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Mass splittings and the finiteness problem of mass shifts in the type-II superstring at one-loop order

Kaoru Amano and Akihiko Tsuchiya
Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152, Japan
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The on-shell two-point amplitudes in the type-II superstring are constructed for some states in the NS-NS (where NS denotes Neveu-Schwarz) sector in the old covariant formalism. The result indicates that the second and higher mass levels split at one-loop corrections. Also it is noted that the integral representations do not readily give on-shell amplitudes and there is some difficulty in demonstrating the finiteness of the two-point amplitudes despite their on-shell modular invariance.

I. INTRODUCTION

Loop corrections in two-point amplitudes lead to the divergence of on-shell $N(\geq 3)$-point amplitudes in string theories in the same manner as in particle theories. The divergences caused by one-loop corrections seem to be removable by renormalizing vertices by infinite amounts under the unitarity condition.[1] In superstring theories, one-loop two-point amplitudes, or mass shifts in the lowest order, vanish for massless states, but not for massive states.[2,3] In Ref. 3, examples are calculated in the SO(32) type-I open superstring for the bosons lying on the leading and the next-to-leading Regge trajectories and there the mass shifts were found to be the same at each mass level for the two trajectories, suggesting the tree mass degeneracies being maintained to one-loop order.

In this article, we take the type-II theory and deal with two problems concerning two-point amplitudes. (1) *Mass degeneracies.* The free string spectrum, determined by the conformal invariance of tree amplitudes, is highly degenerate; the number of states at a mass level grows exponentially as we go to higher levels. In the absence of general arguments as to what aspects of the string spectrum should remain to loop orders, we ask if any of the mass degeneracies are lifted at loop corrections. In Sec. III we show that the second and higher mass levels split at one-loop order. This means the spectrum receives a substantial change. In particular, the mass shifts cannot be absorbed in the string tension as formerly noted for the type-I open string based on the apparently nonlinear dependence of the mass shifts on the mass level.[3] Also we must say the mass degeneracies observed in Ref. 3 for the examples in the type-I theory are not a general phenomenon in strings. From a practical point of view, the splittings add some complexity to the evaluation of mass shifts (Sec. IV) and so to the application of the renormalization procedure. (2) *Finiteness of mass shifts* (Sec. V). If we are to employ the vertex renormalization, the finiteness of mass shifts is essential for massive amplitudes to be obtainable within the present framework of on-shell theory. We try to demonstrate the finiteness by a direct analysis of one-loop two-point amplitudes. The finiteness arguments based on the on-shell modular invariance[4] are only of heuristic value because the naive on-shell integral representations are not by themselves proper amplitudes. The imaginary part of the transition matrix should be obtained by the analytic continuation in the external momenta from somewhere off shell and we need proper off-shell extensions to demonstrate the finiteness.

II. CONSTRUCTION OF AMPLITUDES

We first sketch the construction of one-loop amplitudes for the bosons in the Neveu-Schwarz (NS-NS) sector. We work in the old covariant operator formalism and get around proper treatments of ghosts by adjusting the power of the partition functions by hand. We shall study later only the amplitudes that receive no contributions from the parity-violating part, so we omit that part from the beginning. We follow the notation in Ref. 5 throughout unless noted otherwise.

We start with the sum of traces:

$$A_M=\left[\frac{\kappa}{2}\right]^M \sum_{A,B}\sum_{\substack{\text{cyclic}\\ \text{ordering}}}\int dp\,\mathrm{Tr}\left[\pi_A\tilde{\pi}_B\prod_{j=1}^{M}[V_j(0)\Delta_{AB}]\right]. \tag{1}$$

Here the integral is over the loop momentum, the sum is over the cyclic ordering of $j=1,2,\ldots,M$ and over the sector A-B=NS-NS, NS-R, R-NS, R-R (where R denotes the Ramond sector). Under the trace for each sector are

$$\Delta_{AB}=\frac{1}{4\pi}\int_{|z|<1}\frac{d^2z}{|z|^2}z^{L_0-a_A}\bar{z}^{\tilde{L}_0-a_B} \tag{2}$$

with the intercepts a_A and a_B ($a_{\rm NS}=\frac{1}{2}$, $a_R=0$),

$$\overset{(\sim)}{\pi}_{\rm NS}=\tfrac{1}{2}(1+\overset{(\sim)}{G}),\quad \overset{(\sim)}{\pi}_R=-\tfrac{1}{4} \tag{3}$$

with the G-parity operator $\overset{(\sim)}{G}$ and the vertices $V_j(0)$, for which we temporarily substitute

