New perturbative approximation applied to a self-interacting scalar field theory

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A recently proposed new perturbation technique is applied to a $\lambda(\phi^2)^{1+\delta}$ self-interacting scalar field theory in D-dimensional space-time. We calculate, to second order in powers of δ, not only the two-point and the four-point Green's functions but also the $(2+2\delta)$-point Green's function, which can be defined in this approximation. After the renormalization procedures are carried out to this order, we find that the theory is not fully nontrivial for $D \geq 2$.

I. INTRODUCTION

A new perturbative method was proposed in a recent series of papers by Bender *et al.*[1–4] The method introduces an artificial perturbation parameter and expands the theory with respect to it. A scalar polynomial field theory (e.g., $\lambda\phi^4$) is written as $\lambda(\phi^2)^{1+\delta}$ and δ is regarded as the perturbation parameter. Following the rules in Ref. 1, several theories are expanded successfully; a zero- and a two-dimensional field theory,[1] a two-dimensional supersymmetric quantum field theory,[2] and a scalar field theory in arbitrary dimensions.[4]

Since all physical parameters such as the coupling constant and the mass need not be small in this expansion, we have hope that it is a useful tool to solve the triviality problem of $\lambda(\phi^4)_4$ theory[5] which cannot be studied in the standard weak-coupling perturbation. Bender and Jones[4] already examined the problem using the δ expansion and had results that a $\lambda(\phi^2)_D^{1+\delta}$ theory is nontrivial for $D<4$ and it is suggested to be a free theory for $D \geq 4$, which is consistent with a common belief in $\lambda(\phi^4)_D$. The purpose of this paper is to examine $\lambda(\phi^2)_D^{1+\delta}$ theory from another point of view in order to see whether the theory is trivial and whether this new perturbation is useful to solve the triviality problem. We calculate Green's functions to second order of δ. We introduce $(2+2\delta)$-point Green's functions and define a corresponding coupling constant. The renormalizations can be done using this coupling constant. Contrary to a claim of Ref. 4, we find that the theory is suggested to be trivial for all space-time dimensions higher than two. It seems that this conclusion is not directly applied to $\lambda(\phi^4)_D$ theory.

One disadvantage of this approach is that the diagrammatic rules developed in Ref. 1 are complicated and we are not able to know *a priori* how to calculate an arbitrary order in powers of δ; it is not shown in Ref. 1 how to make calculations higher than δ^4. Here we present simpler rules which makes any order calculations possible and which does not require a provisional Lagrangian introduced in Ref. 1. Other simpler rules were also proposed recently in Ref. 3. Both our rules and those of Ref. 2 give the same results.

II. RULES FOR THE δ EXPANSION

We consider the D-dimensional scalar field theory defined by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2-\frac{m_0^2}{2}\phi^2-\frac{\lambda_0}{4}\mu^D(\mu^{2-D}\phi^2)^{1+\delta} \;, \tag{2.1}$$

where μ is a dimensional parameter introduced to make λ_0 dimensionless and where m_0 and λ_0 are the bare mass and the bare coupling constant, respectively. It should be noted that λ_0 is the coupling constant of $(2+2\delta)$-point vertex. In order to do the δ expansion, it is convenient to divide (2.1) into two parts, unperturbative $\mathcal{L}_0$ and perturbative ones $\mathcal{L}_I$:

$$\mathcal{L}_0=\tfrac{1}{2}(\partial\phi)^2-\frac{m^2}{2}\phi^2 \;, \tag{2.2}$$

$$\mathcal{L}_I=-\frac{\lambda_0}{4}\mu^2\phi^2[(\mu^{2-D}\phi^2)^\delta-1] \;, \tag{2.3}$$

where

$$m^2=m_0^2+\lambda_0\mu^2/2 \;. \tag{2.4}$$

Using the relation

$$y^\delta=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\delta\frac{d}{dk}\right]^n y^k\Bigg|_{k=0}\equiv e^{\delta\partial_k}y^k|_{k=0} \;, \tag{2.5}$$

we rewrite $\mathcal{L}_I$ such as

$$\mathcal{L}_I=-D_k\phi^{2k+2}|_{k=0}$$

with (2.6)

$$D_k=\frac{\lambda_0}{4}\mu^2(e^{\delta\partial_k}-1)(\mu^{2-D})^k \;.$$

When δ is small, i.e., D_k can be regarded as a small "coupling constant," then we can make an ordinary perturbation if k is an integer. Thus rules for calculating the n-point Green's function

$$G^{(n)}(x_1,x_2,\ldots,x_n)=\prod_{p=0}^{\infty}\frac{1}{p!}\int d^Dy_1d^Dy_2\cdots d^Dy_p\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)$$
$$\times D_{k_1}D_{k_2}\cdots D_{k_p}[\phi^2(y_1)]^{k_1+1}[\phi^2(y_2)]^{k_2+1}\cdots[\phi^2(y_p)]^{k_p+1}|0\rangle_c|_{k=0} \quad (2.7)$$

are given as the following steps.

Step 1. First, regard $k_1,k_2,\ldots,k_p$ as integers with the same value. Draw all diagrams contributing to $G^{(n)}$. Then count an overall symmetry factor.

Step 2. Regard $k_1,k_2,\ldots,k_p$ as integers with $k_i\neq k_j$ for $i\neq j$. Apply ordinary diagrammatic perturbation[6] (regard $D_{k_1},D_{k_2},\ldots,D_{k_p}$ as small) and calculate $G^{(n)}$.

Step 3. Regard $k_1,k_2,\ldots,k_p$ as continuous with $k_i\neq k_j$ for $i\neq j$. Apply derivative operators D_{k_i} and finally set all k_i as zero.

