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Yang-Mills formulation of interacting strings

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A suggestion that the theory of the interacting open bosonic string be reformulated as a generalized Yang-Mills theory is further elucidated. Moreover, a serious reservation regarding the ordering of operators in the earlier "proof" of equivalence between the new and standard formulations is now removed.

I. INTRODUCTION

String theory has one unusual and somewhat unsatisfactory aspect. Although it is supposed to contain as special cases both relativity and gauge theories,[1] both of which are founded on geometry, one is still uncertain on what physical principles string theory is based.[2] As an attempt toward more understanding in that general direction, it has been suggested that string theory be considered as a generalized Yang-Mills theory of an extended object.[3] The purpose of this paper is to elucidate further the relationship between these two theories, and to make more precise the conclusions obtained in the earlier work.

Suppose we consider generalizations of basic gauge theory concepts in two directions.

First, let us take as fundamental objects not points, but strings having a linear extension, which are to be described by wave functionals $\Phi[\mathbf{X}]$, where $\mathbf{X}$ are functions $\mathbf{X}(\sigma)$, say, with $\sigma=0\rightarrow\pi$. Because of its extension in actual physical space, a string possesses "intrinsic" internal degrees of freedom in addition to and different in nature from the usual "extrinsic" internal degrees of freedom dealt with in standard Yang-Mills theory, which are in a separate internal space having nothing to do with the original physical space. One ought therefore to generalize for strings the gauge concepts so as to incorporate these "intrinsic" internal degrees of freedom into the theory as part of an extended gauge invariance.

Second, standard Yang-Mills theory is in a sense a degenerate case in that the base space (namely, space-time) is both the space over which the wave functions ϕ are defined and also the translation group space over which the gauge potential A_μ provides a criterion for parallel transport. One normally thinks of a wave function $\phi(x)$ as assigning to every point x in space-time a value for the state vector ϕ. Alternatively, however, one can also think of it as a prescription of how ϕ changes under translation, or in other words, as an assignment of ϕ to a representation of the translation group. The gauge potential A_μ then is what gives the additional change in the "phase" of ϕ through parallel transport under translation. Such a dual interpretation of the base space in standard Yang-Mills theory is possible because the group space of space-time translations is the same as space-time itself. However, one can easily imagine a situation where the translation group is generalized to some other group of transformations acting on ϕ, whose group space bears no obvious relation to the space over which ϕ is defined. Indeed, for string theory, it appears that one has to take as base, in place of the translation group, the conformal group,[4] which is of course not the same as the space of functions $\mathbf{X}(\sigma)$, over which the functionals $\Phi[\mathbf{X}]$ are defined.

We cannot claim to have understood any deep physical reason why one should generalize gauge theory in the directions indicated, especially the second. However, it seems that once these generalizations are made, then one gets very close to standard string theory. In what follows we shall attempt to formulate a theory incorporating such generalizations of the Yang-Mills concepts by developing the analogy with the standard theory. It will appear then that the theory of interacting open bosonic strings in the form proposed by Witten[5] will emerge as the pure Yang-Mills theory of the new version in which the open-string functional plays the role of the generalized Yang-Mills gauge potential. The equivalence between the new formulation and standard string theory was established only formally in our earlier work where ordering of operators has not been taken into account. This serious shortcoming is now removed through the introduction of an operator formalism with proper ordering criterion, giving us a better understanding of the relation between this new formulation and the conventional one. When fully developed, this operator formalism will allow us, we hope, to go in the future beyond the standard string theory, which corresponds only to the pure Yang-Mills case here, to a string theory with sources, opening thus perhaps new possibilities for the construction of more realistic models.

II. IDENTIFICATION OF THE GAUGE GROUP AND BASE

Yang-Mills theory has two fundamental ingredients: a gauge group and a base. By developing the analogy with particles along a particular line we are led to specific choices for both as the appropriate generalizations to strings. Our arguments here will be heuristic, and will use a functional formalism, which though less rigorous mathematically, seems to us to give one a much clearer physical picture.

Consider first a classical free particle. The system is invariant under shifts in both position and momentum as well as in time. The generators of these shifts, say t_i^q, t_j^p, and t^τ together form a closed algebra. When the system is quantized, however, the generators t are replaced by operators T on a Hilbert space. Then, T_i^q and T_j^p no longer commute,

$$[T_i^q, T_j^p] = i\delta_{ij} , \tag{2.1}$$

although all the other commutators remain the same.

In parallel now consider a classical free string described by $x^\mu(\sigma)$ and its conjugate $p^\mu(\sigma)$, or equivalently by their Fourier modes x_n^μ and p_n^μ. The system is invariant under conformal reparametrizations generated by l_n, satisfying

$$[l_n, l_m] = (n-m) l_{n+m} . \tag{2.2}$$

Alternatively we may choose as generators

$$\begin{aligned} k_r &= l_r - (-1)^r l_{-r}, \quad r > 0 , \\ m_r &= l_r + (-1)^r l_{-r}, \quad r > 0 , \\ m_0 &= 2 l_0 . \end{aligned} \tag{2.3}$$

They satisfy the commutation relations

$$\begin{aligned} [k_r, m_s] &= (r-s) m_{r+s} + (-1)^s (r+s) m_{r-s} , \\ [k_r, k_s] &= (r-s) k_{r+s} - (-1)^s (r+s) k_{r-s} , \\ [m_r, m_s] &= (r-s) k_{r+s} + (-1)^s (r+s) k_{r-s} , \\ [k_r, m_0] &= [1+(-1)^r] r m_r , \\ [m_r, m_0] &= [1+(-1)^r] r k_r . \end{aligned} \tag{2.4}$$

When this system is quantized in the standard fashion by replacing x and p with operators $\mathbf{X}$ and $\mathbf{P}$ on the Hilbert space of wave functionals such that

$$[\mathbf{X}_m^\mu, \mathbf{P}_n^\nu] = -i\delta_{mn}\eta^{\mu\nu} , \tag{2.5}$$

the generators l_n are replaced by the Virasoro operators

$$\mathbf{L}_n = \tfrac{1}{2} \sum_k :: \alpha_k \alpha_{n-k} :: \tag{2.6}$$

(where :: :: denotes normal ordering with respect to the mode creation-annihilation operators α_n) which satisfy the commutation relations

$$[\mathbf{L}_n, \mathbf{L}_m] = (n-m)\mathbf{L}_{n+m} + \frac{d}{12}(n^3 - n)\delta(n+m) , \tag{2.7}$$

differing from (2.2) by an anomalous central charge term depending on the space-time dimension d. Equivalently, in terms of the combinations

$$\begin{aligned} \mathbf{K}_r &= \mathbf{L}_r - (-1)^r \mathbf{L}_{-r} , \\ \mathbf{M}_r &= \mathbf{L}_r + (-1)^r \mathbf{L}_{-r} , \\ \mathbf{M}_0 &= 2\mathbf{L}_0 , \end{aligned} \tag{2.8}$$

one finds that of the commutation relations in (2.4) only the first one is changed, giving

$$\begin{aligned} [\mathbf{K}_r, \mathbf{M}_s] = {} & (r-s)\mathbf{M}_{r+s} + (-1)^s (r+s) \mathbf{M}_{r-s} \\ & + (-1)^s \frac{d}{6}(s^3 - s)\delta(r-s) \end{aligned} \tag{2.9}$$

which acquires the anomalous central charge term. All other commutation relations are left unaltered. Thus the effect of quantization is similar to that in (2.1) for particles, in that (apart from $m_0 \sim t^\tau$) the generators separate into two sets in such a way that quantization only adds a c-number term to commutators between members of different sets.

