D}\Phi \, e^{-S[\phi]} \ . \tag{4.6}$$

If $\phi(x)$ is an isolated stationary point (except for zero modes which we discuss below) then we may approximate Z by

$$Z \simeq Z_0 + Z_1 \ , \tag{4.7}$$

where Z_0 is the contribution to Z from the (perturbative) vacuum solution $\Phi = \phi_0$. In the Gaussian (semiclassical) limit

$$Z_1 = e^{-\beta E[\phi]} \det{}^{-1/2} \mathcal{G} = e^{-\beta F_1} \ , \tag{4.8}$$

where formally

$$F_1 = E[\phi] + \frac{1}{2\beta}\,\mathrm{Tr}(\ln \mathcal{G}) \ . \tag{4.9}$$

The trace (4.9) is over all eigenvalues labeled in (4.5) by n and m. For fixed ϵ_m the contribution of the mode ψ_m to (4.9) is just that of a simple harmonic oscillator with frequency ϵ_m (provided $\epsilon_m^2 > 0$).

This contribution to (4.9) is

$$-\frac{1}{\beta}\ln\left[\sum_{l=0}^{\infty} e^{-(l+1/2)\beta\epsilon_m}\right] = \frac{1}{\beta}\ln\left[2\sinh\left[\frac{\beta\epsilon_m}{m}\right]\right]$$
$$= \frac{\epsilon_m}{2} + \frac{1}{\beta}\ln(1-e^{-\beta\epsilon_m}) \ . \tag{4.10}$$

The first term is the zero-point energy of the oscillator while the second is the finite-temperature contribution to the free energy coming from the mode m.

Now if the classical solution is well localized and approaches the vacuum solution ϕ_0 fast enough as $|x| \to \infty$, H will have a continuous spectrum and the corresponding scattering solutions ψ will have the asymptotic forms (for $d=1$)

$$\psi_{\epsilon(p)}(x) \to A^{(\pm)}(p)e^{ipx} + B^{(\pm)}(p)e^{-ipx} \tag{4.11}$$

as $x \to \pm\infty$. In fact we shall be interested specifically in the case that the coefficients $B^{(\pm)}$ vanish identically. In one dimension this occurs only for some very special potentials. If $d>1$ and $\phi(x)$ is spherically symmetric, we need to reformulate the scattering problem in terms of radially outgoing partial waves for unit flux incident from the left.[20] For such scattering wave solutions the coefficients analogous to $B^{(\pm)}$ always vanish.

Thus, all important information about the scattering solutions resides in the phase shift $\delta(p)$ defined by the transmission coefficient corresponding to unit flux incident from the left in the Schrödinger Eq. (4.4):

$$\delta(p) = \arg\left[\frac{A^{(+)}(p)}{A^{(-)}(p)}\right] \ . \tag{4.12}$$

By differentiating (4.4) with respect to ϵ it is not difficult to show that

$$\mathrm{Tr} f(H) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(\epsilon^2(p))\psi_{\epsilon(p)}(x)\psi^*_{\epsilon(p)}(x)$$
$$= \int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{1}{2\epsilon(p)} f(\epsilon^2(p)) W\left[\frac{d\psi_{\epsilon(p)}}{d\epsilon},\psi^*_{\epsilon(p)}\right]_{-\infty}^{\infty} , \tag{4.13}$$

where $W[u,v]_a^b$ is the Wronskian of the functions u,v evaluated between a and b. As $a \to -\infty$, $b \to +\infty$, (4.4) and (4.11)–(4.13) with $B^{(\pm)}=0$ give

$$\mathrm{Tr} f(H) = (b-a)\int_{-\infty}^{\infty}\frac{dp}{2\pi} f(\epsilon^2(p))$$
$$+ \int_{-\infty}^{\infty}\frac{dp}{2\pi} f(\epsilon^2(p))\frac{d\delta(p)}{dp} \tag{4.14}$$

for any function $f(\epsilon^2)$. In deriving Eq. (4.14) we have used the fact that $d\epsilon^2 = dp^2$, which holds provided $\phi(x) \to \phi_0$ fast enough as $|x| \to \infty$.

We now apply (4.14) to (4.9) and (4.10), i.e., we take

$$f(\epsilon^2) = \frac{\epsilon}{2} + \frac{1}{\beta}\ln(1-e^{-\beta\epsilon}) \tag{4.15}$$

and subtract the same quantity for the vacuum case, $\Phi = \phi_0$. Then the linear volume divergence in (4.14) cancels and we obtain

$$\mathrm{Tr}[f(H) - f(H_0)] = \frac{1}{\pi}\int_0^{\infty} dp\, f(\epsilon^2(p))\frac{d\delta(p)}{dp} \ . \tag{4.16}$$