In order to see how these rules work, we show, as a simple example, how to obtain the free energy up to $O(\delta^2)$ in the zero-dimensional field theory given in Ref. 1, where the partition function is

$$Z=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}\exp[-(x^2)^{1+\delta}]$$
$$=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}}\exp[-x^2-x^2(x^{2\delta}-1)] . \quad (2.8)$$

The Lagrangian is written by

$$\mathcal{L}^{(0)}=x^2+D_k^{(0)}x^{2k+2}|_{k=0} \text{ with } D_k^{(0)}=e^{\delta\partial_k}-1 . \quad (2.9)$$

The diagrams of order $O(\delta)$ and $O(\delta^2)$ contributing to the energy E are given in Fig. 1. Note that the overall symmetry factor of the diagram in Fig. 1(b) is $\frac{1}{2}$ since we regard $k_1=k_2$ following step 1 of the rules. The Feynman rules in step 2, which are explained in detail in Ref. 1, give E_1 and E_2, the contributions from Figs. 1(a) and 1(b), as

$$E_1=\frac{D_k(2k+2)!}{2^{2+2k}(k+1)!}=\frac{D_k\Gamma(k+\frac{3}{2})}{2\Gamma(\frac{3}{2})} , \quad (2.10)$$

$$E_2=-\frac{1}{2}\sum_{l=1}^{l_{\max}}\frac{D_{k_1}D_{k_2}(2k_1+2)!(2k_2+2)!4^l}{4^{k_1+k_2+2}(2l)!(k_1+1-l)!(k_2+1-l)!}$$
$$=-\frac{1}{2\pi}D_{k_1}D_{k_2}[\Gamma(k_1+k_2+\tfrac{5}{2})\sqrt{\pi}-\Gamma(k_1+\tfrac{3}{2})\Gamma(k_2+\tfrac{3}{2})] , \quad (2.11)$$

where $l_{\max}=\min(k_1+1,\ k_2+1)$. The first factor $\frac{1}{2}$ on the right-hand side of the first line of (2.11) is the overall symmetry factor derived in step 1. In the second line of (2.11), we have used the formula

$$\sum_{l=1}^{l_{\max}}\frac{4^l}{\Gamma(2l+1)\Gamma(\alpha-l+2)\Gamma(\beta-l+2)}=\frac{1}{\Gamma(\alpha+2)\Gamma(\beta+2)}\left[\frac{\Gamma(\alpha+\beta+\frac{5}{2})\sqrt{\pi}}{\Gamma(\alpha+\frac{3}{2})\Gamma(\beta+\frac{3}{2})}-1\right] .$$

Step 3 leads to

$$E_1=\left[\delta\partial_k+\frac{\delta^2\partial_k^2}{2}\right]\frac{\Gamma(k+3/2)}{2\Gamma(3/2)}$$
$$\overset{k\to 0}{\Longrightarrow}\frac{\delta}{2}\psi(\tfrac{3}{2})+\frac{\delta^2}{4}[\psi'(\tfrac{3}{2})+\psi^2(\tfrac{3}{2})] , \quad (2.12)$$

$$E_2=-\frac{\delta^2}{2}[\tfrac{3}{4}\psi'(\tfrac{3}{2})+\tfrac{1}{2}\psi^2(\tfrac{3}{2})+\psi(\tfrac{3}{2})] , \quad (2.13)$$

where $\psi(z)$ is the ψ function, $\psi(z)\equiv\Gamma'(z)/\Gamma(z)$. Then we eventually have

$$E=E_1+E_2=\frac{\delta}{2}\psi(\tfrac{3}{2})-\frac{\delta^2}{2}[\psi(\tfrac{3}{2})+\psi'(\tfrac{3}{2})] , \quad (2.14)$$

which agrees with the exact solution up to this order. Likewise, we can obtain the Green's function at any order in powers of δ if we use the rules described here and if we are careful and patient enough to accomplish lengthy calculations.

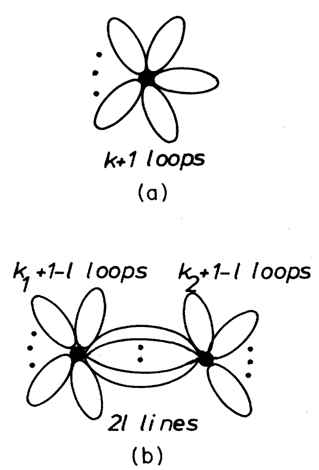

FIG. 1. (a) The leading-order diagram contributing to the energy. The diagram consists of one vertex with $k+1$ loops. (b) The δ^2-order diagram contributing to the energy. The diagram consists of two vertices with k_1+1-l and k_2+1-l loops and of $2l$ internal lines.

III. SELF-INTERACTING SCALAR FIELD THEORY

A. The $(2k+2)$-point Green's function

Let us calculate the $(2+2k)$-point Euclidean Green's function up to order δ^2. Here k is an arbitrary integer which we, however, regard as continuous in the later stage. The diagrams we need are drawn in Fig. 2. According to steps 1 and 2 in the previous section, the leading-order contribution $G^{(2k+2)}_{(1)}$ is easy to estimate:

$$G^{(2k+2)}_{(1)} = -D_{k_1} \frac{(2k_1+2)!}{2^{k_1-k}(k_1-k)!}[I_0(m)]^{k_1-k} . \tag{3.1}$$

At this stage, k_1 is assumed to be an integer with $k_1 > k$. $I_0(m)$ is the loop integral given by

$$I_{-n}(m) = \int \frac{d^{D-1}q}{2(2\pi)^{D-1}} \frac{1}{(\mathbf{q}^2+m^2)^{n+1/2}} . \tag{3.2}$$

This is the divergent integral for $n \le [(D-2)/2]$ which is regularized by an introduction of the ultraviolet cutoff Λ. Following step 3, we have