Suppose that we are minded to take the conformal generators l_n as the analogues for the string of the shifts t for the particle, as was suggested in the Introduction; we would be tempted to take m_0 as the analogue of t^τ, and either k_r or m_s as corresponding to t_i^q or t_j^p. The operators m_s, however, do not form a closed algebra. We are thus led to the choice of k_r as the generators of the base group for string which is to be the analogue of the group of spatial translations for particles generated by t_i^q. A major difference between the two cases, of course, is that the generators k_r in (2.3) do not commute, in contrast with the generators t_i^q of the Abelian group of spatial translations.

Next, we attempt to advance from the free theory to a gauge theory and try to identify the gauge group for a string by drawing again on the analogy with a particle. Take a standard Yang-Mills theory with unitary symmetry, say, for example, SU(2). This is described by a field ψ^i, $i=1,2$, belonging to the fundamental (doublet) representation, and a gauge potential A^{ij}, $i,j=1,2$, belonging to the adjoint (triplet) representation of the gauge group. To what classical system does it correspond? The question was answered by Wong—one obtains in the classical limit in place of ψ a particle in an isotriplet (adjoint) representation interacting with the gauge field.[6]

Consider now the string. We wish to construct a gauge theory in a similar fashion by treating the string extension itself as the gauge degree of freedom. In other words, we want the coordinates $\mathbf{X}(\sigma)$, $\sigma = 0 \to \pi$, to play the role of internal indices. By analogy with Wong's result we shall put the classical string in the "adjoint" representation of the gauge group. Suppose we were to introduce a Ψ field in analogy with ψ above: on what indices should Ψ depend? Ψ should be in the "fundamental" representation and depend on only half the number of indices which label the classical string: namely, $\mathbf{X}(\sigma)$, $\sigma = 0 \to \pi$. One simple solution is obviously that we take Ψ to be a functional of only half a string: namely, $\Psi = \Psi[X]$, $X = X(\sigma)$, $\sigma = 0 \to \pi/2$.

In standard Yang-Mills theory an inner product between two fields ψ_1 and ψ_2 is defined as

$$(\psi_1,\psi_2)=\sum_j \psi_1^{*j}\psi_2^j \ . \tag{2.10}$$

The gauge group is the group of unitary transformations preserving this inner product. In analogy, let us define the inner product between the Ψ functionals also as a sum of $\Psi_1^*\Psi_2$ over all internal indices, thus,

$$(\Psi_1,\Psi_2)=\int \delta X\, \Psi_1^*[X]\Psi_2[X] \ , \tag{2.11}$$

and define also the gauge group as the group of all unitary transformations,

$$\Psi[X_1]\rightarrow\tilde{\Psi}[X_1]=\int \delta X_2\, U[X_1,X_2]\Psi[X_2] \ , \tag{2.12}$$

which preserve the inner product (2.11): namely,

$$\int \delta X_2\, U^\dagger[X_1,X_2]U[X_2,X_3]=\delta[X_1-X_3] \ , \tag{2.13}$$

where, by definition, as usual for a matrix,

$$U^\dagger[X_1,X_2]=U^*[X_2,X_1] \ . \tag{2.14}$$

Next, the gauge potential is what defines parallel transport of the "phase" of Ψ under a displacement induced by an element of the "translation" algebra, and belongs to the adjoint representation as the classical string. It should thus be a matrix and carry twice the number of indices as Ψ. Thus $A=A[X_1,X_2]$, $X_i=X_i(\sigma)$, $\sigma=0\rightarrow\pi/2$. Alternatively, one can consider the potential to be a functional $\mathbf{A}[\mathbf{X}]$ of the full string $\mathbf{X}(\sigma)$, $\sigma=0\rightarrow\pi$, with $\mathbf{X}(\pi-\sigma)=X_2(\sigma)$, where $X_2(\sigma)$ is the "column index" of the matrix $A[X_1,X_2]$ introduced above. Further, like the potential A_μ^{ij} in standard Yang-Mills theory, the potential for the string should have as many "space-time" components as there are generators in the "translation" group. Thus $A[X_1,X_2]=A_r[X_1,X_2]$, where the integer $r>0$ labels the generators $\mathbf{K}_r$ defined in (2.8) of the analogue "translation" group as suggested above. Alternatively, we can take as generators of the "translation" group the Fourier transforms of $\mathbf{K}_r$, namely,

$$\mathbf{K}_{\pm\sigma}=\mathbf{L}_{\pm\sigma}-\mathbf{L}_{\pm(\pi-\sigma)}, \quad \sigma=0\rightarrow\pi/2 \ , \tag{2.15}$$

where $\mathbf{L}_\sigma$ is the Fourier transform of $\mathbf{L}_n$,

$$\mathbf{L}_{\pm\sigma}=\frac{1}{2}::\left[-i\pi\frac{\delta}{\delta\mathbf{X}(\sigma)}\pm\mathbf{X}'(\sigma)\right]^2::, \quad \sigma=0\rightarrow\pi \ , \tag{2.16}$$

and write $A[X_1,X_2]=A_\sigma[X_1,X_2]$, $\sigma=0\rightarrow\pi/2$. Furthermore, the gauge group being unitary, $A_\sigma[X_1,X_2]$ must therefore be Hermitian: thus,

$$A^\dagger[X_1,X_2]=A[X_1,X_2] \ . \tag{2.17}$$

In terms of the full-string notation when A is considered as a functional $\mathbf{A}[\mathbf{X}]$ of the full string $\mathbf{X}(\sigma)$, $\sigma=0\rightarrow\pi$, (2.17) is the same as

$$\mathbf{A}^*[\mathbf{X}]=\mathbf{A}[\overline{\mathbf{X}}], \quad \overline{\mathbf{X}}(\sigma)=\mathbf{X}(\pi-\sigma) \ , \tag{2.18}$$

which is the reality condition imposed on the string functional by Witten[7] and others in the standard formulation of string theory. Here in our case, (2.18) appears as a unitarity condition in half-string space.

Finally, the "translation" generators (2.15) act on "adjoint" representations; how do they act on Ψ? Define L_σ as

$$L_{\pm\sigma}=\frac{1}{2}:\left[-i\pi\frac{\delta}{\delta X^\mu(\sigma)}\pm X'^\mu(\sigma)\right]^2: \ , \qquad 0\le\sigma\le\pi/2 \ , \tag{2.19}$$

where : : means normal ordering with respect to a criterion yet to be specified. The transformation

$$\Psi\rightarrow(1+iv^\sigma L_\sigma)\Psi \tag{2.20}$$

on half-string functionals will induce on the matrix elements $(\Psi_1|M|\Psi_2)$ of any operator M in half-string space, the corresponding transformation

$$\Delta(\Psi_1|M|\Psi_2)=(\Psi_1|iv^\sigma[L_\sigma,M]|\Psi_2) \ . \tag{2.21}$$

Using then the definition (2.11) of inner products and performing some simple partial functional integrations, while ignoring for the moment all questions concerned with normal ordering, one can easily see that in terms of the full-string notation this is the same as

$$\Delta\mathbf{M}[\mathbf{X}]=iv^\sigma\mathbf{K}_\sigma\mathbf{M}[\mathbf{X}] \ . \tag{2.22}$$

In other words we have shown that the "translations" (2.15) are represented by the operators L_σ in (2.19) acting on Ψ.