It is clear that formula (4.8) breaks down if $\mathcal{G}$ has zero or negative eigenvalues. This can only be the case if H does. The zero eigenmodes of H are easy to treat, since they are just harmonic-oscillator modes with zero oscillator frequency, i.e., free modes. The contribution to Z_1, of each free mode is thus just the factor

$$\int \frac{dq\,dp}{2\pi\hbar} e^{-p^2/2kT} = \frac{1}{\hbar}\left[\frac{kT}{2\pi}\right]^{1/2}\int dq \ , \tag{4.17}$$

where q is the coordinate in this direction and p the corresponding canonical momentum (we take the mass to be unity). That is, the projection of the general linear fluctuation $\delta\Phi$ onto its zero-mode subspace is given

$$(\delta\Phi)_0 = (\Phi - \phi)_0 = g\,\psi_0(x)q(t) \tag{4.18}$$

and a factor of the coupling constant $g \sim \hbar^{1/2}$ has been exhibited explicitly. If $(\delta\Phi)_0$ can be related to some symmetry of the action S, not shared by the solution $\phi(x)$, then

$$(\delta\Phi)_0 = \frac{\partial\phi}{\partial a}\delta a \ , \tag{4.19}$$

where a is the parameter that breaks the symmetry (such as translational invariance). If $\psi_0(x)$ is normalized by

$$\int d^dx\, |\psi_0(x)|^2 = 1 \ , \tag{4.20}$$

we make use of Eqs. (4.18)–(4.20) to secure

$$\int dq = \Delta q = \frac{\Delta a}{g}\left[\int d^dx\left|\frac{\partial\phi}{\partial a}\right|^2\right]^{1/2} . \tag{4.21}$$

Hence, the zero-mode factor (4.17) is

$$\frac{\Delta a}{g}\left[\frac{kT}{2\pi\hbar}\int d^dx\left|\frac{\partial\phi}{\partial a}\right|^2\right]^{1/2} . \tag{4.22}$$

If in addition H has a negative eigenvalue $\epsilon_-^2 = -|\epsilon_-^2| < 0$ then $\mathcal{G}$ does as well. If

$$kT > \frac{|\epsilon_-|}{2\pi} \ , \tag{4.23}$$

$\mathcal{G}$ has only one negative eigenvalue. Then we may interpret the negative mode as giving rise to an imaginary part in Z, according to the prescription

$$\frac{1}{2\sinh\left[\dfrac{\beta\epsilon_-}{2}\right]} \to \frac{1}{2}\frac{1}{2i\sin\left[\dfrac{\beta|\epsilon_-|}{2}\right]} \ . \tag{4.24}$$

The additional factor of $\frac{1}{2}$ in (4.24) arises from the distortion of the non-Gaussian contour over half of its range.[19] This means the free energy $\mathcal{F}$ defined by (4.6) picks up an imaginary part from the unstable stationary point $\Phi=\phi(x)$:

$$\mathrm{Im}\mathcal{F} = -\frac{1}{\beta}\,\mathrm{Im}\ln Z \simeq -\frac{1}{\beta}\frac{1}{Z_0}\,\mathrm{Im}Z_1 \; . \tag{4.25}$$

Reassembling the various contributions (4.16), (4.22), and (4.24), we find

$$\mathrm{Im}\mathcal{F} = +\frac{1}{4\beta}\frac{1}{\sin\left[\dfrac{\beta|\epsilon_-|}{2}\right]}\mathcal{N}\mathcal{V}e^{-\beta(F_1-F_0)} \; , \tag{4.26}$$

where

$$F_1-F_0 = E[\phi]-E[\phi_0]+\mathrm{Tr}[f(H)-f(H_0)] \tag{4.27}$$

and $\mathcal{N}\mathcal{V}$ is the product of the normalized volume factors (4.22) for each zero mode.

According to Langer,[19] the imaginary part of the free energy function $\mathcal{F}$ is to be interpreted as giving rise to a decay rate of the perturbative vacuum ϕ_0 according to

$$\Gamma_0 = \frac{|\kappa|}{\pi kT}\,\mathrm{Im}\mathcal{F} \; , \tag{4.28}$$

where κ is a damping constant: namely, the real-time rate of decay of the configuration $\Phi=\phi(x)$ in the heat bath. All the dynamics of the heat bath are buried in this one quantity. For a weakly coupled theory the interaction with the heat bath does not affect the decay of the configuration $\phi(x)$, which is determined purely by its negative eigenvalue ϵ_-^2. That is, if $g^2 \ll 1$ we are always in the underdamped limit and

$$|\kappa| = |\epsilon_-| \; . \tag{4.29}$$

Hence,

$$\Gamma_0 = \frac{1}{4\pi}\frac{|\epsilon_-|}{2\pi\sin\left[\dfrac{\beta|\epsilon_-|}{2}\right]}\mathcal{N}\mathcal{V}e^{-\beta(F_1-F_0)} \; . \tag{4.30}$$