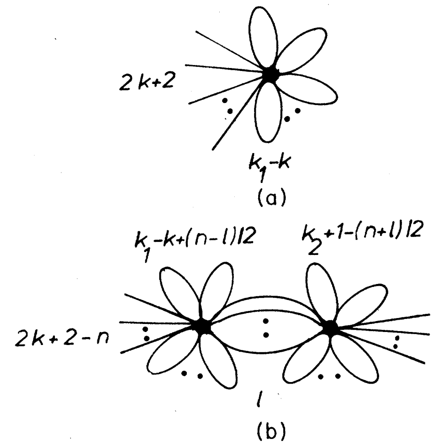

FIG. 2. (a) The leading-order diagram contributing to the $(2+2k)$-point Green's functions. The diagram has one vertex with k_1-k loops. (b) The δ^2-order diagram contributing to the $(2+2k)$-point Green's functions. The diagram has two vertices with $k_1-k+(n-l)/2$ and $k_2+1-(n+l)/2$ loops and l internal lines.

$$G^{(2k+2)}_{(1)} = -\frac{\delta}{2}\lambda_0\mu^2 \left[\frac{2}{I_0(m)}\right]^k \frac{1}{\Gamma(1-k)} \left[\psi(\tfrac{3}{2})+\psi(2)-\psi(1-k)+L + \frac{\delta}{2}\{[\psi(\tfrac{3}{2})+\psi(2)-\psi(1-k)+L]^2+\psi'(\tfrac{3}{2})+\psi'(2)-\psi'(1-k)\} \right] , \tag{3.3}$$

where

$$L = \ln[2\mu^{2-D}I_0(m)] . \tag{3.4}$$

The next order contribution $G^{(2k+2)}_{(2)}(p)$ is estimated from the diagram drawn in Fig. 2(b), where p is the external momentum. It follows from steps 1 and 2 of the previous rules that

$$G^{(2k+1)}_{(2)}(p) = \frac{1}{2}\sum_{n=0}^{2k+2} \frac{(2k+2)!}{(2k+2-n)!n!} D_{k_1} D_{k_2} \sum_{l=2}^{l_{\max}} \frac{(2k_1+2)![I_0(m)]^{k_1-k+n/2--l/2}}{2^{k_1-k+n/2-l/2}(k_1-k+n/2-l/2)!} \times \frac{(2k_2+2)![I_0(m)]^{k_2+1-n/2-l/2}}{2^{k_2+1-n/2-l/2}(k_2+1-n/2-l/2)!} \frac{J_l(\tilde{p})}{l!} , \tag{3.5}$$

where the sums over l and n must be taken under the constraint

$$l+n = \text{even} . \tag{3.6}$$

Here some explanations of notation are necessary. First, $l_{\max}$ is given by

$$l_{\max} = \min(2k_2+2-n, 2k_1+n-2k) . \tag{3.7}$$

It should be noted that we take $l_{\max} \to \infty$, since the factor

$$[(k_1-k+n/2-l/2)!(k_2+1-l/2-n/2)!]^{-1} = [\Gamma(k_1-k+n/2-l/2+1)\Gamma(k_2+1-l/2-n/2+1)]^{-1}$$

in (3.5) vanishes when $l > l_{\max}$. $J(\tilde{p})$ is the contribution from the multiple loop appearing in the middle between two vertices of Fig. 2(b):

After calculations according to step 3 of the rules, we have

$$J_l(\tilde{p}) = \prod_{i=1}^{l} \int \frac{d^D q}{(2\pi)^D} \frac{(2\pi)^D \delta^{(D)}\left[\tilde{p} - \sum_{j=1}^{l} q_j\right]}{q_i^2+m^2}$$
$$= \int d^D x \, e^{-i\tilde{p}x} [I_0(x;m)]^l \,, \tag{3.8}$$

where

$$I_0(x;m) = \int \frac{d^D q}{(2\pi)^D} \frac{e^{iqx}}{q_i^2+m^2} \tag{3.9}$$

and

$$\tilde{p} = \begin{cases} 0 & \text{for } n=0 \text{ or } 2k+2 \,, \\ p & \text{otherwise} \,, \end{cases} \tag{3.10}$$

$$G_{(2)}^{(2k+2)}(p) = \frac{\delta^2}{2}(\lambda_0\mu^2)^2 \sum_{n=0}^{2k+2} \frac{(2k+2)!}{(2k+2-n)!n!}$$
$$\times \sum_{l=2}^{\infty} \frac{2^{k-1-n/2+l/2}}{\Gamma(1-k+n/2-l/2)} \left[\psi(\tfrac{3}{2})+\psi(2)-\psi\left[1-k+\frac{n}{2}-\frac{l}{2}\right]+L\right][I_0(m)]^{-k-l+1}$$
$$\times \frac{2^{-2+n/2+l/2}}{\Gamma(2-n/2-l/2)} \left[\psi(\tfrac{3}{2})+\psi(2)-\psi\left[2-\frac{n}{2}-\frac{l}{2}\right]+L\right] \frac{J_l(\tilde{p})}{l!} \,. \tag{3.11}$$

Taking the fact $1-k+n/2-l/2 \le 1$ and $2-l/2-n/2 \le 1$ and the relation $\psi(-m)/\Gamma(-m)=(-1)^{m-1}m!$ into account, we can rewrite (3.11) as

$$G_{(2)}^{(2k+2)}(p) = \frac{\delta^2}{2}(\lambda_0\mu^2)^2 \left[\sum_{n=0}^{2k+2} \frac{(2k+2)!}{(2k+2-n)!n!} \sum_{l=2}^{\infty} 2^{k+l-3} \left[\frac{n+l}{2}-2\right]! \left[k-1+\frac{l-n}{2}\right]! [-I_0(m)]^{-k-l+1} \frac{J_l(\tilde{p})}{l!} \right.$$
$$\left. \times [1-\delta_{l,2}(\delta_{n,0}+\delta_{n,2k+2})] - 2^{k-1}k!S[-I_0(m)]^{-k-1}J_2(\tilde{p}) \right] \,, \tag{3.12}$$

where we use the notation

$$S = \psi(\tfrac{3}{2})+1+L \,. \tag{3.13}$$

Setting $k=0$ and $k=1$ in (3.12), we have the two-point and the four-point Green's functions (by adding the zeroth-order contribution for the two-point function), respectively. Now it should be emphasized that an introduction of noninteger vertex is possible and is natural in our approximation. We can define $(2+2N\delta)$-point Green's functions (N=integer≥ 1) such as

$$G^{(2+2N\delta)}(p) \equiv e^{N\delta\partial_k} \sum_{n=0}^{2k+2} \frac{(2k+2)!}{(2k+2-n)!n!} \times \langle 0|\phi^n(p)\phi^{2+2k-n}(0)|0\rangle|_{k=0} \,, \tag{3.14}$$

where p is the total momentum of n external lines.