$$V_j(0)=V_{Rj}(\sigma^-)V_{Lj}(\sigma^+)\big|_{\sigma^\pm=0}, \tag{4}$$

$$V_{Rj}(\sigma^-)=:\exp\left[\sum_{m\geq 0}(\xi^j_m)_\mu\partial_-{}^mX_R{}^\mu+\sum_{m\geq 0}(\theta^j_m)_\mu\partial_-{}^m\psi_-{}^\mu\right]:, \tag{5a}$$

$$V_{Lj}(\sigma^+)=:\exp\left[\sum_{m\ge 0}(\tilde{\xi}^j_m)_\mu\partial_+{}^m X_L{}^\mu + \sum_{m\ge 0}(\tilde{\theta}^j_m)_\mu\partial_+{}^m\psi_+{}^\mu\right]: , \qquad (5b)$$

where we have introduced vector-valued parameters ξ^j_m, $\tilde{\xi}^j_m$ (c numbers) and θ^j_m, $\tilde{\theta}^j_m$ (Grassmann numbers). We take the right- and the left-moving parts of the operators to be independent, including the zero modes of X_R, X_L, and impose the relation $\alpha_0=\tilde{\alpha}_0$ as a constraint on states. Accordingly we set $\xi^j_0=\tilde{\xi}^j_0$ ($\equiv ik_j$), and the traces in (1) are actually over the states on which $\alpha_0=\tilde{\alpha}_0=p/2$. We can carry out the trace easily by the coherent-state methods because of the exponential form of the operators (5). After the momentum integration, the result is written in the form

$$A_M=i\left[\frac{\pi\kappa}{2}\right]^M\int d^2\tau\, d^2\nu_1\cdots d^2\nu_{M-1}\left[\frac{2}{\mathrm{Im}\tau}\right]^5 \times\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} f_\alpha(\tau)\overline{f_\beta(\tau)}e^{S_{\alpha\beta}(\nu,\tau)} . \qquad (6)$$

Here the τ integral is over the region where $\mathrm{Im}\tau>0$ and $|\mathrm{Re}\tau|<\frac{1}{2}$, and each ν_j integral over the parallelogram spanned by 1 and τ in the complex plane. The f_α and $S_{\alpha\beta}$ are given by

$$f_\alpha(\tau)=2^3(-1)^\alpha\left[\frac{\theta_{\alpha+1}(0|\tau)}{\theta_1'(0|\tau)}\right]^4 , \qquad (7)$$

$$S_{\alpha\beta}(\nu,\tau)=\sum_{i=1}^{M}\sum_{j=1}^{M}\left[\frac{1}{4\pi\,\mathrm{Im}\tau}\xi^i_1\tilde{\xi}^j_1+\sum_{m\ge 0}\sum_{n\ge 0}[\xi^i_m\xi^j_n G^{mn}_{ij}+\tilde{\xi}^i_m\tilde{\xi}^j_n(-1)^{m+n}\overline{G^{mn}_{ij}}+\theta^i_m\theta^j_n P^{mn}_{\alpha ij}+\tilde{\theta}^i_m\tilde{\theta}^j_n(-1)^{m+n}\overline{P^{mn}_{\beta ij}}]\right] , \qquad (8)$$

where

$$G^{00}_{ii}=0 , \qquad (9a)$$

$$G^{mn}_{ij}=-\tfrac{1}{4}(-1)^n\left[\frac{1}{\pi}\frac{\partial}{\partial\nu}\right]^{m+n}\ln\chi\Big|_{\nu=\nu_i-\nu_j} \quad (i\ne j) , \qquad (9b)$$

$$\chi(\nu|\tau)=2\pi\exp\left[-\frac{\pi(\mathrm{Im}\nu)^2}{\mathrm{Im}\tau}\right]\left|\frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)}\right| , \qquad (10)$$

$$P^{mn}_{\alpha ij}=\frac{i}{4}(-1)^n\frac{\theta_1'(0|\tau)}{\theta_{\alpha+1}(0|\tau)}\left[\frac{1}{\pi}\frac{\partial}{\partial\nu}\right]^{m+n}\times\frac{\theta_{\alpha+1}(\nu|\tau)}{\theta_1(\nu|\tau)}\Big|_{\nu=\nu_i-\nu_j} \quad (i\ne j) . \qquad (11)$$

In Eqs. (10) and (11), θ_α ($\alpha=1,2,3,4$) are the Jacobi theta functions and we omitted the definition of the coefficients in (8) with $i=j$ except for $m=n=0$. In Eq. (9b), χ is differentiated as a function of independent variables $\nu,\bar{\nu}$. Equation (6) is interpreted as a sum (integral) over the torus with the complex Teichmüller parameter τ and the spin structure designated here by two numbers α,β. Then the ν_j's denote the locations of the fields on the torus and each term in $S_{\alpha\beta}$ defines a correlation function on it. In our construction, the correlations of the derivatives of the fields do not have δ-function terms since we treated the right- and left-moving parts as independent in the beginning. We note, however, that the right movers X_R, $\partial_- X_R$ and the left movers X_L, $\partial_+ X_L$ have nonzero correlations arising from the closed-string constraint $\alpha_0=\tilde{\alpha}_0$.

