Curvature-singularity-free solutions for colliding plane gravitational waves with broken u-v symmetry

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We discuss the most general solution describing the collision of plane gravitational waves with constant polarization. Among these solutions there is an infinite-dimensional family of metrics free of curvature singularities and analytically extendable across the "focusing" hypersurface. These regular solutions describe collisions between two incoming plane waves with different amplitudes for which u-v symmetry is broken. Boundary conditions on the null hypersurfaces $u=0$, $v=0$ are discussed and it is shown that any solution describing the scattering of plane gravitational waves with constant polarization has to include at least two solitary terms each of which stabilizes the behavior of the gravitational field on the null boundaries.

I. INTRODUCTION

During the last few years there has been a renewed considerable interest in the solutions of Einstein's equations describing collisions between plane gravitational waves. One of the main features of these solutions is the development of a curvature singularity in the interaction region. Recently, however, a few examples were found which do not result in a curvature singularity after the collision.[1,2] These solutions have been obtained mainly by using either analogy with axisymmetric spacetimes or the inverse-scattering techniques.[2,3]

In this paper we develop a new approach to construct solutions and to study the problem of collision between plane gravitational waves with constant polarization. This approach is based on the similarity between spacetimes produced by the collision between plane gravitational waves and that of inhomogeneous cosmological models. This enables us to apply results, accumulated from the studies of mathematical cosmology by various authors during last two decades, to the problem of plane-wave scattering in general relativity.

Since considerable effort has already been devoted to the study of inhomogeneous vacuum cosmological models, the approach developed in this paper becomes a powerful tool to get a new insight and resolve in fact the two basic problems related to the collision of plane gravitational waves. These problems are (1) the study of the most general solution of Einstein's vacuum field equations describing a spacetime with two spacelike commuting Killing vectors which verifies the proper boundary conditions imposed by the physics of collision of two plane gravitational waves with constant polarization and (2) determination of the most general family of solutions among these mentioned in (1) that fail to develop a curvature singularity on the "focusing" hypersurface, and their analytic extension.

An essential new feature to our approach is that it enables us to isolate those terms in the general solution which contribute to the development of the curvature singularity in the interaction region. This allows one to construct models for which the avoidance or evolution of the singularity depends on the relation between the amplitudes of the two incoming plane waves. In Sec. II we discuss the behavior of the gravitational field in the interaction region near the "focusing" hypersurface. In Sec. III the boundary conditions on the null hypersurfaces are considered.

II. THE INTERACTION REGION IN CANONICAL COORDINATES

We consider the metric describing the interaction region produced by the collision between plane gravitational waves with constant polarization in the form

$$ds^2=e^F du\, dv+G(e^p dx^2+e^{-p} dy^2), \quad u>0, \quad v>0\ , \tag{1}$$

where F, p, and G depend on the null coordinates u and v alone.

It follows from Einstein's vacuum field equations that the transitivity surface area G has to satisfy the classical wave equation

$$G_{uv}=0 \tag{2}$$

and the solution of this equation relevant to the problem of colliding waves might be initially taken in the form[4]

$$G=a(u)+b(v)=1-(\alpha u)^n-(\beta v)^m\ , \tag{3}$$

where

$$a(u)\equiv\tfrac{1}{2}-(\alpha u)^n, \quad b(v)\equiv\tfrac{1}{2}-(\beta v)^m\ . \tag{4}$$

Here α and β are arbitrary positive constants and n and m are determined by boundary conditions.

In order to interpret the spacetime defined by Eqs. (1)–(4) as the interaction region which results in the collision of two gravitational plane waves one has to impose proper boundary conditions on the null hypersurfaces $u=0$ and $v=0$. On these hypersurfaces one matches the metric given by (1) to those with higher exactly plane symmetries which describe incoming pure advanced and retarded plane waves.[5]

We will first study the evolution of the gravitational field near the "focusing" hypersurface

$$1=(\alpha u)^n+(\beta v)^m \ , \tag{5}$$

and return to the problem of null boundary conditions in Sec. III.

Along the lines of general singularity theorems[6,7] predicting one or another kind of singularity to be evolved in the interaction region, it is usually believed that the spacetime resulting in the collision of plane gravitational waves is inextendable across the focusing hypersurface. Recently, however, a few "degenerate" examples of solutions were constructed in which the curvature singularity does not evolve in the interaction region.[2,3,8] Two of these solutions have been shown to be extendable.[2,3] In what follows we will show that there exists an infinite-dimensional family of solutions describing collisions between two plane gravitational waves, regular on the hypersurface $1=(\alpha u)^n+(\beta v)^m$ and admitting an analytic extension across it.

To study the behavior of the gravitational field in the interaction region $u>0$, $v>0$ we have found it convenient to use canonical coordinates t and z, rather than the null ones. We define these coordinates as

$$\begin{aligned} t&=a(u)+b(v)=1-(\alpha u)^n-(\beta v)^m \ , \\ z&=a(u)-b(v)=(\beta v)^m-(\alpha u)^n \ . \end{aligned} \tag{6}$$

The line element (1) then takes the form

$$ds^2=e^f(-dt^2+dz^2)+t(e^p dx^2+e^{-p}dy^2) \ , \tag{7}$$

where f and p are treated as functions of t and z.