B. The renormalization

From the Green's function $G^{(2k+2)}(p^2=0)$, we define the renormalized parameter, the mass m_R, the $(2+2\delta)$-vertex coupling constant λ_R, the $(2+2N\delta)$-vertex coupling constant $\lambda_R^{(N\delta)}$, and the four-point vertex coupling constant $\lambda_R^{(4)}$ as follows:

$$Z^{-1}m_R^2 = [G^{(2)}(p^2=0)]^{-1} \,, \tag{3.15}$$

$$Z^{-1-\delta}\delta\lambda_R = G^{(2+2\delta)}(p^2=0)/\mu^{2+(2-D)\delta} \,, \tag{3.16}$$

$$Z^{-1-N\delta}\delta\lambda_R^{(N\delta)} = G^{(2+2N\delta)}(p^2=0)/\mu^{2+(2-D)N\delta} \,, \tag{3.17}$$

$$Z^{-2}\delta\lambda_R^{(4)} = G^{(4)}(p^2=0)/\mu^{4-D} \,. \tag{3.18}$$

The wave-function renormalization constant Z is defined by

$$Z^{-1} = 1 + d[G^{(2)}(p^2)]^{-1}/dp^2|_{p^2=0} \,. \tag{3.19}$$

Some calculations shown in Appendix A lead to

$$Z^{-1}m_R^2 = m^2 + \frac{\delta}{2}\lambda_0\mu^2\left[S+\frac{\delta}{2}[S^2+\psi'(\tfrac{3}{2})-1]\right] - \frac{\delta^2}{4}\lambda_0^2\mu^4\left[\frac{I_{-1}(m)}{I_0(m)}S+I_0(m)F_1(0)\right] \,, \tag{3.20}$$

$$Z^{-1-\delta}\lambda_R=\frac{\lambda_0}{2}S+\frac{\delta}{4}\lambda_0[S^2+\psi'(\tfrac{3}{2})-1]+\frac{\delta}{2}\lambda_0\{-S^2+S[\psi(\tfrac{3}{2})+\ln 2+1+\psi(1)]+\psi'(1)\}$$
$$-\frac{\delta}{4}\lambda_0^2\mu^2\left[\frac{I_{-1}(m)}{I_0(m)}S+I_0(m)F_1(0)\right], \tag{3.21}$$

$$Z^{-1-N\delta}\lambda_R^{(N\delta)}=\frac{\lambda_0}{2}S+\frac{\delta}{4}\lambda_0[S^2+\psi'(\tfrac{3}{2})-1]+\frac{\delta}{2}N\lambda_0\{-S^2+S[\psi(\tfrac{3}{2})+\ln 2+1+\psi(1)]+\psi'(1)\}$$
$$-\frac{\delta}{4}\lambda_0^2\mu^2\left[\frac{I_{-1}(m)}{I_0(m)}S+I_0(m)F_1(0)\right], \tag{3.22}$$

$$Z^{-2}\lambda_R^{(4)}=\frac{\lambda_0\mu^{D-2}}{I_0(m)}(1+\delta S)+\frac{\delta}{2}\lambda_0^2\mu^D\frac{I_{-1}(m)}{I_0^2(m)}S+\frac{\delta}{4}\lambda_0^2\mu^D F_2(0), \tag{3.23}$$

where

$$Z^{-1}=1-\frac{\delta^2}{4}(\lambda_0\mu^2)^2 I_0(m)\left.\frac{dF_1}{dp^2}\right|_{p^2=0}, \tag{3.24}$$

and

$$F_1(0)=\int d^Dx\int_0^1 dt\frac{\sqrt{1-t}}{t^2}[(1-\sqrt{z}\,t)\ln(1-zt)+tz], \tag{3.25}$$

$$\left.\frac{dF_1}{dp^2}\right|_{p^2=0}=\frac{1}{2D}\int d^Dx\, x^2\sqrt{z}\int_0^1 dt\frac{\sqrt{1-t}}{t}\ln(1-zt), \tag{3.26}$$

$$F_2(0)=\int d^Dx\int_0^1 dt\left[\frac{3\ln(1-zt)}{t\sqrt{1-t}}+(4z^{3/2}-2z^2)\frac{\sqrt{1-t}}{1-t^2}\right], \tag{3.27}$$

with

$$z\equiv I_0^2(x;m)/I_0^2(m). \tag{3.28}$$

Our results of m_R^2, $\lambda_R^{(4)}$, and Z agree with those of Ref. 4. Taking into account of the Λ dependences of F_1, dF_1/dp^2, and F_2 which are examined in Appendix B, we write the most divergent parts of (3.20)–(3.32) as

$$m_R^2=m^2+\frac{\delta}{2}\lambda_0\mu^2S+\frac{\delta^2}{4}\lambda_0\mu^2[S^2+\psi'(\tfrac{3}{2})-1]+O\left[\frac{\delta^2 I_{-1}(m)}{I_0(m)}S\right], \tag{3.29}$$

$$\lambda_R=\frac{\lambda_0}{2}S+\frac{\delta}{4}\lambda_0\{-S^2+2S[\psi(\tfrac{3}{2})+\ln 2+1+\psi(1)]+2\psi'(1)+\psi'(\tfrac{3}{2})-1\}+O\left[\frac{\delta I_{-1}(m)}{I_0(m)}S\right], \tag{3.30}$$

$$\lambda_R^{(N\delta)}=\lambda_R+\frac{\delta}{2}\lambda_0(N-1)\{-S^2+S[\psi(\tfrac{3}{2})+\ln 2+1+\psi(1)]+\psi'(1)\}+O\left[\frac{\delta I_{-1}(m)}{I_0(m)}S\right], \tag{3.31}$$

$$\lambda_R^{(4)}=\frac{\lambda_0\mu^{D-2}}{I_0(m)}(1+\delta S)+O\left[\frac{\delta I_{-1}(m)}{I_0(m)}SF_2\right]. \tag{3.32}$$