To obtain a string amplitude, we have to replace the V_j in Eq. (1) with the F1-picture vertices for the external lines. This can be achieved by applying an appropriate differential operator on A_M and set the parameters $\xi,\tilde{\xi},\theta,\tilde{\theta}$ equal to zero except $\xi^j_0=\tilde{\xi}^j_0=ik_j$, where k_j denotes the momentum carried by the external state j. The process is equivalent in effect to contracting the string coordinate fields from the vertices using the correlation functions defined by $S_{\alpha\beta}$. In this work we concentrate on the physical states written

$$|\zeta\rangle=P_\zeta(b_{-1/2},\alpha_{-1};\tilde{b}_{-1/2},\tilde{\alpha}_{-1})|0;k\rangle \qquad (12)$$

with

$$P_\zeta(b_{-1/2},\alpha_{-1};\tilde{b}_{-1/2},\tilde{\alpha}_{-1})=\frac{1}{l!}\zeta_{\beta\gamma_1\cdots\gamma_l;\lambda\mu_1\cdots\mu_l}\times b^\beta_{-1/2}\alpha^{\gamma_1}_{-1}\cdots\alpha^{\gamma_l}_{-1}\times\tilde{b}^\lambda_{-1/2}\tilde{\alpha}^{\mu_1}_{-1}\cdots\tilde{\alpha}^{\mu_l}_{-1} , \qquad (13)$$

where l is an arbitrary non-negative integer, $k^2=-8l$, and the polarization tensor ζ is transverse and traceless symmetric in the first half of the indices (right-moving part) and in the latter half (left-moving part) separately. So the state $|\zeta\rangle$ is the tensor product of two open-string states of level l on the leading trajectory.[3] The F2-picture vertex to emit (or absorb if $k^0_j>0$) the state $|\zeta\rangle$ is

$$W=:P_\zeta(\psi_-,\partial_- X_R;\psi_+,\partial_+ X_L)\exp[ik(X_R+X_L)]: , \qquad (14)$$

where P_ζ is the general polynomial (13). We have to supertransform the vertices W^j ($j=1,2,\ldots,M$) into V_j (F1-picture vertices) before using the formula (6). Operationally, the whole process of obtaining an M-point amplitude of states of the type (12) can be summarized as follows.

(1) Introduce new Grassmannian parameters $\phi^j,\tilde{\phi}^j$ and make the following change of variables in Eq. (6):

$$\xi^j_n\to\xi^j_n+\theta^j_{n-1}\phi^j,\quad \theta^j_n\to\theta^j_n-\frac{i}{2}\xi^j_n\phi^j ,$$
$$\tilde{\xi}^j_n\to\tilde{\xi}^j_n+\tilde{\theta}^j_{n-1}\tilde{\phi}^j,\quad \tilde{\theta}^j_n\to\tilde{\theta}^j_n-\frac{i}{2}\tilde{\xi}^j_n\tilde{\phi}^j . \qquad (15)$$

(2) Then apply

$$\prod_{j=1}^{M}\int d\tilde{\phi}^j\int d\phi^j P_{\zeta_j}\left[\frac{\partial}{\partial\theta_1^j},\frac{\partial}{\partial\xi_1^j},\frac{\partial}{\partial\tilde{\theta}_1^j},\frac{\partial}{\partial\tilde{\xi}_1^j}\right] \quad (16)$$

and set $\xi_0^j=\tilde{\xi}_0^j=ik_j$ and other parameters zero.

Below we study the two- and three-point amplitudes obtained by the method described here. The expressions are considerably simplified upon the summation over the spin structure by using the Riemann theta formula.

III. MASS SPLITTINGS

The two-point amplitude of incoming states $|\zeta_1,k\rangle$, $|\bar{\zeta}_2,-k\rangle$ $(k^2=-8l)$ is

$$A_2(\bar{\zeta}_2,-k;\zeta_1,k)=i\left[\frac{\pi\kappa}{2}\right]^2\int d^2\tau d^2\nu\left[\frac{2}{\mathrm{Im}\tau}\right]^5\chi(\nu|\tau)^{-k^2/2}$$
$$\times\sum_{p+q+r=l-1}(\bar{\zeta}_2,\zeta_1)_{p+2,q,r}\left[\frac{l!}{p!q!r!}\right]^2\left[\frac{1}{4\pi\,\mathrm{Im}\tau}\right]^{2(q+r)}\left|\frac{1}{2}\left[\frac{1}{\pi}\frac{\partial}{\partial\nu}\right]^2\ln\chi(\nu|\tau)\right|^{2p} . \quad (17)$$