III. FUNCTIONAL FORMULATION OF THEORY

Having identified the gauge group as the group of unitary transformations and the base group as the group generated by L_σ, both acting on $\Psi[X]$, one can now proceed to construct a gauge-covariant theory along familiar lines. This is the point at which we started in our earlier work.[3] For convenience we shall continue to refer to it as the "comma" formulation. Thus, introduce in analogy to the differentials dx^μ in standard (point) theories, the anticommuting differentials η^σ dual to the "translation" generators L_σ. These are the so-called Becchi-Rouet-Stora-Tyutin (BRST) ghosts.[8–10,5] From them one constructs the potential 1-form

$$A=\int_{-\pi/2}^{\pi/2} d\sigma\, A_\sigma\eta^\sigma \tag{3.1}$$

and the exterior derivative

$$Q=\int_{-\pi/2}^{\pi/2} d\sigma([L_\sigma,\ ]\eta^\sigma+4i\pi\eta^\sigma\eta'^\sigma\{\bar{\eta}^\sigma,\ \}) \tag{3.2}$$

which are entirely analogous, respectively, to the potential 1-form $A=A_\mu dx^\mu$ and the exterior derivative $\partial=[\partial_\mu,\ ]dx^\mu$ of standard Yang-Mills theory, except that in (3.2) there is an additional term due to the fact that the base group here is non-Abelian.[3,4]

Using A and Q, one can build other gauge-covariant quantities such as the field tensor (curvature) 2-form

$$F=QA+A\cdot A \ , \tag{3.3}$$

where the dot denotes matrix multiplication. Explicitly, if we introduce, via bosonization, a matrix representation of the "ghosts" η^σ; thus,

$$\eta^\sigma=\frac{1}{\sqrt{\pi}}:e^{\xi(\sigma)}:,\quad \bar{\eta}^\sigma=\frac{1}{\sqrt{\pi}}:e^{-\xi(\sigma)}: \tag{3.4}$$

with

$$\xi(\sigma)=\int_0^\sigma d\sigma'\left[\frac{\delta}{\delta\phi(\sigma')}+i\pi\phi'(\sigma')\right] \tag{3.5}$$

treating $\phi(\sigma)$ as an additional matrix index, then, for any two matrices M and N,

$$M\cdot N=\int\delta X_2\delta\phi_2 M[X_1,\phi_1;X_2,\phi_2]N[X_2,\phi_2;X_3,\phi_3]\ . \tag{3.6}$$

Furthermore, as usual, by taking traces

$$\mathrm{Tr}M=\int\delta X\,\delta\phi\,M[X,\phi;X,\phi] \tag{3.7}$$

of such covariant quantities, one can construct gauge invariants to serve as candidate actions. For example, the trace of the Chern-Simons 3-form

$$\mathcal{A}=\mathrm{Tr}(A\cdot QA+\tfrac{2}{3}A\cdot A\cdot A) \tag{3.8}$$

apart from some details, such as "ghost insertion factors" already spelled out in Ref. 3, and the serious ambiguity of ordering that we shall discuss later, is the same as the Witten field action for the interacting open bosonic string.[5]

The action (3.8) is by construction invariant under the gauge transformations:

$$\delta A=Q\epsilon+A\cdot\epsilon-\epsilon\cdot A \tag{3.9}$$

for any infinitesimal Hermitian matrix ϵ, corresponding to a local change in "phase" under an infinitesimal unitary transformation $U=1+i\epsilon$ of the Ψ field as in (2.12). In addition, (3.8) is invariant under the variations

$$\Delta A=iv^\sigma[\hat{L}_\sigma,A]\ , \tag{3.10}$$

where

$$\hat{L}_\sigma=L_\sigma+L_\sigma^{\mathrm{gh}}\ . \tag{3.11}$$

In (3.11), L_σ is as given in (2.19), which generates "translations" on $\Psi[X]$, and L_σ^{gh} is the corresponding generator in the ghost coordinate ϕ (Ref. 11).

$$L_{\pm\sigma}^{\mathrm{gh}}=\frac{1}{2}:\left[\frac{\delta}{\delta\phi(\sigma)}\pm i\pi\phi'(\sigma)\right]^2: -\frac{3}{2}i\frac{d}{d\sigma}\left[\frac{\delta}{\delta\phi(\sigma)}\pm i\pi\phi'(\sigma)\right],\quad 0\le\sigma\le\pi/2\ . \tag{3.12}$$

The second term in (3.8), being a trace of a product of A's, is obviously invariant under (3.10), while the first is also invariant because $\hat{L}_\sigma$ commutes with Q, or in other words, because the exterior derivative Q is "translational" invariant. Writing (3.10) in terms of full-string notation and performing some simple partial functional integrations (ignoring ordering for the moment) similar to those leading to (2.22), one obtains

$$\Delta\,\mathbf{A}[\mathbf{X}]=iv^\sigma\hat{\mathbf{K}}_\sigma\mathbf{A}[\mathbf{X}]\ , \tag{3.13}$$

where

$$\hat{\mathbf{K}}_\sigma=\hat{\mathbf{L}}_\sigma-\hat{\mathbf{L}}_{\pi-\sigma} \tag{3.14}$$

and

$$\hat{\mathbf{L}}_\sigma=\mathbf{L}_\sigma+\mathbf{L}_\sigma^{\mathrm{gh}} \tag{3.15}$$

for $\mathbf{L}_\sigma$ as given in (2.16) and

$$\mathbf{L}_{\pm\sigma}^{\mathrm{gh}}=\frac{1}{2}::\left[\frac{\delta}{\delta\boldsymbol{\phi}(\sigma)}\pm i\pi\boldsymbol{\phi}'(\sigma)\right]^2:: -\tfrac{3}{2}i\frac{d}{d\sigma}\left[\frac{\delta}{\delta\boldsymbol{\phi}(\sigma)}\pm i\pi\boldsymbol{\phi}'(\sigma)\right],\quad 0\le\sigma\le\pi\ . \tag{3.16}$$

But the Fourier transform of (3.14) is

$$\hat{\mathbf{K}}_r=\hat{\mathbf{L}}_r-(-1)^r\hat{\mathbf{L}}_{-r},\quad r>0\ . \tag{3.17}$$

So we have deduced that the action $\mathcal{A}$ in (3.8) is invariant under the algebra generated by $\hat{\mathbf{K}}_r$. Therefore, it appears that the invariance of interacting string theory under $\hat{\mathbf{K}}_r$, first discovered by Witten in Ref. 7, means in our present language just "translational" invariance.