Now we show that the theory can be renormalized in terms of m_R^2 and λ_R. We begin by writing the bare coupling constant λ_0 as

$$\lambda_0\approx\frac{2\lambda_R}{S_R+\delta(-S_R^2/2+S_R[\psi(\tfrac{3}{2})+\ln 2+1+\psi(1)]+\psi'(1)+\psi'(\tfrac{3}{2})/2-1/2)}$$
$$\approx\frac{2\lambda_R}{S_R}+\delta\lambda_R\left[1-\frac{2}{S_R}[\psi(\tfrac{3}{2})+\psi(1)+1+\ln 2]+O\left[\frac{1}{S_R^2}\right]\right], \tag{3.33}$$

which can be derived from (3.30) by noting $S=S_R+O(\delta I_{-1}/I_0)$, where

$$S_R=\psi(\tfrac{3}{2})+1+\ln[2\mu^{2-D}I_0(m_R)]. \tag{3.34}$$

Substituting (3.33) into (3.29), we have

$$m^2=m_R^2-\delta\lambda_R\mu^2+\delta^2\lambda_R\mu^2\left[-S_R+\psi(\tfrac{3}{2})+\psi(1)+1+\ln 2+O\left[\frac{1}{S_R}\right]\right]. \tag{3.35}$$

Using (3.33) and (3.35) into (3.31), we obtain

$$\lambda_R^{(N\delta)}=\lambda_R+\frac{\delta(N-1)\lambda_R\{-S_R^2+S[\psi(\frac{3}{2})+\ln 2+1+\psi(1)]+\psi'(1)\}}{S_R+\delta(-S_R^2/2+S_R[\psi(\frac{3}{2})+\ln 2+1+\psi(1)]+\psi'(1)+\psi'(\frac{3}{2})/2-1/2)}+O\left[\frac{\delta I_{-1}}{I_0}S\right]. \tag{3.36}$$

We can eliminate the divergences from (3.36) because the most divergent terms S_R^2 appear in both numerator and denominator on the right-hand side of (3.36). We have a finite $\lambda_R^{(N\delta)}$:

$$\lambda_R^{(N\delta)}\approx\lambda_R+2(N-1)\lambda_R=(2N-1)\lambda_R+O(1/S_R). \tag{3.37}$$

Thus when we choose the bare parameters as (3.33) and (3.35), it seems that we finish the renormalization program successfully up to $O(\delta^2)$ and that we get an evidence that the theory is nontrivial. However, this statement is not completely correct. To obtain (3.37), we implicitly assume $\delta S_R\gg 1$ which contradicts with $\delta\to 0$. Usually we mean by the perturbation with respect to δ that δ must be smaller than all other quantities, e.g., $1/\Lambda$ or $1/S_R$. If $\delta\ll 1/S_R$, then (3.37) becomes

$$\lambda_R^{(N\delta)}=\lambda_R+\delta\lambda_R(N-1)[-S_R+\psi(\tfrac{3}{2})+\ln 2+1+\psi(1)]+O\left[\frac{1}{S_R}\right]. \tag{3.38}$$

We cannot eliminate the cutoff-dependent term S_R from (3.38). One may suspect that the theory cannot be renormalized in this way. However, we think that this is not so but this is a signal of the triviality, because the right-hand side of (3.38) is suggested to have the form

$$\lambda_R^{(N\delta)}\approx\frac{\lambda_R}{1+\delta(N-1)[S_R-\psi(\frac{3}{2})-\ln 2-1-\psi(1)]}, \tag{3.39}$$

i.e., which looks like $\lambda_R^{(N\delta)}\to 0$ as $\Lambda\to\infty$. In fact the form of (3.39) can be guessed from the consideration of $\lambda_R^{(4)}$: Eq. (3.32) reads, by using (3.33) and (3.35),

$$\lambda_R^{(4)}=\frac{2\lambda_R\mu^{D-2}}{I_0(m_R)}\left[\frac{1}{S_R}+\delta\right], \tag{3.40}$$

which vanishes as $\Lambda\to\infty$.

IV. CONCLUDING REMARKS

Let us summarize our conclusions with some comments.

(1) The self-interacting scalar $\lambda(\phi^2)_D^{1+\delta}$ theory can be renormalized [at least up to $O(\delta^2)$] in terms of m_R^2 and λ_R, the renormalized coupling constant of $2+2\delta$ vertex.

(2) $\lambda(\phi^2)_D^{1+\delta}$ is not fully nontrivial in the sense that $\lambda_R^{(4)}$ [and $\lambda_R^{(N\delta)}(N>1)$]$\to 0$ as $\Lambda\to\infty$. To be more precise, $\lambda_R^{(N\delta)}$ is nontrivial when the limit $\Lambda\to\infty$ is taken first before $\delta\to 0$ and it looks trivial when $\delta\to 0$ is taken before the limit $\Lambda\to\infty$. On the other hand, $\lambda_R^{(4)}$ becomes always zero for $\Lambda\to\infty$. This holds any dimensions higher than two and perhaps lower than $O(1/\delta)$.