To recapitulate, the weak-coupling limit ensures the validity of the Gaussian approximation used in deriving this formula and also leads to the weak damping limit (4.29). Other than $g^2 \ll 1$ the only additional assumption made in deriving (4.30) is that the stationary point $\Phi=\phi(x)$ is isolated, except for zero modes related to symmetries in the theory. If the solution is not isolated in this sense, there will be additional "accidental" zero modes of $\mathcal{G}$ which will cause (4.30) to break down. This is just what happens as $kT \to |\epsilon_-|/2\pi$, for instance. For temperatures not satisfying (4.23) the static solution $\Phi=\phi(x)$ does not contribute to Im $\mathcal{F}$ or the decay rate Γ_0, which are dominated by *nonstatic*, instantonlike configurations. It is in this way that the high-temperature analysis matches onto the low-temperature instanton analysis. A quantitative method of implementing this matching has been described by Affleck.[19]

We have elected to present this path-integral derivation of the rate because of its compactness. However, there is no need to resort to path integrals or the analytic continuation in the negative mode direction implied in (4.24). Equation (4.30) could have been derived, as in Langer's original paper, by consideration of the probability flow in one direction over the saddle point $\Phi=\phi(x)$. The main point is that (4.30) is a formula based solely on *classical statistical mechanics* and correctly accounts for entropy effects through the free energy function F_1-F_0. If there were something pathological about the sphaleron, such as a large entropy suppression it would have to show up in the expression (4.27). We turn in the next section to an explicit evaluation of (4.26) and (4.27) for the sphaleron solution found in Sec. III.

V. THE FREE ENERGY OF THE SPHALERON

Having found the sphaleron solution of the model in Sec. III we proceed now with the calculation of the one-loop corrections to it by an analysis of the small fluctuations about the classical solution. If there is a suppression due to phase-space or entropy effects, it should show up in the free energy function given by the finite-temperature loop expansion. We begin by parametrizing the fluctuations in a convenient way. Let

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{1+u^2}}(\sin(\xi_{\mathrm{sph}}+v), u, -\cos(\xi_{\mathrm{sph}}+v)) \; . \tag{5.1}$$

Substituting this form for $\hat{\mathbf{n}}$ into the action functional and expanding to quadratic order in (u,v) gives the desired small fluctuation operators. The eigenvalues are determined by solving

$$\begin{aligned} H_1 u &\equiv \left[-\frac{d^2}{dx^2}+\omega^2(1-6\,\mathrm{sech}^2\omega x)\right]u = \epsilon^2 u \; , \\ H_2 v &\equiv \left[-\frac{d^2}{dx^2}+\omega^2(1-2\,\mathrm{sech}^2\omega x)\right]v = \epsilon^2 v \; . \end{aligned} \tag{5.2}$$

It is a special feature of the present model that these equations are just Schrödinger's equations in the Rosen-Morse potentials, $U_0\,\mathrm{sech}^2\omega x$, whose eigenfunctions are known explicitly. Each of the two scattering potentials in (5.2) satisfies all the conditions postulated in the general discussion of the previous section. It is amusing to note as well that the two potentials are supersymmetric partners so that their spectra are closely related.

The first operator H_1 describes fluctuations in $\hat{\mathbf{n}}$ which are perpendicular to the sphaleron. This operator has exactly one negative eigenvalue, namely, $\epsilon_-^2 = -3\omega^2$, associated with the fact that sliding the sphaleron loop on the sphere in the η or u direction must decrease the energy. This we knew already. There is one zero eigenvalue associated with the ability to rotate the sphaleron solution about the $\hat{\mathbf{n}}_3$ axis without changing its energy. The angle that $\hat{\mathbf{n}}_{\mathrm{sph}}$ makes with the x_1 axis is the corresponding parameter a in (4.19) for this zero mode. All the remaining eigenvalues are in the continuum above ω^2.

The second operator, H_2 describes fluctuations in $\hat{\mathbf{n}}$ along the direction of the sphaleron (i.e., η remains fixed at $\pi/2$). Its lowest eigenvalue is zero with the corresponding mode associated with translation of the sphaleron position. All other eigenvalues are positive. The one negative-mode and two zero-mode eigenfunctions are easy to find explicity

$$u_- = \operatorname{sech}^2(\omega x) ,$$
$$u_0 = \sin\xi_{\rm sph} = 2\operatorname{sech}(\omega x)\tanh(\omega x) , \tag{5.3}$$
$$v_0 = \frac{d\xi_{\rm sph}}{dx} = 2\omega\operatorname{sech}(\omega x) .$$

For the positive spectral continuum of each operator above ω^2, we evaluate the finite-temperature determinants by relations (4.12) and (4.16). For the potentials in H_1 and H_2, the transmission coefficients are known and they lead to

$$\frac{d\delta_2(p)}{dp} = -\frac{2\omega}{p^2+\omega^2}$$
and
$$\frac{d\delta_1(p)}{dp} = -\frac{2\omega}{p^2+\omega^2} - \frac{4\omega}{p^2+4\omega^2} , \tag{5.4}$$

where $p^2+\omega^2=\epsilon^2(p)$. We may now apply (4.16) and sum over the two orthogonal sets of modes for H_1 and H_2, respectively. The zero-point energy contributions from the two operators yield the logarithmically divergent integral

$$\frac{-2\omega}{\pi}\int_0^\infty dp\sqrt{p^2+\omega^2}\left[\frac{1}{p^2+\omega^2}+\frac{1}{p^2+4\omega^2}\right] .$$