We have put $\nu=\nu_1-\nu_2$. The sum runs over all the nonnegative integers p,q,r satisfying $p+q+r=l-1$ and we used the following notation for scalars made out of $\bar{\zeta}_2,\zeta_1$ by contractions:

$$(\bar{\zeta}_2,\zeta_1)_{p,q,r}\equiv\bar{\zeta}_{2\alpha^p\sigma^q\mu^r}{}^{;\beta^p\tau^q\mu^r}\zeta_1{}^{\alpha^p}{}_{\tau^q}{}^{\nu^r}{}_{;\beta^p}{}^{\sigma^q}{}_{\nu^r} , \quad (18)$$

where the symbol α^p stands for p indices, say, $\alpha_1\cdots\alpha_p$ collectively; similarly, $\sigma^q=\sigma_1\cdots\sigma_q$, $\mu^r=\mu_1\cdots\mu_r$, and so on. The mixings of the right- and left-moving parts in the contractions of the indices are due to the correlation

$$\langle\partial_- X_R^\mu\partial_+ X_L^\nu\rangle_{\mathrm{torus}}=\left[\frac{1}{4\pi\,\mathrm{Im}\tau}\right]\eta^{\mu\nu} .$$

As noted in Ref. 3 for open-string two-point amplitudes, the amplitude (17) apparently has a nontrivial (nonlinear) dependence on tree mass level l. Here we shall deduce from Eq. (17) that the mass shift varies with the state at a fixed l for $l\geq 2$. Suppose the mass shifts are the same for all the states on a level l, then the A_2 is proportional to the inner product $\langle\ |\ \rangle$ on that level: with some δm^2,

$$A_2(\bar{\zeta}_2,-k;\zeta_1,k)=-2(2\pi)^{10}i\delta m^2\langle\zeta_2k|\zeta_1k\rangle \quad (19)$$

for any states $|\zeta_1k\rangle$, $|\zeta_2k\rangle$ of level l. Apparently this is not the case with the amplitude (17) for $l\geq 2$ if we note $\langle\zeta_2k|\zeta_1k\rangle=(\bar{\zeta}_2,\zeta_1)_{l+1,0,0}$ in our notation. We shall now establish the nonproportionality by giving for each $l(\geq 2)$ a pair of states which are mutually orthogonal but have a nonvanishing two-point amplitude. Using a Lorentz transformation if necessary, we take the time in the k direction so that $k^0=\sqrt{8l}$, $k^i=0$ $(i=1,2,\ldots,9)$. We define $(l-1)$-rank symmetric traceless tensors B_1,B_2 by the components

$$B_1^{\sigma_1\cdots\sigma_{l-1}}=\sum_{n=0}^{l-1}i^n\frac{1}{n!(l-n-1)!}\times\delta_1^{(\sigma_1}\cdots\delta_1^{\sigma_n}\delta_2^{\sigma_{n+1}}\cdots\delta_2^{\sigma_{l-1})} , \quad (20a)$$

$$B_2^{\sigma_1\cdots\sigma_{l-1}}=\sum_{n=0}^{l-1}i^n\frac{1}{n!(l-n-1)!}\times\delta_3^{(\sigma_1}\cdots\delta_3^{\sigma_n}\delta_4^{\sigma_{n+1}}\cdots\delta_4^{\sigma_{l-1})} , \quad (20b)$$

and then tensors ζ_1,ζ_2 by

$$\zeta_j^{\alpha_1\cdots\alpha_{l+1};\beta_1\cdots\beta_{l+1}}=B_j^{(\alpha_1\cdots\alpha_{l-1}}\delta_5^{\alpha_l}\delta_6^{\alpha_{l+1})}\times\bar{B}_j^{(\beta_1\cdots\beta_{l-1}}\delta_5^{\beta_l}\delta_6^{\beta_{l+1})}\quad(j=1,2) . \quad (21)$$

In Eqs. (20) and (21), the parentheses enclosing the indices denote symmetrizations. The tensors ζ_1,ζ_2 define physical states $|\zeta_1k\rangle$, $|\zeta_2k\rangle$ with Eq. (12). We easily see that $(\bar{\zeta}_2,\zeta_1)_{2,0,l-1}>0$, but $(\bar{\zeta}_2,\zeta_1)_{p+2,q,r}=0$ for other $p,q,r\geq 0$. So, if $l\geq 2$, then $\langle\zeta_2k|\zeta_1k\rangle=0$, but $A_2(\bar{\zeta}_2-k;\zeta_1k)\neq 0$ from Eq. (17). Since the inner product $\langle\ |\ \rangle$ is not identically zero on the states at each mass level, the above examples show that Eq. (19) cannot be true for $l\geq 2$, proving that the levels split by one-loop corrections. As for the first excited level $(l=1)$ there is no splitting as far as the NS-NS sector is concerned. All the independent physical states in this sector are accounted for by the states (12) and the ones obtained from them by replacing one or both of the right- and the left-moving parts by open string states on the next-to-leading trajectory. The relation (19) is confirmed by calculating amplitudes with the replacements in the external states, and comparing the results. In fact, such replacements do not affect the general form of the integral (17), corresponding to the mass degeneracies observed for the leading and the next-to-leading trajectories in the type-I super open string.[3]