The requirement of gauge invariance and invariance under "translations," though stringent, does not by itself determine the action uniquely. Instead of (3.8), one could have chosen as action, for example, the trace of a higher Chern-Simons form, provided one includes an appropriate ghost insertion factor to make the trace nonvanishing, as explained in Ref. 3. The result will still keep both invariances, although of course, it may be excluded by other physical requirements.[2] Furthermore, the action (3.8), as well as the other possibilities with higher Chern-Simons forms, involves only the gauge-potential 1-form A. In the standard Yang-Mills language, they are therefore pure gauge theories with only gauge bosons and nothing else. In principle, as far as gauge invariance is concerned, there seems nothing to stop us from extending the theory to include the Ψ field, as one does in ordinary Yang-Mills theory.

IV. CHOICE OF BASIS IN FUNCTIONAL SPACE AND EXPANSION IN OSCILLATOR MODES

The equivalence claimed in Ref. 3 between the action (3.8) and that of conventional string theory was only formal because, in the "proof" of equivalence, one had to ignore the normal ordering of operators, such as L_σ in (2.19) and η^σ in (3.4), without which the operators would be singular and undefined. Actually, the difficulties encountered in incorporating ordering in the operators L_σ and η^σ differ somewhat in significance. Whereas for η^σ, they appear to be merely technical, for L_σ the problem runs deeper and has its origin in the fact that one has not defined the half-string functional space properly. Indeed, without a more precise specification of the half-string functional space, one does not even know what ordering means for such operators as L_σ.

To understand the problem better, let us return to the conventional formulation of string theory.[1] There one usually works with an operator formalism. One starts with a function $\mathbf{X}^{\mu}(\tau,\sigma)$, $\sigma=0\rightarrow\pi$, which obeys open boundary conditions at both ends: namely,

$$\mathbf{X}'^{\mu}(\tau,\sigma)=\frac{\partial}{\partial\sigma}\mathbf{X}^{\mu}(\tau,\sigma)=0, \quad \sigma=0,\pi . \tag{4.1}$$

One then forms the combination

$$\xi^{\mu}(s)=[\pi\dot{\mathbf{X}}^{\mu}(\tau,\sigma)\pm\mathbf{X}'^{\mu}(\tau,\sigma)], \quad s=\tau\pm\sigma , \tag{4.2}$$

which, because of (4.1), is periodic for s in the range $[-\pi,\pi]$. On quantization, one replaces $\dot{\mathbf{X}}(\tau,\sigma)$ by $-i\delta/\delta\mathbf{X}(\sigma)$, obtaining

$$\mathbf{P}(\pm\sigma)=\left[-i\pi\frac{\delta}{\delta\mathbf{X}(\sigma)}\pm\mathbf{X}'(\sigma)\right], \quad \sigma=0\rightarrow\pi . \tag{4.3}$$

Taking the Fourier expansion of $\mathbf{P}(\sigma)$ one has

$$\mathbf{P}(\sigma)=\sum_{n}e^{-in\sigma}\alpha_n, \quad \sigma=-\pi\rightarrow\pi , \tag{4.4}$$

where α_n obey the commutation relations

$$[\alpha_n^{\mu},\alpha_m^{\nu}]=-n\delta(n+m)\eta^{\mu\nu} \tag{4.5}$$

behaving thus as mode creation and annihilation operators. Using the creation operators α_{-n}, $n>0$, one then constructs a Fock space by repeated application on the vacuum; thus,

$$\|r_1,r_2,\ldots,r_n,\ldots\rangle=\alpha_{-1}^{r_1}\alpha_{-2}^{r_2}\cdots\alpha_{-n}^{r_n}\cdots\|0\rangle . \tag{4.6}$$

Folding in the motion of the center of mass (zero mode) one has

$$\exp(i\mathbf{P}_0\cdot\mathbf{X}_0)\|r_1,r_2,\ldots,r_n,\ldots\rangle \tag{4.7}$$

which, for varying $\mathbf{P}_0$ and r_n, forms a basis for our Fock space.

The state vector (4.7) can of course be considered as a functional of $\mathbf{X}(\sigma)$, or equivalently of $\mathbf{X}_n$. The vacuum state $\|0\rangle$ is the Gaussian functional

$$\|0\rangle=\exp\left[-\sum_{n>0}\mathbf{X}_n^2\right] . \tag{4.8}$$

By repeated application of the creation operators α_{-n}, $n>0$, on $\|0\rangle$, one obtains then, as usual, just a bunch of Hermite polynomials in $\mathbf{X}_n$ multiplying the Gaussian functional (4.8), which when considered as functionals of $\mathbf{X}_n$ are always strongly damped for large $\mathbf{X}_n$. This is true for all modes except the zero mode, which is represented in (4.7) by a plane wave $\exp(i\mathbf{P}_0\cdot\mathbf{X}_0)$. The choice of (4.7) as a basis for our Fock space is convenient since the points on a string are held together by the string tension and cannot wander too far from its neighbor. The same condition, however, does not apply to the zero mode, which represents the motion of the center of mass of the string, and is permitted to vary freely.

Let us now turn to the "comma" formulation. Instead of choosing the coordinates $\mathbf{X}_0$ of the center of mass and the coordinates $\mathbf{X}_n\sim\mathbf{X}'(\sigma)$ of all the other points relative to it as variables to describe the string, we can of course equally well choose as alternative variables the coordinates of the midpoint

$$x(\tau)=\mathbf{X}(\tau,\pi/2) \tag{4.9}$$

and the coordinates of the other points relative to x

$$\begin{aligned}\chi_L(\tau,\sigma)&=\mathbf{X}(\tau,\sigma)-x(\tau) ,\\ \chi_R(\tau,\sigma)&=\mathbf{X}(\tau,\pi-\sigma)-x(\tau), \quad \sigma=0\rightarrow\pi/2 .\end{aligned} \tag{4.10}$$

Again, x should be allowed to vary freely, but $\chi_{L,R}$ should be restricted in such a way that neighboring points on the string may not wander too far from each other. To implement this, introduce for the left half string the operators

$$\Pi(\pm\sigma)=P(\pm\sigma)+i\pi\frac{\partial}{\partial x}, \quad \sigma=0\rightarrow\pi/2 , \tag{4.11}$$

where

$$P(\pm\sigma)=-i\pi\frac{\delta}{\delta X(\sigma)}\pm X'(\sigma), \quad \sigma=0\rightarrow\pi/2 . \tag{4.12}$$

Being antiperiodic in the range $-\pi/2\rightarrow\pi/2$, $\Pi(\sigma)$ will have only odd modes in their Fourier expansions:

$$\begin{aligned}\Pi(\sigma)&=\sum_{k\ \mathrm{odd}}\sqrt{2}\,e^{-ik\sigma}\beta_k ,\\ \sigma&=-\pi/2\rightarrow\pi/2 .\end{aligned} \tag{4.13}$$

Furthermore, with respect to the inner product (2.11), $\Pi(\sigma)$ is Hermitian so that again

$$\beta_k^{\dagger}=\beta_{-k} \tag{4.14}$$

and they satisfy the commutation relations

$$[\beta_k^{\mu},\beta_l^{\nu}]=-k\delta(l+k)\eta^{\mu\nu} \tag{4.15}$$

analogous to (4.5) of α_n^{μ} for the full string. Similar operators can be introduced also for the right half string leading to essentially the same mode expansion.