(3) Comparing λ_R or $\lambda_R^{(N\delta)}$ with $\lambda_R^{(4)}$, we note that the cutoff Λ dependences are quite different. For $\delta\to 1$, λ_R approaches the four-point vertex coupling. However, λ_R has a totally different Λ behavior from that of $\lambda_R^{(4)}$ even when $\delta\to 1$. This is because our calculations are only up to $O(\delta^2)$ though the higher-order corrections to λ_R become important for $\delta\to 1$. Namely, effects of higher-order terms are significant and cannot be ignored when we study the triviality problem of field theory.

(4) The previous statement means that our conclusion (2) does not necessarily imply that $\lambda(\phi^4)_D$ is trivial. In fact it is known to be nontrivial for $D=2,3$. Thus, we are led to a claim that the δ-expansion method is not a useful tool to get an insight into the triviality problem of $\lambda(\phi^4)_D$ theory.

APPENDIX A

Here we show how to derive (3.20)–(3.23) from $G^{(2+2k)}(p)$ given by (3.3) and (3.12). Since the sums over l and n in $G_{(2)}^{(2+2k)}(p)$ should be taken under the constraint (3.6); $l+n=$even, it is convenient to write $G_{(2)}^{(2+2k)}(p)$ as $G_{\rm even}^{(2+2k)}(p)$ or $G_{\rm odd}^{(2+2k)}(p)$ when both l and n are even or odd, respectively. In (3.12), we replace l and n with $2l$ and $2n$ ($2l+1$ and $2n+1$) for even (odd) l and n:

$$G_{(2)}^{(2k+2)}(p)=-\delta^2(\lambda_0\mu^2)^2 2^{k-2}k!S[-I_0(m)]^{-k-1}J_2(\tilde{p})+G_{\rm even}^{(2k+2)}(p)+G_{\rm odd}^{(2k+2)}(p), \tag{A1}$$

where

$$G_{\rm even}^{(2k+2)}(p)=\delta^2 2^{k-4}(\lambda_0\mu^2)^2[-I_0(m)]^{-k+1}$$
$$\times\sum_{l=1}^{\infty}\sum_{n=0}^{k+1}\begin{bmatrix}2k+2\\2n\end{bmatrix}\frac{\Gamma(k+2l-1)}{\Gamma(2l+1)}B(n+l-1,k+l-n)\int d^Dx\,e^{-ipx}(4z)^l[1-\delta_{l,1}(\delta_{n,0}+\delta_{n,k+1})] \quad \text{(A2)}$$

and

$$G_{\rm odd}^{(2k+2)}(p)=-\delta^2 2^{k-4}(\lambda_0\mu^2)^2[-I_0(m)]^{-k+1}\sum_{l=1}^{\infty}\sum_{n=0}^{k}\begin{bmatrix}2k+2\\2n+1\end{bmatrix}\frac{\Gamma(k+2l)}{\Gamma(2l+2)}B(n+l,k+l-n)\int d^Dx\,e^{-ipx}(4z)^{l+1/2}\,, \quad \text{(A3)}$$

$B(a,b)$ is the β function; $B(a,b)\equiv\Gamma(a)\Gamma(b)/\Gamma(a+b)$ whose integral representation is given by

$$B(a,b)=\int_0^1 ds\,s^{a-1}(1-s)^{b-1}\quad\text{for } a,b>0\,. \quad \text{(A4)}$$

Using this and the relations

$$\sum_{n=0}^{k+1}\begin{bmatrix}2k+2\\2n\end{bmatrix}X^{2n}=\tfrac{1}{2}[(1+X)^{2+2k}+(1-X)^{2+2k}] \quad \text{(A5)}$$

and

$$\sum_{n=0}^{k}\begin{bmatrix}2k+2\\2n+1\end{bmatrix}X^{2n}=\frac{1}{2X}[(1+X)^{2+2k}-(1-X)^{2+2k}]\,, \quad \text{(A6)}$$

we can write (A2) and (A3) as

$$G_{\rm even}^{(2k+2)}(0)=\delta^2 2^{k-5}(\lambda_0\mu^2)^2[-I_0(m)]^{-k+1}\left[\sum_{l=2}^{\infty}\frac{\Gamma(k+2l-1)}{\Gamma(2l+1)}\right.$$
$$\times\int_0^1 ds\,s^{l-2}(1-s)^{k+l-1}\left\{\left[1+\left[\frac{s}{1-s}\right]^{1/2}\right]^{2+2k}+\left[1-\left[\frac{s}{1-s}\right]^{1/2}\right]^{2+2k}\right\}\int d^Dx\,(4z)^l$$
$$\left.+4\theta(k-1)\Gamma(3+2k)\sum_{n=1}^{k}\frac{\Gamma(n)\Gamma(k+1-n)}{\Gamma(2k+3-2n)\Gamma(2n+1)}\int d^Dx\,z\right]\,, \quad \text{(A7)}$$

$$G_{\rm odd}^{(2k+2)}(0)=-\delta^2 2^{k-5}(\lambda_0\mu^2)^2[-I_0(m)]^{-k+1}$$
$$\times\left[\sum_{l=1}^{\infty}\frac{\Gamma(k+2l)}{\Gamma(2l+1)}\int_0^1 ds\,s^{l-1}(1-s)^{k+l-1}\left[\frac{1-s}{s}\right]^{1/2}\left\{\left[1+\left[\frac{s}{1-s}\right]^{1/2}\right]^{2+2k}-\left[1-\left[\frac{s}{1-s}\right]^{1/2}\right]^{2+2k}\right\}\right.$$
$$\left.\times\int d^Dx\,(4z)^{l+1/2}\right]\,, \quad \text{(A8)}$$

where $\theta(x)$ is the step function. We have written these expressions at $p^2=0$ for simplicity. It is also possible to give the Green's function at $p\neq 0$, which, however, needs longer expressions.