Introducing an ultraviolet cutoff Λ and defining the renormalized coupling constant by

$$\frac{1}{g^2_{\rm ren}(\omega)} = \frac{1}{g_0^2} - \frac{1}{2\pi}\ln(\Lambda/\omega) , \tag{5.5}$$

we observe that this zero-point contribution may be absorbed into the classical sphaleron energy (3.6), provided that we replace the bare $1/g^2$ appearing there by the renormalized running coupling evaluated at ω: $1/g^2(\omega)$. Then we are left with only the second term of (4.16), which gives the finite-temperature corrections to the sphaleron's statistical weight. This is summarized succintly by the function

$$h(a) = \frac{-4a}{\pi}\int_0^\infty dx\left[\frac{1}{x^2+a^2}+\frac{1}{x^2+4a^2}\right] \times \ln(1-e^{-\sqrt{x^2+a^2}}) \geq 0 , \tag{5.6}$$

where $a=\omega/T$. The limiting forms of this function for $a\to\infty$ and $a\to 0$ are, respectively,

$$h(a) \to \frac{5}{\sqrt{2\pi a}}e^{-a}$$

and

$$h(a) \to -3\ln a + \frac{4}{\pi}(\arctan a + \tfrac{1}{2}\arctan 2a - 3a)\ln a - C + O(a) ,$$

where

$$C = \frac{2}{\pi}\int_0^\infty dx\left[\frac{1}{x^2+1}+\frac{1}{x^2+4}\right]\ln(x^2+1) = 2.251\,585\,2 . \tag{5.7}$$

Turning to the evaluation of the zero-mode factors $\mathcal{N}\mathcal{V}$ required, we find that the mode u_0 contributes the factor

$$\frac{4}{g}\left[\frac{\pi kT}{3\omega}\right]^{1/2} \tag{5.8}$$

since the range in the parameter corresponding to a in the general formula (4.22) is 2π for rotations about the $\hat{\mathbf{n}}_3$ axis. The translational zero mode contributes the factor

$$\frac{2L}{g}\left[\frac{\omega kT}{\pi}\right]^{1/2} . \tag{5.9}$$

We are now in a position to give a closed-form answer for the rate per unit volume L of thermal activation over the energy barrier between two topologically distinct vacuum configurations, the height of which is the classical sphaleron energy, $E_{\rm sph}=8\omega/g^2$. The result of substituting (5.4)–(5.9) into the general formula (4.30), derived in the previous section is

$$\frac{\Gamma_0}{L} = \frac{2}{\pi g^2}\frac{\omega T}{\sin\left[\frac{\sqrt{3}\omega}{2T}\right]}\exp\left[-\frac{8\omega}{g^2T} - h\left[\frac{\omega}{T}\right]\right] , \tag{5.10}$$

where we set $\hbar=k=1$.

This transition rate does not lead to any violation of chiral fermion number unless there is an initial asymmetry in fermion number. We may introduce such an asymmetry by adding a chemical potential to the Hamiltonian[21]

$$H \to H - \mu N_{\rm CS} , \tag{5.11}$$

where $N_{\rm CS}$ is the Chern-Simons number introduced in Eq. (2.19). The vacuum state which is unique in the gauge-invariant description, $\hat{\mathbf{n}}_V=(0,0,-1)$ corresponds to an infinitely degenerate set of states labeled by the topological winding number $N_{\rm CS}$. This quantity is not gauge invariant but changes in it are.

We take $\mu/T \ll 1$ so that we may expand in this small quantity in all that follows. First-order perturbation theory then gives $-(\mu/T)\Gamma_0$ for the transition rate from a state with $N_{\rm CS}=1$ to one with $N_{\rm CS}=0$, i.e.,

$$\frac{d\langle N_{\rm CS}\rangle}{dt} = -\frac{\mu}{T}\Gamma_0 . \tag{5.12}$$

The chemical potential induces the asymmetry in N_5 given by

$$\langle N_5\rangle = \frac{4L}{\pi}\mu \tag{5.13}$$

to first order in μ. Substituting this relation for μ into Eq. (5.12) and using (5.10) and (2.18) gives finally

$$\frac{d\langle N_5\rangle}{dt}=-\Gamma_5\langle N_5\rangle \tag{5.14}$$

with

$$\Gamma_5=\frac{\pi}{2T}\frac{\Gamma_0}{L}$$
$$=\frac{\omega}{g^2}\frac{1}{\sin\left[\frac{\sqrt{3}\omega}{2T}\right]}\exp\left[-\frac{8\omega}{g^2(\omega)T}-h\left[\frac{\omega}{T}\right]\right]. \tag{5.15}$$

In the temperature range where $T \gg \omega$ so that the sphaleron-induced transitions are dominant compared to those caused by instantons, but $T \ll \omega/g^2$ so that the semiclassical expansion around a single sphaleron solution is justified, we may employ relations (2.22) and (5.7) to obtain

$$\Gamma_5=K\frac{\omega}{g^2(T)}\left[\frac{\omega}{T}\right]^2 e^{-8\omega/g^2(T)T} \tag{5.16}$$

with

$$K=\frac{2}{\sqrt{3}}e^C=10.971\,766 \tag{5.17}$$

and $g^2(T)$ the temperature-dependent running coupling constant evaluated at the temperature T. Thus, the initial asymmetry (5.13) decays exponentially with a rate that is considerably greater than the instanton inferred rate, at temperatures large compared to ω.