Construct now a Fock space by repeated application of $\beta_{-k}(k>0)$ on the vacuum; thus,

$$|r_1r_3\cdots r_k\cdots\rangle=\beta_{-1}^{k_1}\beta_{-3}^{k_3}\cdots\beta_{-k}^{r_k}\cdots|0\rangle . \tag{4.16}$$

We have two copies of this Fock space corresponding, respectively, to the left and right halves of the string, the tensor product of which we shall denote by matrices; thus,

$$|r_1 r_3 \cdots r_k \cdots \rangle\langle s_1 s_3 \cdots s_l \cdots| = \beta_{-1}^{r_1}\beta_{-3}^{r_3}\cdots\beta_{-k}^{r_k}\cdots|0\rangle\langle 0|\cdots\beta_{-l}^{s_l}\cdots\beta_{-3}^{s_3}\beta_{-1}^{s_1} . \tag{4.17}$$

Folding in further the motion of the midpoint $x = \mathbf{X}(\pi/2)$,

$$\exp(ipx)|r_1 r_3 \cdots r_k \cdots \rangle\langle s_1 s_s \cdots s_l \cdots | , \tag{4.18}$$

we have again a basis for the Fock space of our string in which the overall motion of the string is unrestricted but neighboring points on the string are not permitted to wander far from each other.

The change of basis from (4.7) to (4.17) is not a simple one. The explicit transformations relating them will be reported in a later paper.[12] The commonly used basis (4.7) has of course the merit of being simple for constructing solutions of the free-string equation of motion. On the other hand, the new basis (4.17) has the advantage of being more transparent in questions connected with gauge invariance. Thus, for example, Witten's "star" products[5] defined in terms of half-string functional integrals are awkward to express in terms of the basis (4.7), but are just matrix multiplications in terms of the basis (4.17). Indeed, although we have not yet done so, one may hope in the future to avoid altogether the ambiguities associated with functional integrals by defining matrix products such as (3.6) and inner products such as (2.11) directly in terms of the β oscillator modes. For the present we shall restrict ourselves to using β for defining the ordering for half-string operators, and to reexamine thereby the question of equivalence between the conventional and the "comma" formulations of interacting string theory.

V. THE "TRANSLATION" GENERATORS L_σ

For operators to be finite in the half-string Fock space defined in Sec. IV, they have to be normal ordered with respect to β_k. Thus, for the "translation" generators L_σ in (2.19), we now specify that, by the symbol ::, we mean normal ordering with respect to β_k. Then using (4.11), since $\partial/\partial x$ commutes with $\Pi(\sigma)$, we can write

$$L_\sigma = \frac{1}{2}\left[:[\Pi(\sigma)]^2: - 2i\pi\Pi(\sigma)\frac{\partial}{\partial x} - \pi^2\frac{\partial^2}{\partial x^2}\right] . \tag{5.1}$$

Notice that since $\Pi(\sigma)$ is antiperiodic in the range $[-\pi/2,\pi/2]$, the first term in (5.1) is periodic in the same range, but L_σ itself is neither periodic nor antiperiodic. This means that in its Fourier expansion

$$L_\sigma = \sum_n e^{-in\sigma}L_n \tag{5.2}$$

there will be both odd and even integer modes. Explicitly,

$$\begin{aligned} L_n &= \sum_{k\ \mathrm{odd}} :\beta_k\beta_{n-k}:, \quad n \ \text{even} \neq 0 , \\ L_k &= -i\sqrt{2}\,\pi\frac{\partial}{\partial x}\beta_k, \quad k \ \text{odd} , \\ L_0 &= \sum_{k\ \mathrm{odd}} :\beta_k\beta_{-k}: - \pi^2\frac{\partial^2}{\partial x^2} . \end{aligned} \tag{5.3}$$

The operators $L_n \equiv \Lambda_n$, n even $\neq 0$, together with

$$\Lambda_0 = \sum_{k\ \mathrm{odd}} :\beta_k\beta_{-k}: \tag{5.4}$$

and the identity, generate a closed algebra. This can be shown by evaluating the commutators between them using standard techniques,[13] giving

$$[\Lambda_n,\Lambda_m] = (n-m)\Lambda_{n+m} + \frac{d}{12}(n^3+2n)\delta(n+m) . \tag{5.5}$$

Indeed, the algebra generated is just the standard Virasoro algebra as can be seen by simply shifting Λ_0 by a constant; thus,

$$\Lambda_0 \to \Lambda_0 + d/4 . \tag{5.6}$$

We shall call this algebra $\mathbf{\Lambda}$.

The odd components L_k, k odd, in (5.3), together with $\partial^2/\partial x^2$, also generate a closed algebra

$$[L_k,L_l] = -k\delta(k+l)\left[-2\pi^2\frac{\partial^2}{\partial x^2}\right] \tag{5.7}$$

which we shall call $\mathbf{\Lambda}_\perp$.

Taking a generator from $\mathbf{\Lambda}$ and one from $\mathbf{\Lambda}_\perp$, we have

$$[\Lambda_n, L_k] = -kL_{n+k} \tag{5.8}$$

which is in $\mathbf{\Lambda}_\perp$. The whole algebra $\mathcal{L}$ generated by L_σ is thus a semidirect product of $\mathbf{\Lambda}$ and $\mathbf{\Lambda}_\perp$. Namely, as a linear space

$$\mathcal{L} = \mathbf{\Lambda} \oplus \mathbf{\Lambda}_\perp \tag{5.9}$$

and as an algebra

$$[\mathbf{\Lambda},\mathbf{\Lambda}] \subset \mathbf{\Lambda}, [\mathbf{\Lambda}_\perp,\mathbf{\Lambda}_\perp] \subset \mathbf{\Lambda}_\perp, [\mathbf{\Lambda},\mathbf{\Lambda}_\perp] \subset \mathbf{\Lambda}_\perp . \tag{5.10}$$

The structure of $\mathcal{L}$ is vaguely reminiscent of the Poincaré algebra, with $\mathbf{\Lambda}$ playing the role of the Lorentz algebra, and $\mathbf{\Lambda}_\perp$, which, according to (5.7), is almost commutative, playing the role of the space-time translations. In the present context, $\mathcal{L}$ represents the algebra of conformal reparametrizations of the world surface swept out by the comma with the head of the comma at $x = \mathbf{X}(\pi/2)$, fixed in both space *and* time, while $\mathbf{\Lambda}_\perp$ represents the effect due to the change in x.