IV. MIXINGS AND DIAGONALIZATION

In Sec. III we gave examples of amplitudes mixing orthogonal states. The existence of mixings at a mass level is equivalent to the splitting of the level. For possible mixings in a larger scale than implied by the previous two-point amplitudes, we now turn to the three-point amplitude with one massive and two massless states and consider its factorization. The three-point amplitude for a massive state (state 1, with arbitrary l) and two massless states (states 2,3) is

$$A_3=i\left[\frac{\pi\kappa}{2}\right]^3\int d^2\tau d^2\nu_1 d^2\nu_2\left[\frac{2}{\mathrm{Im}\tau}\right]^5 f(\nu_1\nu_2\nu_3|\tau) \quad (22)$$

with

$$f(\nu_1\nu_2\nu_3|\tau)=\sum_{q+r=l-1}\frac{l!}{r!(q!)^2}\left[\frac{1}{4\pi\,\mathrm{Im}\tau}\right]^r K^{(q)}_{\lambda_1\cdots\lambda_q;\mu_1\cdots\mu_q}\prod_{s=1}^{q}\left[\frac{1}{4\pi^2}\sum_{i\neq 1}k_i^{\lambda_s}\frac{\partial}{\partial\nu_1}\ln\chi_{1i}\sum_{j\neq 1}k_j^{\mu_s}\frac{\partial}{\partial\nu_1}\ln\chi_{1j}\right]\prod_{i<j}\chi_{ij}^{k_i\cdot k_j/2}\,, \tag{23}$$

where $\chi_{ij}=\chi(\nu_i-\nu_j|\tau)$. In the sum, $q=0, 1, \ldots, l-1$, and for $q=0$ we set $\prod_{s=1}^{q}(\cdots)\equiv 1$. The tensors $K^{(q)}$ are defined by

$$K^{(q)}_{\lambda_1\cdots\lambda_q;\mu_1\cdots\mu_q}=k_2^{\rho}k_3^{\tau}k_2^{\tilde\rho}k_3^{\tilde\tau}\eta_{\alpha[\rho}\eta_{\beta][\tau}\eta_{\gamma]\sigma}\eta_{\tilde\alpha[\tilde\rho}\eta_{\tilde\beta][\tilde\tau}\eta_{\tilde\gamma]\tilde\sigma}\zeta_1^{\alpha\sigma}{}_{\lambda_1\cdots\lambda_q\kappa_1\cdots\kappa_r}{}^{;\tilde\alpha\tilde\sigma}{}_{\mu_1\cdots\mu_q}{}^{\kappa_1\cdots\kappa_r}\zeta_2{}^{\beta;\tilde\beta}\zeta_3{}^{\gamma;\tilde\gamma}\,, \tag{24}$$

where the indices in each set of square brackets are antisymmetrized. Note that all the indices of $K^{(q)}$ come from the massive polarization tensor ζ_1 and that a larger q means fewer contractions between the indices in ζ_1.

To factorize the three-point amplitude into a massive two-point amplitude and a coupling, we consider the integral over a region where ν_2 and ν_3 are close to each other,[6] or $|\nu_2-\nu_3|<\epsilon$ for some $\epsilon>0$. We introduce variables $\nu_0\equiv(\nu_2+\nu_3)/2$ and $\delta\equiv\nu_{32}=\nu_3-\nu_2$, and expand f in δ and $\bar\delta$ with $\nu_{10}=\nu_1-\nu_0$ fixed:

$$f=|\delta|^{(k_2\cdot k_3)/2-2}\sum_{m,n\geq 1}\delta^m\bar\delta^n h_{mn}(\nu_{10}|\tau)\,. \tag{25}$$