VI. THE HIGH-TEMPERATURE LIMIT: DISCUSSION

Equations (5.14)–(5.17) are the principal results of this paper. They show that the rate of chiral fermion-number violation in the O(3) model is unsuppressed at temperatures high compared to the symmetry-breaking scale ω. From the point of view of the index theorem, this conclusion may seem quite surprising. Fermion-number violation must be accompanied by topological winding number on the sphere, but we knew right from the outset that in the $Q=1$ sector the Euclidean action satisfies the bound (2.4). Does it follow that all such fermion-number-violating processes are suppressed by at least one factor of $\exp(-8\pi/g^2)$?

The critical element in this objection is the bound on the Euclidean action, which is positive semidefinite. The process described by Eq. (5.14) is a classical thermal process (not a quantum vacuum to vacuum transition) and the rate is a real time rate. In real time the action is not positive semidefinite and the Euclidean bound need not apply. Stated a different way, topology changing amplitudes in real time need not correspond to topology changing amplitudes in Euclidean time. The decomposition of the path integral into discrete sectors with distinct topological winding number Q may be carried out in either real or imaginary time, but this decomposition transforms quite nontrivially under analytic continuation. In particular, there is no reason why a given Q sector in Euclidean time must transform into the *same* Q sector in real time. The simple pendulum or even the free particle on a ring illustrate this point very clearly, quite apart from any basic differences they may have vis-à-vis more realistic field theories. If the arguments based on bounds of the Euclidean action were generally valid, even the unsuppressed, real-time response of a free particle on a ring in a thermal bath would be incorrect.[22] Clearly, this is not the case.

It is instructive to see explicitly what goes wrong with the argument based on the bound of the Euclidean action in the case of the simple pendulum. By forming the quantity

$$\tfrac{1}{2}\int d\tau[\dot{\theta}-2\omega\sin(\theta/2)]^2$$

we easily deduce that the Euclidean action (1.2) obeys the bound

$$S\geq 2\omega\int_0^\beta d\tau\,\dot{\theta}\sin(\theta/2)\ . \tag{6.1}$$

Periodicity in Euclidean time β fixes the limits of the integration in (6.1). However, if this integral over τ is converted to one over θ, we cannot immediately infer what the correct limits in θ are. These depend on the turning points of the motion in the potential and are complicated functions of the temperature. If, nonetheless, we were to *insist* upon the limits $0<\theta<2\pi$, corresponding to a vacuum-to-vacuum transition, we would then obtain a bound similar to (2.6): namely,

$$S\geq 8\omega\ . \tag{6.2}$$

Of course, this is incorrect at high temperatures where the "sphaleron" solution dominates, since the range of θ traversed in *Euclidean* time by the trajectory from $\theta=0$ to $\theta=2\pi$ goes to *zero* at sufficiently high temperatures. Winding certainly occurs in real time at high temperatures, but the Euclidean bound is irrelevant, since the process is classically allowed. Repeating the analysis for the σ model, or non-Abelian gauge theory in 3 + 1 dimensions, we see that the argument that the Euclidean action is bounded from below is equally flawed. One has simply *assumed* in this argument that the correct boundary conditions on the field variables are those which yield $Q=1$ in *Euclidean* time. At zero temperature the argument is correct, since the system is in its ground state and must make a full vacuum-to-vacuum transition in Euclidean time. But at finite temperatures, the fields are thermally excited and may be very far from the perturbative vacuum. Then the probability for traversal of most, or all of the trajectory from one vacuum sector to another in real time may be quite large. In that case, as in the pendulum example, $Q=1$ in *real* time, but this implies no bound on the Euclidean action and no suppression of the rate of violation of the associated anomalous charge. Such a situation has been anticipated by previous authors.[23]

As mentioned in the Introduction, there is another objection to unsuppressed fermion-number violation that is often raised. It is that something very strange happens to the sphaleron at very high temperatures, and in the symmetric theory where no sphaleron solution exists at all, it cannot possibly contribute to fermion-number violation. If we examine the loop expansion about the sphaleron in the limit of large T, we observe that the fluctuations dominate the classical energy for $T \gg \omega/g^2$. Clearly, it is no longer a good approximation to expand around a single sphaleron configuration at these temperatures, and the loop expansion fails. Indeed, at these very high temperatures, the symmetry-breaking term we added should become negligible, and the model reverts to the *symmetric* O(3) model. Do we then recover the instanton suppression of the rate at these very high temperatures?