VI. EQUIVALENCE TO STANDARD STRING THEORY

Having specified the ordering of the operators L_σ, we are now ready to reexamine the question of equivalence between the "comma" formulation and the standard formulation of string theory.[3]

One crucial step in the "proof" of equivalence between the comma formulation and the standard formulation is the demonstration that the operator Q in (3.2) acting on the potential 1-form A in the matrix notation of (3.1) is equivalent to the BRST charge

$$\mathbf{Q}=\int_{-\pi}^{\pi} d\sigma \left[\eta^\sigma \mathbf{L}_\sigma + 4i\pi\eta^\sigma\eta'^\sigma \frac{\delta}{\delta\eta^\sigma} \right] \quad (6.1)$$

operating on the full-string wave functional $\mathbf{A}$ in the standard formulation. Whence it would follow that the action (3.8) is indeed equivalent to Witten's action. Notice however that the operators L_σ occurring in Q of (3.2) are ordered with respect to the half-string oscillators β_k, while the corresponding full-string operators $\mathbf{L}_\sigma$ occurring in $\mathbf{Q}$ of (6.1) are ordered with respect to the full-string oscillators α_n. This fact was ignored in our earlier treatment because we did not know then what ordering meant in half-string space, but it has now to be taken into account.

We proceed as follows. Starting from the expansion (4.11) for $\Pi(\sigma)$ we can write

$$\Pi^\mu(\sigma)\Pi^\nu(\sigma')=:\Pi^\mu(\sigma)\Pi^\nu(\sigma'):+\eta^{\mu\nu}D(\sigma,\sigma') \quad (6.2)$$

for

$$D(\sigma,\sigma')=\frac{2yz(y^2+z^2)}{(y^2-z^2)^2}, \quad y=e^{i\sigma}, \quad z=e^{i\sigma'} , \quad (6.3)$$

provided that

$$|y|>|z| . \quad (6.4)$$

Hence, since $\partial/\partial x$ commutes with $\Pi(\sigma)$, one can write also, for $P^\mu(\sigma)$ defined in (4.12),

$$P^\mu(\sigma)P^\nu(\sigma')=:P^\mu(\sigma)P^\nu(\sigma'):+\eta^{\mu\nu}D(\sigma,\sigma') \quad (6.5)$$

and for the "translation" generators L_σ in (2.19):

$$L_\sigma=\lim_{\sigma'\to\sigma}[P^\mu(\sigma)P^\mu(\sigma')-dD(\sigma,\sigma')] \quad (6.6)$$

so long as σ and σ' are in the range (6.4).

Consider now the operator Q in (3.2) acting on the gauge potential A. We shall encounter in the resulting expression the commutator

$$[L_\sigma, A_{\sigma''}]=L_\sigma A_{\sigma''}-A_{\sigma''}L_\sigma , \quad (6.7)$$

where both L_σ and $A_{\sigma''}$ are themselves operators on the wave functionals $\Psi[X]$. Thus, using (6.7),

$$L_\sigma A_{\sigma''}\Psi=\lim_{\sigma'\to\sigma}\int \delta X_2[P_1^\mu(\sigma)P_1^\mu(\sigma')-dD(\sigma,\sigma')]A_{\sigma''}[X_1,X_2]\Psi[X_2] , \quad (6.8)$$

$$A_{\sigma''}L_\sigma\Psi=\lim_{\sigma'\to\sigma}\int \delta X_2 A_{\sigma''}[X_1,X_2][P_2^\mu(\sigma)P_2^\mu(\sigma')-dD(\sigma,\sigma')]\Psi[X_2] , \quad (6.9)$$

where

$$P_i^\mu(\pm\sigma)=-i\pi\frac{\delta}{\delta X_i^\mu(\sigma)}\pm X'^\mu_i(\sigma), \quad 0\le\sigma\le\pi/2 . \quad (6.10)$$

Notice that the singular terms proportional to $D(\sigma,\sigma')$ in (6.8) and (6.9) are just c numbers in functional space; they therefore commute with A and cancel in the commutator (6.7). The remaining term in (6.9) involves only an ordinary product of operators $P_2(\sigma)$ and $P_2(\sigma')$ on which we can perform partial functional integrations with respect to $X_2(\sigma)$, obtaining for the commutator just

$$[L_\sigma, A_{\sigma''}]=\lim_{\sigma'\to\sigma}[P_1^\mu(\sigma)P_1^\mu(\sigma') -P_2^\mu(\sigma')P_2^\mu(\sigma)]A_{\sigma''}[X_1,X_2] . \quad (6.11)$$

Change now to full-string notation as in Ref. 3; thus,

$$A_{\sigma''}[X_1,X_2]\to \mathbf{A}_{\sigma''}[\mathbf{X}] ,$$
$$X_1(\sigma)\to\mathbf{X}(\sigma), \quad 0\le\sigma\le\pi/2 , \quad (6.12)$$
$$X_2(\sigma)\to\mathbf{X}(\pi-\sigma), \quad 0\le\sigma\le\pi/2 .$$

We obtain, for $\sigma>0$ or $\sigma<0$, respectively,

$$[L_\sigma, A_{\sigma''}]=\lim_{\sigma'\to\sigma}[\mathbf{P}^\mu(\sigma)\mathbf{P}^\mu(\sigma') -P^\nu(\pm\pi-\sigma')\mathbf{P}^\nu(\pm\pi-\sigma)]\mathbf{A}_{\sigma''}[\mathbf{X}] . \quad (6.13)$$

Using next the expansion (4.4) for $\mathbf{P}(\sigma)$ in terms of the full-string oscillator modes α_n, we can rewrite the first term within brackets in (6.13) as an ordered product ordered with respect to α_n in accordance with the standard formulation. Thus,

$$\mathbf{P}^\mu(\sigma)\mathbf{P}^\nu(\sigma')=::\mathbf{P}^\mu(\sigma)\mathbf{P}^\nu(\sigma')::+\eta^{\mu\nu}\mathbf{D}(\sigma,\sigma'') , \quad (6.14)$$

where

$$\mathbf{D}(\sigma,\sigma'')=\frac{yz}{(y-z)^2}, \quad y=e^{i\sigma}, \quad z=e^{i\sigma'} , \quad (6.15)$$

the formula (6.14) being valid again for y and z satisfying the condition (6.4). For the second term in (6.13) we note that

$$|\exp[i(\pm\pi-\sigma')]|=\frac{1}{|z|}>\frac{1}{|y|}=|\exp[i(\pm\pi-\sigma)]| , \quad (6.16)$$

so that we can write

$$\mathbf{P}^\nu(\pm\pi-\sigma')\mathbf{P}^\nu(\pm\pi-\sigma)=::\mathbf{P}^\nu(\pm\pi-\sigma')\mathbf{P}^\nu(\pm\pi-\sigma):: +\eta^{\mu\nu}\mathbf{D}(\pm\pi-\sigma',\pm\pi-\sigma) . \quad (6.17)$$

But, by (6.15),

$$\mathbf{D}(\pm\pi-\sigma',\pm\pi-\sigma)=\mathbf{D}(\sigma-\sigma') . \quad (6.18)$$

Hence the singular term in (6.17) cancels with that of the first term in (6.13), as exhibited in (6.14). We therefore obtain for the commutator, by (2.16), just

$$[L_\sigma, A_{\sigma''}]=\mathbf{L}_\sigma \mathbf{A}_{\sigma''}-\mathbf{L}_{\pm\pi-\sigma}\mathbf{A}_{\sigma''} \quad (6.19)$$

for, respectively, $\sigma > 0$ or $\sigma < 0$, which is exactly the same as that obtained in Ref. 3. Thus here, at least, it happens that taking proper account of ordering has not changed our earlier conclusion.