Integrating the expansion over $\delta(|\delta|<\epsilon)$, we obtain

$$A_3\sim\frac{4}{k_1^2+8l}\left[\frac{2}{\mathrm{Im}\tau}\right]^5\int d^2\tau\, d^2\nu_1\epsilon^{(k_1^2+8l)/4}h_{ll}\,, \tag{26}$$

where we evaluated the integral by an analytic continuation in k_1^2 from the Euclidean region and left only the part with a pole at $k_1^2=-8l$, dropping some numerical factors and the coupling constant. (The on-shell propagator makes the three-point amplitude divergent,[1] but we shall not linger on this subject.) By the factorizability of string amplitudes, the residue must be the two-point amplitude for state 1 and the state obtained by superposing the physical states coupled to states 2,3 using the tree couplings as the weights in the superposition. If we carry out the expansion (25), we see h_{ll} can involve the derivatives of $\ln\chi$ of even orders high up to the $l-1$ or lth order, not counting mixed derivatives

$$\left[\frac{\partial^2}{\partial\nu\,\partial\bar\nu}\ln\chi,\ \frac{\partial^4}{\partial\nu^2\partial\bar\nu^2}\ln\chi,\ \text{etc.}\right].$$

The maximum order derivatives come from the terms with $q=0$ or 1 under the sum in (23). For $q\geq l-2$, the derivatives are only of second order (note that $K^{(q)}$ vanish when contracted with $-k_1=k_2+k_3$). So, if l is large ($l\geq 4$) and the massive polarization tensor is such that $K^{(q)}$ is nonvanishing for some small q ($q\leq l-3$), then the h_{ll} is different in form from the integrand of the two-point amplitude (17). To reproduce the integrand with high derivatives of $\ln\chi$ by the correlations of vertices, we shall have to think of vertices with high derivatives of X^μ (or maybe of ψ^μ, with F2-picture vertices). Thus, if we leave out the possibility of the high derivatives vanishing or reducing to the terms with only the second-order derivatives by integrations, the residue of (26) has nontrivial contributions from the amplitude for state 1 and some other state that corresponds to a vertex with high derivatives and so not of the type (14). Then the states (12) can mix not only with states of the same type but with states of some other types.

As suggested by the examples of mixings, the two-point amplitude will have off-diagonal elements if we choose the "wrong" basis for the states on a mass level splitting at the loop corrections. So the diagonalization of the two-point amplitude is necessary for the evaluation of the mass shifts in much the same way as the diagonalization of the perturbing Hamiltonian is necessary for the evaluation of the energy levels in the perturbation method for degenerate states in ordinary quantum mechanics. Since mass shifts are scalars, the diagonalization is partly done by decomposing the states into the bases for irreducible representations of the little group. Especially the totally symmetric traceless tensors form at each mass the unique irreducible representation with the highest dimension and so do not mix with other states. The highest spin states are protected from mixing by the angular momentum conservation. The absence of mixings can be checked in the factorization of the three-point amplitude discussed above. (If we are only concerned with the order of derivatives, we only have to note that $K^{(q)}=0$ for $q<l-1$.) Thus, for a totally symmetric traceless ζ_1, Eq. (19) holds whatever the state 2 is. From Eq. (17) the mass shift in this case is found to be

$$\delta m^2_{\text{tot sym}}=-\frac{1}{2(2\pi)^{10}}\left[\frac{\pi\kappa}{2}\right]^2\int d^2\tau\, d^2\nu\left[\frac{2}{\mathrm{Im}\tau}\right]^5\chi(\nu|\tau)^{-k^2/2}$$

$$\times\sum_{p+q=l-1}\left[\frac{l!}{p!q!}\right]^2\left[\frac{1}{4\pi\,\mathrm{Im}\tau}\right]^{2q}\left|\frac{1}{4}\left[\frac{1}{\pi}\frac{\partial}{\partial\nu}\right]^2\ln\theta_1(\nu|\tau)+\frac{1}{4\pi\,\mathrm{Im}\tau}\right|^{2p}. \tag{27}$$

Generally, the decomposition into irreducible representations will not complete the diagonalization. If it does, the states of type (12), which themselves form a basis for a representation, cannot mix with the states belonging to other representations. The mixings suggested by the three-point amplitude factorization would then be impossible. The free string spectrum allows equivalent irreducible representations at excited levels, and those representations may have different mass shifts at loop corrections.

V. FINITENESS OF TWO-POINT AMPLITUDES

We shall now discuss the finiteness of the two-point amplitudes. If we set k on shell in the amplitude (17) before integration, the ν_j integral converges and for the remaining τ integration the integrand has the right modular weight to make the amplitude modular invariant. In other words, our on-shell amplitudes have the two properties that are believed to guarantee the finiteness of massless one-loop amplitudes.[4] So we expect they should be finite. The fact is, however, the integration of the on-shell integrand gives an infinity even if we restrict the τ integral to one fundamental region of the modular group. To see this, we take the fundamental region to be the one extending to $i\infty$ and note that the $4l$th power of χ comes in the integrand of (17) when k is on shell. Then we look at χ written