Again, the answer is no. Exactly how this takes place is quite interesting. Even though the loop expansion breaks down, we can still analyze what happens at high temperatures by using the fact that the σ model collapses to that of a quantum rotor for $T \gg \omega$ (Ref. 16). In the intermediate range, $\omega \ll T \ll \omega/g^2$, the wave function of the rotor is sharply peaked at $\hat{\mathbf{n}}_V=(0,0,-1)$. In fact, one may carry out the semiclassical approximation scheme on rotor winding-number transitions in this temperature range and show that it precisely duplicates the sphaleron computed rate, (5.10). As T is increased beyond ω/g^2 the system becomes more and more disordered. When the thermal coherence length (also the moment of inertia of the rotor) $1/g^2T$ becomes smaller than the size of the coherent sphaleron configuration ω^{-1}, such configurations become irrelevant. For $T \gg \omega/g^2$, the rotor becomes free and its wave function uniform on S^2. This corresponds to the fact that the state of the field theory is completely disordered. In this limit, it is obvious that winding-number transitions are unsuppressed. It is not even possible to define the fermion number in this limit, since the fermion-number-violating processes are now in equilibrium with all others. If we were artificially to introduce into the system a region where the field $A_1(x)$ has a coherent twist, so that $N_{\rm CS}=1$, say, it would dissappear on a time scale of order $1/g^2T$. This should be clear from elementary uncertainty principle arguments applied to the rotor. For if we try to localize the field in a range $\Delta\theta$, angular momentum components of order $L \sim \hbar/\Delta\theta$ are thereby induced. Since $L \sim \Delta\theta/(g^2T\Delta t)$ the time scale for θ to spread to values of order unity is $\hbar^2/g^2T$, i.e., it is purely the small inertia of the rotor that determines the coherence time, as the barrier between regions of different winding number has become negligible.

Thus, the σ model present an explicit example of how the bounds on fermion-number-violating processes, based on Euclidean instanton configurations are evaded. Instead, a physically appealing picture emerges in which the field theory becomes highly disordered by thermal fluctuations and fermion-number asymmetry cannot persist at high temperatures. The relevant coherence time scale is given by the nonperturbative mass g^2T.

These considerations should apply to the Weinberg-Salam theory, modified only by the fact that the gauge theory has a true phase transition, unlike the (1 + 1)-dimensional model. As the temperature is raised above M_W the sphaleron contributions become important relative to instantons for the first time. As T approaches T_c, the fluctuations become very large as the system begins to disorder and approach symmetry restoration. Correspondingly, the fluctuation expansion about the sphaleron breaks down.[9] Notice that this occurs before the sphaleron energy scale is reached. For $T > T_c$, the Higgs field no longer plays a vital role, as the gauge field becomes highly disordered. On the basis of the O(3) σ model results obtained here, we would expect the relevant configurations for topological winding to be those dominated by large (i.e., nonperturbative) magnetic field strengths on the scale of the magnetic screening length in the high-temperature plasma, of order $1/g^2T$. The contributions of these configurations with large magnetic field energy but small electric field energy (*not* instantons) cannot be calculated by any known analytic method. Nevertheless there is no reason to believe that the O(3) model leads us astray in the high-temperature phase: the rate for baryon- and lepton-number-violating processes would be entirely unsuppressed.

One application of these ideas is to early Universe cosmology. The unsuppresed rate of baryon-plus lepton-number-violating processes wipe out any initial asymmetry in this quantity, generated, for example, by the simplest SU(5) grand unified theory (GUT). However $B-L$ is not affected by the process, since the anomalies cancel in the difference current. Hence, one can keep the GUT scenario for observed B and L asymmetry provided only that the theory generates a $B-L$ asymmetry as well. If this is not the case it seems to be quite difficult to regenerate any asymmetry after the temperature is as low as M_W (Ref. 9), though we believe that results of these preliminary studies of the issue are not completely conclusive.

The other principal question is whether these B- and L-violating processes can be observed at energies typical of the Superconducting Super Collider (SSC). What is required here theoretically is a sensible way to estimate the matrix elements of overlap between the initial *pp* beam and the coherent magnetic gauge field configuration necessary to jump over the barrier.[5,23] If the cross section does indeed rise sharply with beam energy as we might expect from the Boltzmann factor in the sphaleron analysis, it might be possible to cut on a very restrictive subset of scattering events with only a few particles in the final state. Then calorimetry and particle identification might be enough to ensure no baryons were missed. A few events in this restricted set with nonconserved baryon number would be a quite striking confirmation of these ideas. However, if what is required is a concentration of energy of order $E_{\rm sph}$ in a region of size M_W^{-1} to excite the relevant mode, the process may be too improbable for even the SSC to detect.

Finally, the remarks made above about vacuum disorder and thermal coherence lengths apply equally well to QCD. In that context the question becomes whether thermal effects induce large chiral charge-changing processes in QCD. What effects of such unsuppressed chirality-violating rates would be on the value of the θ parameter in QCD or on processes involving axions in the early Universe are intriguing issues worth pursuing. It would be interesting to learn what light can be shed on these issues by lattice Monte Carlo techniques in the high-temperature limit of an unbroken gauge theory such as QCD (Ref. 24).