Consider next the ghost operators η^σ. The question of ordering enters here only in the bosonization (3.4): namely, in the introduction of a matrix representation for the anticommuting operators η^σ and $\bar{\eta}^\sigma$. Such a representation can be achieved by defining the usual expansion of the exponent $\xi(\sigma)$ in (3.5) in terms of oscillator modes $\hat{b}$ of the half-string:

$$\xi(\sigma)=i[\hat{q}+\hat{p}\sigma+\xi_+(\sigma)+\xi_-(\sigma)] ,$$
$$\xi_-(\sigma)=\sum_{n=2}^{\infty}\left[\frac{2}{n}\right]^{1/2}\hat{b}_n e^{in\sigma},\quad n \text{ even} , \tag{6.20}$$
$$\xi_+(\sigma)=\sum_{n=2}^{\infty}\left[\frac{2}{n}\right]^{1/2}\hat{b}_n^\dagger e^{-in\sigma},\quad n \text{ even} ,$$

and specifying that by : : in (3.4) we mean normal ordering with respect to $\hat{b}$.

Take now the first term of Q in (3.2) operating on the gauge potential 1-form A: namely,

$$\int_{-\pi/2}^{\pi/2} d\sigma\, I_\sigma=\int_{-\pi/2}^{\pi/2} d\sigma (L_\sigma \eta^\sigma A + A L_\sigma \eta^\sigma) , \tag{6.21}$$

where we have used the anticommutivity of η^σ with A. In comma-matrix notation,

$$\eta^\sigma A \Psi=\frac{1}{\sqrt{\pi}}\int \delta\phi_2 {:}\exp[\xi_1(\sigma)]{:}\, A[\phi_1,\phi_2]\Psi[\phi_2] , \tag{6.22}$$

$$A\eta^\sigma \Psi=\frac{1}{\sqrt{\pi}}\int \delta\phi_2 A[\phi_1,\phi_2]{:}\exp[\xi_2(\sigma)]{:}\Psi[\phi_2] , \tag{6.23}$$

where

$$\xi_i(\sigma)=\int_0^\sigma d\sigma' \left[\frac{\delta}{\delta\phi_i(\sigma')}+i\pi\phi_i'(\sigma')\right] . \tag{6.24}$$

Following Ref. 3 we wish to rewrite the first term in (6.21) in full-string notation as

$$\eta^\sigma A=\boldsymbol{\eta}^\sigma \mathbf{A} . \tag{6.25}$$

This is not immediate as it was previously since although η^σ has still formally the same expression in terms of ϕ for both the half string and the full string, the ordering conventions are different in the two cases, and there is also a difference in the normalization by a factor of $\sqrt{2}$ due to the change in the range of σ. Indeed, for the full string, we have

$$\boldsymbol{\eta}^\sigma=\frac{1}{\sqrt{2\pi}}{::}e^{\boldsymbol{\xi}(\sigma)}{::} , \tag{6.26}$$

where :: :: means ordering with respect to full-string oscillators $\hat{a}_n$, not $\hat{b}_n$ as in (6.20). To take account of this we use a trick similar to that employed above for L_σ. We write first

$$\eta^\sigma=\lim_{\sigma_i\to\sigma}\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{1}{r!}{:}\xi(\sigma_1)\xi(\sigma_2)\cdots\xi(\sigma_r){:} . \tag{6.27}$$

Then, using the commutation relations between $\xi_+(\sigma_1)$ and $\xi_-(\sigma_2)$ as defined in (6.20), we can write the ordered products in (6.27) as ordinary products. Thus, for example,

$${:}\xi(\sigma_1)\xi(\sigma_2){:}=\xi(\sigma_1)\xi(\sigma_2)-\Delta_{12} , \tag{6.28}$$

where

$$\Delta_{12}=[\xi_-(\sigma_1),\xi_+(\sigma_2)]=\ln(1-z_1^2/z_2^2) , \tag{6.29}$$

the formula (6.28) being valid for

$$|z_1|<|z_2|,\quad z_1=e^{i\sigma_1},\quad z_2=e^{i\sigma_2} . \tag{6.30}$$

Similar formulas for higher products can be obtained using the Wick theorem for ordering, treating Δ_{ij} as the contraction function. One can thus express the whole series in (6.27) in terms of ordinary products of the operators $\xi(\sigma_i)$, which are the same as the full-string operators $\boldsymbol{\xi}(\sigma_i)$ for $0\le\sigma_i\le\pi/2$. We next rearrange these ordinary products in terms of products ordered with respect to the full-string convention, obtaining

$${::}\boldsymbol{\xi}(\sigma_1)\boldsymbol{\xi}(\sigma_2){::}=\boldsymbol{\xi}(\sigma_1)\boldsymbol{\xi}(\sigma_2)-\boldsymbol{\Delta}_{12} , \tag{6.31}$$

with

$$\boldsymbol{\Delta}_{12}=[\boldsymbol{\xi}_-(\sigma_1),\boldsymbol{\xi}_+(\sigma_2)]=\ln(1-z_1/z_2) , \tag{6.32}$$

and similar expressions for all higher products. Although the contraction functions differ in the two cases,

$$\Delta_{12}=\boldsymbol{\Delta}_{12}+\ln\left[\frac{z_1+z_2}{z_2}\right] , \tag{6.33}$$

the difference is nonsingular and gives in the limit $z_1\to z_2$ just a number, ln2. By collecting all these additional terms in the series (6.27), it can be seen that their effect is just to give an overall multiplicative factor $1/\sqrt{2}$, leading thus to exactly (6.26). This then verifies (6.25).

For the second term in (6.21), we need to perform a functional partial integration with respect to ϕ_2 so as to convert η^σ into an operator acting from the left on the second half of the full-string functional $\mathbf{A}$, analogous to what we did above for the operator L in (6.11). This was what we did also in Ref. 3, but had then to ignore the ordering. To take account now of ordering, we first express (6.27) as a series of ordinary operator products as we did above. We then partially integrate term by term in (6.23) with respect to ϕ_2. For example,

$$\int \delta\phi_2 \frac{1}{\sqrt{\pi}} A[\phi_1,\phi_2]\frac{1}{3!}[{:}\xi_2(\sigma_1)\xi_2(\sigma_2)\xi_2(\sigma_3){:}]\Psi[\phi_2] \tag{6.34}$$

becomes, on partial integration,

$$\int \delta\phi_2 \frac{1}{\sqrt{\pi}}\frac{1}{3!}[\tilde{\xi}_2(\sigma_3)\tilde{\xi}_2(\sigma_2)\tilde{\xi}_2(\sigma_1)-\Delta_{12}\tilde{\xi}_2(\sigma_3)$$
$$-\Delta_{13}\tilde{\xi}_2(\sigma_2)-\Delta_{23}\tilde{\xi}_2(\sigma_1)]A[\phi_1,\phi_2]\Psi[\phi_2] , \tag{6.35}$$

where

$$\tilde{\xi}_2(\sigma_i)=\int_0^{\sigma_i} d\sigma \left[-\frac{\delta}{\delta\phi_2(\sigma)}+i\pi\phi_2'(\sigma)\right] . \tag{6.36}$$