$$\chi(\nu|\tau)=e^{\pi\,\mathrm{Im}\nu(1-\mathrm{Im}\nu/\mathrm{Im}\tau)}\left|\left[1-\frac{w}{\rho}\right](1-\rho)\right| \times\left|\prod_{m=1}^{\infty}\frac{(1-w^m\rho)(1-w^{m+1}/\rho)}{(1-w^m)^2}\right|, \tag{28}$$

where $w=e^{2\pi i\tau}$, $\rho=e^{2\pi i\nu}$. The third factor is unimportant as $\mathrm{Im}\tau\to\infty$ ($w\to 0$) because it approaches 1(unity) uniformly with respect to ν (recall that $0<\mathrm{Im}\nu<\mathrm{Im}\tau$ in the integration region). The exponent in the first factor is positive and becomes large in the region where $\mathrm{Im}\tau$ is large and $\mathrm{Im}\nu$ is comparable to but appreciably smaller than $\mathrm{Im}\tau$. And in the same region the second factor is comparable to 1. So the ν integral of χ^{4l} ($l>0$) grows exponentially as $\mathrm{Im}\tau\to\infty$. A little more analysis shows that other ingredients in the integrand of (17) do not change this exponential behavior. So the integral diverges.

Part of the reason for the divergence may be traced to the propagator (2). The parameter representation is valid only when L_0-a_A, $\tilde{L}_0-a_B>0$, and to secure this in a loop amplitude, we resort to the Wick rotation of the loop momentum, which in turn forces the external momenta off shell by the momentum conservation. So it is illegitimate to set the external momenta on shell before the integration is completed. We should also note that a Wick-rotated self-energy-type amplitude is purely imaginary as can be seen by setting $V_2(0)=V_1(0)^\dagger$ in the general construction (1) for $M=2$: the traces are real since the Hermitian operators $\pi_A\tilde{\pi}_B\Delta_{AB}$ commute with each other, and the integral is purely imaginary after the Wick rotation. This means that the corresponding diagonal element of the transition matrix has no imaginary part. The imaginary part required for the unitarity of the scattering matrix can only be gained in the continuation in the external momentum to the mass shell. The amplitude should have branch points on the real axis in k^2 plane. So the divergent on-shell integrals do not represent string amplitudes by themselves and the proper interpretation will involve analytic continuations in k^2. The same comment applies to the open-string amplitudes in Ref. 3 and similar situations are encountered in particle theories if parameter representations similar to Eq. (2) are used in calculations.

Let us take a close look at the case $l=1$. The amplitude is essentially the integral

$$\begin{aligned}I_F&=\int_F d^2\tau\int d^2\nu\left[\frac{2}{\mathrm{Im}\tau}\right]^5\chi(\nu|\tau)^{-k^2/2}\\&=R+\int_F d^2\tau\int d^2\nu\left[\frac{2}{\mathrm{Im}\tau}\right]^5\\&\qquad\times\exp\left[-\frac{k^2}{2}\pi\,\mathrm{Im}\nu\left[1-\frac{\mathrm{Im}\nu}{\mathrm{Im}\tau}\right]\right].\end{aligned} \tag{29}$$

On the right-hand side we segregated the part coming from the exponential in Eq. (28) and denoted the rest by R. We take the fundamental region F to be the one extending to $i\infty$. We omit the proof here, but the integral R converges for k^2: $-8\le\mathrm{Re}k^2<4$, where the upper bound is due to the singularity of $\chi^{-k^2/2}$ at $\nu=0$. Furthermore the function $R(k^2)$ defined by the convergent integral is continuous, and analytic in the interior of the domain. So the divergence of the on-shell integral solely comes from the latter part in the right-hand side, which is the loop integral of a massless particle in the ten-dimensional field theory except for the ultraviolet cutoff brought by the restriction of the τ integral to the fundamental region F. Therefore, the integral is apparently divergent for $\mathrm{Re}k^2<0$, but by the analytic continuation from the Euclidean region defines a function of k^2 with a branch point at $k^2=0$. It gives a finite complex value at $k^2=-8$ (on shell) with an imaginary part of the right sign for the unitarity if we approach the real axis from the lower plane. So, with our unbounded F, the two-point amplitude defined by the integral $I_F(k^2)$ with the analytic continuation in k^2 is finite. If we change F in I_F for a bounded fundamental region, the integral never converges at real k^2, or equivalently, never converges absolutely at any point in k^2 plane. The analysis is most conveniently done noting the formula

$$I_{\hat{F}}=\int_F d^2\tau\int d^2\nu\left[\frac{2}{\mathrm{Im}\tau}\right]^5\chi(\nu|\tau)^{-k^2/2}|c\tau+d|^{(k^2+8)/2}, \tag{30}$$

where $\hat{F}$ is the image of F by the modular transformation

$$\tau\to\hat{\tau}=\frac{a\tau+b}{c\tau+d},\quad\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathrm{SL}(2,Z). \tag{31}$$