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APPENDIX

In this appendix we discuss the behavior of the broken-O(3) σ model at finite temperature. In particular, we shall see that although the model has no phase transition in the strict sense, there is a continuous change in behavior from that characteristic of symmetry breaking at low temperature to asymptotic full symmetry restoration at very high temperature. The fundamental quantity we consider is the Euclidean generating functional $W[j]$ given by

$$e^{-W[j]}=\int \{\mathcal{D}\hat{\mathbf{n}}\}\delta(\hat{\mathbf{n}}^2-1) \times \exp\left[-S_0-\int d^2x\, j(\hat{\mathbf{n}}_3+1)\right] . \quad (A1)$$

The free energy density of the model with symmetry breaking is obtained by setting

$$j=\omega^2/g^2$$

and computing

$$\mathcal{F}(\omega,T,g^2)=\frac{T}{L}W[j=\omega^2/g^2] . \quad (A2)$$

Consider now the expectation value $\langle\hat{\mathbf{n}}_3\rangle_T$ in thermal equilibrium

$$\langle\hat{\mathbf{n}}_3\rangle_T=-1+g^2\frac{d\mathcal{F}}{d\omega^2} . \quad (A3)$$

For sufficiently low temperatures the symmetry-breaking term dominates and the system is near its ground state, $\langle\hat{\mathbf{n}}_3\rangle=-1$. This state has two degenerate massive excitations, with mass ω. Thus, there are no infrared problems and perturbation theory is valid. To calculate the one-loop contribution to $\mathcal{F}$, parametrize the $\hat{\mathbf{n}}$ field as in Eq. (6.1) of the text with $\xi=0$ replacing $\xi_{\rm sph}$. Then, to lowest order in g, $\mathcal{F}$ is just the free energy of two uncoupled massive particles in one spatial dimension. We find

$$\mathcal{F}=\frac{T}{L}\,\mathrm{Tr}\ln(-\partial^2+\omega^2) =\mathcal{F}(T=0)+2T\int\frac{dp}{2\pi}\ln(1-e^{-\beta\sqrt{p^2+\omega^2}}) , \quad (A4)$$

where

$$\mathcal{F}(T=0)=\int\frac{dp}{2\pi}\sqrt{p^2+\omega^2} \quad (A5)$$

is the zero-point-energy contribution at zero temperature. This quantity contains an irrelevant quadratic divergence and an ω-dependent logarithmic divergence. The logarithmically divergent counterterm must be chosen so that

$$\langle\hat{\mathbf{n}}_3\rangle_{T=0}=-1 . \quad (A6)$$

With this renormalization $\mathcal{F}(T=0)$ becomes a constant which may be taken to vanish. Hence, from (A3) and (A4) we obtain

$$\langle\hat{\mathbf{n}}_3\rangle_T=-1+g^2\int_{-\infty}^{\infty}\frac{dx}{\pi}\frac{1}{\sqrt{x^2+\omega^2\beta^2}} \times\frac{1}{\exp(\sqrt{x^2+\omega^2\beta^2})-1} +O(g^4) . \quad (A7)$$

The low- and high-temperature forms of (A7) are

$$\langle\hat{\mathbf{n}}_3\rangle_T\to-1+\frac{g^2}{\pi}e^{-\omega/T}(1+T/\omega+\cdots)+O(e^{-2\omega/T}) , \quad T\ll\omega , \quad (A8)$$

and

$$\langle\hat{\mathbf{n}}_3\rangle_T\to-1+g^2\left[\frac{T}{2\omega}-\frac{1}{2\pi}\ln(T/\omega)+C'+O\left[\frac{\omega}{T}\right]\right] , \quad \omega\ll T\ll\frac{\omega}{g^2} , \quad (A9)$$

where

$$C'=\frac{1}{\pi}\int_0^{1/2}\frac{dx}{x}\left[-\frac{1}{2}+\frac{1}{x}-\frac{1}{e^x-1}\right] +\frac{1}{\pi}\int_{1/2}^{\infty}\frac{dx}{x}\left[\frac{1}{x}-\frac{1}{e^x-1}\right] =0.310\,968\,3 . \quad (A10)$$

As expected perturbation theory is excellent up to $T=\omega$. Above the symmetry-breaking scale, the thermal bath begins to excite the massive modes and $\langle\hat{\mathbf{n}}_3\rangle_T$ grows linearly in T. At T of order $2\omega/g^2$ the corrections to $\langle\hat{\mathbf{n}}_3\rangle_T$ become of order 1 and perturbation theory fails completely.