To get the two-point Green's function, we take $k=0$ in (A7) and (A8). Now the integration over s in (A7) can be done by the use of (A4). Then we have to sum the following over l:

$$\sum_{l=2}^{\infty}\frac{2}{2l(2l-1)}\frac{\Gamma^2(l-1)}{\Gamma(2l-2)}(4z)^l=\sum_{l=2}^{\infty}\frac{4\sqrt{\pi}\,\Gamma(l-1)}{l\Gamma(l+1/2)}z^l$$
$$=\sum_{l=0}^{\infty}\frac{4\sqrt{\pi}\,\Gamma(l+1)}{(l+2)\Gamma(l+5/2)}z^{l+2}=4\sqrt{\pi}\int_0^z dz\,z\sum_{l=1}^{\infty}\frac{\Gamma(l+1)}{\Gamma(l+5/2)}z^l\,, \quad \text{(A9)}$$

where the first equality follows from $\Gamma(2l-2)=2^{2l-3}\Gamma(l-1)\Gamma(l-\frac{1}{2})/\sqrt{\pi}$ and the second one from shifting $l\to l+2$. The sum (A9) is given in terms of the hypergeometric function

$$\text{(A9)}=\frac{4\sqrt{\pi}}{\Gamma(\frac{5}{2})}\int_0^z dz\,zF(1,1,\tfrac{5}{2};z)$$
$$=8\int_0^z dz\,z\int_0^1 dt\frac{\sqrt{1-t}}{1-tz}=-8\int_0^1 dt\frac{\sqrt{1-t}}{t^2}[\ln(1-tz)+tz]\,, \quad \text{(A10)}$$

where the integral representation of the hypergeometric function has been used. Therefore (A7) becomes

$$G^{(2)}_{\text{even}}(0)=\frac{\delta^2}{4}(\lambda_0\mu^2)^2 I_0(m)\int d^Dx\int_0^1 dt\frac{\sqrt{1-t}}{t^2}[\ln(1-tz)+tz] . \tag{A11}$$

Similarly we have

$$G^{(2)}_{\text{odd}}(0)=-\frac{\delta^2}{4}(\lambda_0\mu^2)^2 I_0(m)\int d^Dx\sqrt{z}\int_0^1 dt\frac{\sqrt{1-t}}{t}\ln(1-tz) . \tag{A12}$$

Gathering (3.3), (A1) [(A11), (A12)], and the zeroth-order contribution m^{-2}, we can obtain $G^{(2)}(0)$. Then Eq. (3.20) follows from $[G^{(2)}(0)]^{-1}$. Note $J_2(0)=I_{-1}(m)$. There are similar arguments used to derive λ_R, $\lambda_R^{(N\delta)}$, and $\lambda_R^{(4)}$ from $e^{\delta\partial_k}G^{(2+2k)}(0)|_{k=0}$, $e^{N\delta\partial_k}G^{(2+2k)}(0)|_{k=0}$ and $G^{(4)}(0)$.

APPENDIX B

We examine the cutoff Λ dependences of $F_1(0)$, $dF_1/dp^2|_{p^2=0}$, and $F_2(0)$ defined by (3.25), (3.26), and (3.27), respectively. We begin by noticing

$$I_0(x;m)=\frac{1}{(2\pi)^{D/2}}\left[\frac{m}{r}\right]^{D/2-1}K_{1-D/2}(mr) , \tag{B1}$$

where $r=(x^2)^{1/2}$ and $K_\nu(x)$ is the modified Bessel function of order ν. $I(x;m)$ has an asymptotic behavior as $r\to 0$:

$$I_0(x:m)\to\begin{cases} r^{2-D}(4\pi)^{-D/2} & \text{for } D>2 ,\\ -\ln(mr)/2\pi & \text{for } D=2 .\end{cases} \tag{B2}$$

The cutoff Λ is used to regularize the divergent integrals such that the integral domain of r is restricted by $\infty>r>1/\Lambda$.

Expanding the logarithm $\ln(1-zt)=-\sum_{n=1}^{\infty}(zt)^n/n$ and using (A4), we can rewrite $F_1(0)$ as

$$F_1(0)=\int_{r>1/\Lambda}d^Dx\sum_{n=1}^{\infty}\frac{\Gamma(n)\Gamma(\frac{3}{2})}{\Gamma(n+\frac{3}{2})}\left[-\frac{z^{n+1}}{n+1}+\frac{z^{n+1/2}}{n}\right] . \tag{B3}$$

The sum over n in (B1) can be expressed as a complex integral by the Sommerfeld-Watson transform:

$$F_1(0)=\frac{i}{2}\int_{r>1/\Lambda}d^Dx\int_{\alpha+i\infty}^{\alpha-i\infty}d\zeta\frac{(-1)^\zeta\Gamma(\zeta)\Gamma(\frac{3}{2})}{\sin(\pi\zeta)\Gamma(\zeta+\frac{3}{2})}\times\left[-\frac{z^{\zeta+1}}{\zeta+1}+\frac{z^{\zeta+1/2}}{\zeta}\right] , \tag{B4}$$

where α is a real number with $0<\alpha<1$. In Ref. 6 we can find the condition under which transformation from (B3) to (B4) is justified. We now evaluate the x integral in (B4) which can be written as

$$\int_{r>1/\Lambda}d^Dx\, z^\nu=\frac{2\pi^{D/2}}{[I_0(m)]^{2\nu}\Gamma(D/2)}\times\int_{1/\Lambda}dr\, r^{D-1}[I_0(x;m)]^{2\nu} . \tag{B5}$$

Since we are interested in an asymptotic Λ behavior, it is sufficient to use (B2) instead of (B1) if there is no infrared divergence. For $D>2$, (B5) becomes

$$\int_{r>1/\Lambda}d^Dx\, z^\nu\approx\frac{2\pi^{D/2}}{\Gamma(D/2)}\left[\frac{1}{(4\pi)^{D/2}I_0(m)}\right]^{2\nu}\times\frac{\Lambda^{2\nu(D-2)-D}}{2\nu(D-2)-D}\quad\text{for } \nu\geq\frac{D}{2(D-2)} . \tag{B6}$$