We take an unbounded F as before. For $k^2<0$, the integral $I_{\hat{F}}$ diverges for the reason noted for the on-shell integrals. For $k^2>0$, the contributions from the region where $\mathrm{Im}\nu\sim\mathrm{Im}\tau$ or ~ 0 add up to be infinite unless $c=0$; i.e., the $\hat{F}$ also extends to $i\infty$. To justify the restriction of the τ integral to a fundamental region, we will have to show that the integral gives the same value for any choice of the region, but now it seems difficult to achieve this with analytic continuation alone. Instead of investigating for what k^2 the integral converges (conditionally) for each choice of F, we read Eq. (30) the other way: take $\hat{F}$ to be the region extending to $i\infty$, then the integral can be defined with an analytic continuation as we have seen, that is, for any F the insertion of the function $g(\tau;k^2)=|c\tau+d|^{(k^2+8)/2}$ $(\tau\in F)$ makes the integral finite. Reflecting the on-shell modular invariance, g becomes 1 as k comes back on shell, so g tells how we should extend the integrand towards the Euclidean region for the definition of the amplitude by analytic continuation. By definition g depends on $\hat{F}$, but defines a unique amplitude.

For $l>1$, it becomes more difficult to find the region in k^2 plane where the integral (17) converges. Every term under the sum diverges for $k^2<0$ and the terms with nonzero p diverges also for $k^2\geq 0$ for any choice of F because of the singularities of the integrands at $\nu=0$. Generalizing the procedure for $l=1$, we may introduce a function, which we again denote by $g(\tau;k^2)$, to be inserted in the integral to make it finite and define a function of k^2 which allows a continuation to the mass shell.

We suggest that the factorization of massless $N(\geq 4)$-point amplitudes might be used to justify the continuation scheme and determine the class of functions to which g should belong. For instance, if we consider the s channel of the four-point amplitude and follow the conventional factorization procedure as we did for the massive three-point amplitudes, we will restrict the ν_j integration to the region where $|\nu_{12}|,|\nu_{34}|\leq\epsilon(\tau)$, and replace the integrand with some of the first terms in its expansion in ν_{12} and ν_{34} which will develop into the desired double pole after integration. Factoring out the trees and propagators, we will have the integrand in Eq. (17) multiplied by $\epsilon^{(k^2+8l)/2}$ (Ref. 6), which we identify as $g(\tau;k^2)$. So the problem is now replaced with the one of finding the right $\epsilon(\tau)$ for the factorization procedure. We have not fully investigated the problem yet, but one thing for certain is that the ϵ should be chosen so that the integral converges for k^2 in some region. The reason is that its role in the factorization is to determine what part of the integral is segregated to study the double pole by a continuation in k^2 from somewhere the integral converges although the segregated part is not necessarily required to converge in the same region where the original integral converges since we can continuate the latter step by step as we segregate other divergent parts. Obviously we can adjust $\epsilon(\tau)$ to define an off-shell extension equivalent to the insertion of $g(\tau;k^2)=|c\tau+d|^{(k^2+8)/2}$ for $l=1$. And for a general l, we can choose $\epsilon(\tau)$ so that the integral is convergent in some region where $-8l<\mathrm{Re}k^2<-8l+12$. We assumed in the factorization that the integral representation found in the literature[7] converges for some configurations of the external momenta and defines the massless amplitude after all.

As noted earlier, the divergence of the two-point amplitudes is hardly conceivable in the context of the finiteness argument for massless one-loop amplitudes. The string amplitudes with the τ integral restricted to a fundamental region of the modular group should have no more divergences than the particle field-theoretic amplitudes in ten dimensions with an ultraviolet cutoff. What we have found is that the demonstration of this notion is nontrivial in conventional calculational methods since the proper definition of the string amplitudes requires something more than simple continuations in the external momenta from the Euclidean region, unlike the particle counterparts calculated with the parameter representations similar to Eq. (2). Finally, we comment on the assumption we made in the proof of mass splittings (Sec. III). In the examples of mixings we assumed the terms with $r=l-1$ under the sum in Eq. (17) do not vanish when integrated, and obviously there are no mixings if all the terms vanish except the ones with $p=l-1$. Since our continuation scheme is incomplete, we are not ready to prove the nonzeroness. However, each of the terms is divergent when (wrongly) integrated with the external momentum on-shell and, as suggested by the $l=1$ case, the apparent divergence is related to the imaginary part of the integral, so it is unlikely that the integral of each term vanishes altogether with its imaginary part.

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[7]For example, *Superstring Theory* (Ref. 5), Sec. 9.2.