Fortunately, a complementary expansion to the perturbation series exists in the model. If $T\gg\omega$, we may expand the field variables $\hat{\mathbf{n}}$ in a Fourier series in Euclidean time and neglect all but the zero-frequency mode. Then the partition function (A1) collapses to

$$\int\{\mathcal{D}\hat{\mathbf{n}}\}\delta(\hat{\mathbf{n}}^2-1) \times\exp\left\{-\frac{1}{g^2T}\int dx\left[\frac{1}{2}\left[\frac{d\hat{\mathbf{n}}}{dx}\right]^2+\omega^2(1+\hat{\mathbf{n}}_3)\right]\right\} . \quad (A11)$$

This is the same as the Euclidean path integral (at zero temperature) of a particle constrained to move on a sphere in the potential

$$V=\frac{\omega^2}{g^2T}(1+\hat{\mathbf{n}}_3) . \quad (A12)$$

In the intermediate-temperature range, where the potential dominates over the "kinetic" term, the particle remains localized near the minimum of the potential. The wave function of this "particle" is Gaussian in θ with a width $\omega/2g^2T$. Thus, taking account of the two independent harmonic oscillators near $\theta=0$, we find that

$$\langle \hat{\mathbf{n}}_3 \rangle_T \rightarrow -\langle \cos\theta \rangle \simeq -1+\frac{\langle \theta^2 \rangle}{2} = -1+\frac{g^2T}{2\omega} \tag{A13}$$

in this approximation. This matches smoothly onto the form (A9) from perturbation theory in the full field theory.

For higher temperatures the kinetic term in the effective quantum-mechanical problem becomes more and more important, and the wave function explores more and more of the full S^2. For $T \gg 2\omega/g^2$ the potential becomes negligible and the effective system may be regarded as a *free* quantum rotor with moment of inertia $1/g^2T$. Hence, the wave function of the lowest eigenstate of the system becomes uniform and constant over S^2, so that clearly $\langle \hat{\mathbf{n}} \rangle_T \rightarrow 0$.

The approach to zero may be found by treating the potential as a small perturbation. We find

$$\langle \hat{\mathbf{n}}_3 \rangle_T \rightarrow -\frac{2}{3}\left[\frac{\omega}{g^2T}\right]^2 \quad \text{as } T\rightarrow\infty \; . \tag{A14}$$

Putting the low-, intermediate-, and high-temperature types of behavior together, we arrive finally at a complete picture of how the expectation value $\langle \hat{\mathbf{n}}_3 \rangle_T$ in the original field theory behaves over all temperatures. This is illustrated in Fig. 3. The crossover region around $T=2\omega/g^2$ may be described quantitatively, if desired by solving the Schrödinger equation

$$\left[-\frac{g^2T}{2}\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}\right]+\frac{\omega^2}{g^2T}(1-\cos\theta)\right]\psi_0(\theta) = E_0\psi_0(\theta) \tag{A15}$$

for the lowest eigenstate E_0 and computing

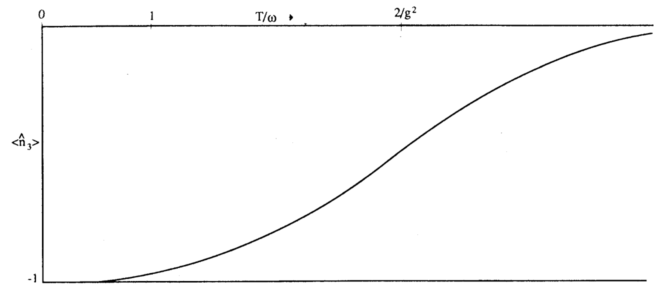

FIG. 3. The schematic behavior of the expectation value $\langle \hat{\mathbf{n}}_3 \rangle_T$ as a function of temperature T. For small T, $\langle \hat{\mathbf{n}}_3 \rangle_T$ is given by (A7) and (A8). In the intermediate range, $\omega \ll T \ll \omega/g^2$, $\langle \hat{\mathbf{n}}_3 \rangle_T$ rises linearly with T according to (A9) or (A13). For $T \gg \omega/g^2$, $\langle \hat{\mathbf{n}}_3 \rangle_T$ approaches zero according to (A14), corresponding to restoration of the full O(3) group symmetry in the limit $T\rightarrow\infty$.

$$\langle \hat{\mathbf{n}}_3 \rangle_T = \frac{-\int_0^{\pi}\cos\theta\sin\theta\, d\theta\, |\psi_0(\theta)|^2}{\int_0^{\pi}\sin\theta\, d\theta\, |\psi_0(\theta)|^2} \; . \tag{A16}$$

The important feature of this analysis is that perturbation theory around the symmetry-broken ground state $\hat{\mathbf{n}}_V=(0,0,-1)$, valid at low temperatures, matches smoothly onto the high-temperature limit where $\langle \hat{\mathbf{n}}_3 \rangle_T \rightarrow 0$. Thus, the full O(3) symmetry is restored asymptotically as $T\rightarrow\infty$ though there is no sharp phase transition at any finite temperature. The broken-O(3) model behaves qualitatively like a spontaneously broken local gauge theory in $3+1$ dimensions, when account is taken of the obvious quantitative difference that Coleman's theorem forbids a true phase transition in $1+1$ dimensions. Also, at high temperatures when $\langle \hat{\mathbf{n}}_3 \rangle_T \rightarrow 0$ it is clear that the system will not support coherent field configurations on time scales longer than $1/g^2T$. Therefore, the rate for transitions from $N_{CS}=1$ to $N_{CS}=0$ is of order g^2T in this high-temperature limit.

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