When $D=2$, (B5) is given by

$$\int_{r>1/\Lambda}d^Dx\, z^\nu\approx\frac{\text{const}}{m^2[I_0(m)]^{2\nu}} , \tag{B7}$$

here we have used (B2) and

$$I_0(x;m)\to e^{-mr}(2\pi mr)^{-1/2}2^{-1}$$

for $r\to\infty$. A substitution of (B6) or (B7) into (B4) gives

$$F_1(0)\approx\frac{i\pi^{D/2}}{\Lambda^D\Gamma(D/2)}\int_{\alpha+i\infty}^{\alpha-i\infty}d\zeta\frac{(-1)^\zeta\Gamma(\zeta)\Gamma(\frac{3}{2})}{\sin(\pi\zeta)\Gamma(\zeta+\frac{3}{2})}\left[\frac{\xi^{2\zeta+2}}{(\zeta+1)[2\zeta(D-2)+D-4]}+\frac{\xi^{2\zeta+1}}{\zeta(2\zeta(D-2)-2)}\right]\quad\text{for } D>2 , \tag{B8}$$

$$F_1(0)\approx\frac{i}{m^2}\int_{\alpha+i\infty}^{\alpha-i\infty}d\zeta\frac{(-1)^\zeta\Gamma(\zeta)}{\sin(\pi\zeta)\Gamma(\zeta+\frac{3}{2})}\left[\frac{\text{const}\times[I_0(m)]^{-2\zeta-2}}{\zeta+1}+\frac{\text{const}\times[I_0(m)]^{-2\zeta-1}}{\zeta}\right]\quad\text{for } D=2 , \tag{B9}$$

where

$$\xi=\Lambda^{D-2}/[(4\pi)^{D/2}I_0(m)] . \tag{B10}$$

Note that $\xi\sim 1$ when $m\ll\Lambda$.

In the case of $D=4$, (B8) is written by

$$F_1(0)=\frac{i\pi^2}{4\Lambda^4}\int_{\alpha+i\infty}^{\alpha-i\infty}d\zeta\frac{(-1)^\zeta\Gamma(\zeta)\Gamma(\frac{3}{2})}{\sin(\pi\zeta)\Gamma(\zeta+\frac{3}{2})\zeta}\left[-\frac{\xi^{2\zeta+2}}{\zeta+1}+\frac{\xi^{2\zeta+1}}{\zeta-\frac{1}{2}}\right] .$$

Since the integrand has poles at $\zeta=1,2,3,\ldots$ in the complex ζ plane, the above integral reduces to the sum of these residues (we can exclude the pole at $\zeta=\frac{1}{2}$ by taking $1>\alpha>\frac{1}{2}$):

$$F_1(0)=\frac{\pi}{2\Lambda^4}\sum_{n=1}^{\infty}\frac{\Gamma(\frac{3}{2})\Gamma(n)}{\Gamma(n+\frac{3}{2})n}\left[\frac{\xi^{2n+2}}{n+1}-\frac{\xi^{2n+1}}{n-\frac{1}{2}}\right]$$
$$=\frac{\pi}{2\Lambda^4}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{3}{2})\Gamma(n+1)}{\Gamma(n+\frac{5}{2})(n+1)}\left[\frac{\xi^{2n+4}}{n+2}-\frac{\xi^{2n+3}}{n+\frac{1}{2}}\right]$$
$$=\frac{\pi}{2\Lambda^4}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{3}{2})\Gamma(n+1)}{\Gamma(n+\frac{5}{2})}\left[\int_0^{\xi^2}d\xi_0^2\int_0^{\xi_0^2}d\xi_1^2\xi_1^{2n}-4\xi\int_0^{\xi}d\xi_0\int_0^{\xi_0}d\xi_1\xi_1^{2n}\right]$$
$$=\frac{\pi}{3\Lambda^4}\left[\int_0^{\xi^2}d\xi_0^2\int_0^{\xi_0^2}d\xi_1^2-4\xi\int_0^{\xi}d\xi_0\int_0^{\xi_0}d\xi_1\right]F(1,1,\tfrac{5}{2};\xi_1^2)\ .$$

Making use of the integral representation of the hypergeometric function and carrying out ξ_1,ξ_0 integrations, we have

$$F_1(0)=\frac{\pi}{2\Lambda^4}\int_0^1 dt\frac{\sqrt{1-t}}{t^2}\left[(1-t\xi^2)\ln(1-t\xi^2)-t\xi^2+t\xi\ln(1-t\xi^2)+t^{3/2}\xi^2\ln\left[\frac{1+\sqrt{t}\,\xi}{1-\sqrt{t}\,\xi}\right]\right]\quad\text{for } D=4\ . \tag{B11}$$

When $\xi\sim 1$, it comes from the above expression that $F_1(0)=\text{const}/\Lambda^4$. For other dimensions, we can follow the similar arguments and we eventually obtain, when $\xi\sim 1$,

$$F_1(0)=\begin{cases}\text{const}/\Lambda^D & \text{for } D>2\ ,\\ \text{const}/[m^2\ln^3(\Lambda/M)] & \text{for } D=2\ .\end{cases} \tag{B12}$$

Likewise we can see the Λ dependences of $dF_1(p)/dp^2|_{p^2=0}$ and $F_2(0)$. We, however, do not repeat the details of derivations and present here results only:

$$\left.\frac{dF_1(0)}{dp^2}\right|_{p^2=0}=\begin{cases}\text{const}/\Lambda^{D+2} & \text{for } D>2\ ,\\ \text{const}/[m^4\ln^3(\Lambda/m)] & \text{for } D=2\ ,\end{cases} \tag{B13}$$

$$F_2(0)=\begin{cases}\text{const}/\Lambda^D & \text{for } D>2\ ,\\ \text{const}/[m^2\ln^2(\Lambda/m)] & \text{for } D=2\ .\end{cases} \tag{B14}$$

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