Change now to full-string notation:

$$A[\phi_1,\phi_2]\to \mathbf{A}[\boldsymbol{\phi}] ,$$
$$\phi_1(\sigma)\to\boldsymbol{\phi}(\sigma),\quad \phi_2(\sigma)\to\boldsymbol{\phi}(\pi-\sigma),\quad 0\le\sigma\le\pi/2 . \tag{6.37}$$

We have

$$\tilde{\xi}_2(\sigma_i)=-\int_{\pi-\sigma_i}^{\pi} d\sigma \left[\frac{\delta}{\delta\boldsymbol{\phi}(\sigma)}+i\pi\boldsymbol{\phi}'(\sigma)\right]$$
$$=\xi(\pi-\sigma_i) . \tag{6.38}$$

Noting then that

$$|z_1|<|z_2|<|z_3|,\quad z_i=e^{i\sigma_i} \tag{6.39}$$

implies

$$|\exp[i(\pi-\sigma_3)]<|\exp[i(\pi-\sigma_2)]|$$
$$<|\exp[i(\pi-\sigma_1)]| \tag{6.40}$$

and that, for $\boldsymbol{\Delta}_{12}=\boldsymbol{\Delta}(\sigma_1,\sigma_2)$ as given in (6.32),

$$\boldsymbol{\Delta}(\pi-\sigma_2,\pi-\sigma_1)=\boldsymbol{\Delta}(\sigma_1,\sigma_2) \tag{6.41}$$

we can resum the expression within brackets in (6.35) as the ordered product:

$$::\xi(\pi-\sigma_1)\xi(\pi-\sigma_2)\xi(\pi-\sigma_3):: \tag{6.42}$$

plus terms proportional to ln2, which eventually give just a factor $1/\sqrt{2}$ as above. The same can be done for every term in the expansion (6.27). Then resumming and taking the limit $\sigma_i\to\sigma$, we obtain

$$A\eta^{\sigma}=\boldsymbol{\eta}^{\pi-\sigma}\mathbf{A} . \tag{6.43}$$

In other words, for I_σ in (6.21), we have

$$I_\sigma=\mathbf{L}_\sigma\boldsymbol{\eta}^{\sigma}\mathbf{A}+\mathbf{L}_{\pi-\sigma}\boldsymbol{\eta}^{\pi-\sigma}\mathbf{A} \tag{6.44}$$

which is again the same as in Ref. 3 before we took ordering into account.

The same analysis can be repeated for the second term of Q in (3.2), leading to the desired conclusion that the exterior derivative Q operating on the gauge potential 1-form A in the comma formulation is, in fact, the same as the BRST charge operator $\mathbf{Q}$ operating on the string functional $\mathbf{A}[\mathbf{X}]$ in the standard full-string formulation. It appears therefore, at least to the present limited degree of rigor in our demonstration, that our former reservation in Ref. 3 about the equivalence being only formal between the two formulations is now removed.

VII. REMARKS

Apart from the conceptual value of viewing string theory as a more or less direct generalization of Yang-Mills theory, which may go some way toward achieving a final understanding of the theory's geometrical or physical significance, our considerations here possess some points of interest which can lead to practical applications.

First, although we have not here done so explicitly, it is clear that internal unitary symmetry of the usual "extrinsic" type, such as color SU(3), can be incorporated into our scheme simply by taking the gauge potential A to be a matrix, not only in the continuum "comma" indices $X(\sigma)$ and $\phi(\sigma)$, but also in the discrete indices corresponding to the "extrinsic" symmetry. For example, color SU(3) can be incorporated by taking A to be a Hermitian 3×3 matrix, which may be written

$$A=\sum_i A_i\lambda^i . \tag{7.1}$$

To make the action (3.8) invariant under color SU(3), all that is needed is to extend the definition of the trace Tr in there to include a trace over SU(3) indices. For instance, for the interaction term $\mathrm{Tr}(A\cdot A\cdot A)$ in (3.8), the substitution of (7.1) gives

$$\sum_{ijk}\mathrm{Tr}(A_iA_jA_k)\mathrm{Tr}(\lambda^i\lambda^j\lambda^k) \tag{7.2}$$

so that the interaction vertex for three strings of colors, respectively, i, j, and k is proportional to the trace factor $\mathrm{Tr}(\lambda^i\lambda^j\lambda^k)$.

Now this was exactly how internal symmetry was incorporated in the dual model of 20 years ago.[14] What is needed in a vertex to make a theory invariant under SU(3) is that it should be proportional to a Clebsch-Gordan coefficient coupling the strings of different colors together into an SU(3) scalar. The particular virtue of writing it as a trace is that it can be trivially extended to n strings in a dual fashion. The interesting point here is that it seems that history is repeated. In the formulation of string theory presented here, the basic symmetry is the invariance under unitary comma gauge transformations, as exhibited in (2.4). To preserve this invariance, the interaction vertex has presumably to be proportional to a "Clebsch-Gordan" coefficient of the huge comma gauge group. The gauge potential A is in the "adjoint" representation of this group and is represented as a comma matrix, just as the potential in SU(3) was represented by the matrix of (7.1), and again the solution is that we take the trace with respect to the comma-matrix indices. This suggests that in coupling n strings together to form a comma gauge invariant in an explicitly dual fashion, one may again take the trace with respect to comma indices of a product of n-string fields. Conceivably, therefore, the representation of A as comma matrices may facilitate the evaluation of n-string vertices and hence eventually also of string amplitudes in general.

Second, as already mentioned in Sec. III, it seems natural in the present context to consider alongside the pure Yang-Mills theory embodied in the action (3.8) a theory involving also the comma field Ψ. In analogy to QCD, for example, the theory of (3.8) contains only "gluons," while the comma field Ψ is the analogue of the fundamental quark matter field. With the covariant derivative $\mathcal{L}_\sigma=(L_\sigma-A_\sigma)$ acting on Ψ, one can readily construct comma gauge invariants from Ψ to serve as candidate actions, as attempted already in Ref. 3. Now that we have introduced an operator formalism for the comma, giving thus a more precise meaning to operations on Ψ, there is hope that a concrete theory containing Ψ may be formulated. It is possible that the new freedom gained in this way can be useful in attempts to construct realistic physical models which have so far proved elusive.

Our considerations above may conceivably also be of interest in a wider context in that in order to reformulate string theory in Yang-Mills language we have been forced to generalize the usual gauge theory concepts in some rather unfamiliar directions. The extension of the gauge group to an enormous group of unitary functional transformations is dramatic, but is perhaps not so unexpected for an extended object. What is more surprising is the necessity also to generalize the concept of the base; first, from the usual translation group in space-time to a non-Abelian group and, second, to working with representations of such a group on a space other than the space of functions of the points in the base manifold. It appears thus that even the concept of locality has to be generalized and, as pointed out in Ref. 5, this takes us to the physically unfamiliar realm of noncommutative geometry.[15] Although one is still far from clear what this means in physical terms, one may well consider whether such a generalization of gauge concepts may not be useful also in other physical systems.

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