The Einstein vacuum field equations now take a very simple form:

$$\ddot{p}+\frac{1}{t}\dot{p}-p''=0 \ , \tag{8}$$

$$\dot{f}=-\frac{1}{2t}+\frac{t}{2}(\dot{p}^2+p'^2) \ , \tag{9a}$$

$$f'=t\dot{p}p' \ , \tag{9b}$$

where an overdot denotes $\partial/\partial t$ and a prime denotes $\partial/\partial z$.

It follows from Eqs. (8) and (9) that the entire dynamics of the gravitational field in the interaction region is determined by the single "transversal" degree of freedom p.

In the new coordinates the "focusing" hypersurface $1=(\alpha u)^n+(\beta v)^m$ corresponds to $t=0$. The hypersurface $u=0$ corresponds to $t=-z+1$ whereas the hypersurface $v=0$ corresponds to $t=z+1$ (see Fig. 1).

To understand the evolution of the gravitational field near the focusing hypersurface $t=0$ it is important to clarify the behavior of different solutions of Eq. (8). These solutions in fact were studied during the last two decades in the cosmological context by various authors.[9–16] Although one might write the general solution of Eq. (8) in terms of a line integral, it is instructive to sort out different terms in the following form:

$$\begin{aligned} p=k\ln(t)&+L\{A_\omega\cos[\omega(z+z_0)]J_0(\omega t)\} \\ &+L\{B_\omega\cos[\omega(z+z_0)]N_0(\omega t)\} \\ &-\sum_i d_i \operatorname{arccosh}\frac{z+z_i}{t} \ . \end{aligned} \tag{10}$$

Here $L\{\ \}$ stands for arbitrary linear combinations of the terms in curly brackets including terms of the form

$$\begin{aligned} &\int_0^\infty B_\omega\cos[\omega(z+z_0)]N_0(\omega t)d\omega \ , \\ &\int_0^\infty A_\omega\cos[\omega(z+z_0)]J_0(\omega t)d\omega \end{aligned} \tag{11}$$

and $J_0(\omega t)$ and $N_0(\omega t)$ are Bessel and Neumann functions of zero order, respectively.

Each term in the solution given by Eq. (10) differs in its behavior when t approaches 0^+. The first term in Eq. (10) can be thought as producing a homogeneous expansion and is mainly "responsible" for the development of a curvature singularity at $t\to 0^+$. The second term induces regular behavior, whereas the third one also contributes to the divergence of the curvature at $t\to 0^+$ due to the "bad" behavior of the Neumann functions near zero. This term is sometimes referred to as chaotic.[15] The last term in the expansion (10) is composed of the so-called gravitational solitons and was studied in the context of inhomogeneous cosmological models quite recently.[12–14,16] Solitons are singled out since they are not decomposible into Fourier-Bessel integrals. The contribution of the solitary terms to the divergence of the curvature near $t\to 0^+$ is similar to that of homogeneous terms.

One can further show that the curvature invariant $C=C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ behaves at $t\to 0^+$ as[10,17]

$$e^f C^{1/2}\simeq\frac{1}{t^2}-\frac{1}{t^2}\left[k+\sum d_i+L\{B_\omega\sin[w(z+z_0)]\}\right]^2 \ , \tag{12}$$

where $\sum d_i$ is a sum over soliton amplitudes.

Behavior similar to this was already discussed by Moncrief.[10] However, in the case of inhomogeneous vacuum Gowdy universes studied by Berger[9] and Moncrief[10] the soliton terms were excluded by the restrictions posed by $T^3\times R$ topology. It will be shown later that these terms prove to be crucial in the problem of collision of plane

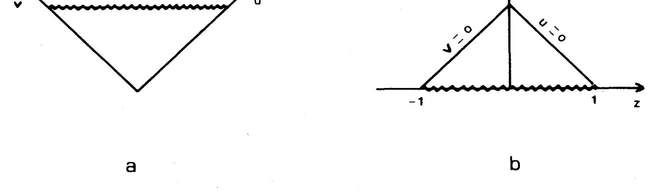

FIG. 1. Interaction region in (a) (u,v) coordinates and (b) (t,z) coordinates.

gravitational waves. Using Eq. (12) and the properties of the metric function f given by Eqs. (9) it can be shown that the following solutions will fail to develop curvature singularities at $t \to 0^+$:

$$p_r = k\ln(t) - \sum d_i \operatorname{arccosh}\frac{z+z_i}{t} + L\{A_\omega \cos[\omega(z+z_0)]J_0(\omega t)\} \tag{13}$$

provided

$$k + \sum d_i = \pm 1 . \tag{14}$$

We would like to stress here that when $\sum d_i = 0$, the evolution of the singularity depends only on the homogeneous "Kasner" term[2] $k\ln(t)$. This corresponds to the situation when either there are no solitonic terms in the general solution (10) or the solitons appear in pairs as discussed in Ref. 16. However, for the case $\sum d_i \neq 0$ (significance of which we will discuss below) the avoidance of the singularity is determined by the difference between the physical structure of incoming plane waves.

One may further show that these solutions are extendible across the $t=0$ hypersurface. To extend the solutions given by Eqs. (9), (13), and (14) we proceed along the lines of the work of Moncrief:[10] the metric for regular solutions can be written as

$$ds^2 = e^f(-dt^2+dz^2) + t^2 e^{\bar{p}} dx^2 + e^{-\bar{p}} dy^2 \tag{15}$$

and

$$\begin{aligned}\bar{p} &\equiv p_r - \ln(t)\\ &= L\{A_\omega \cos[\omega(z+z_0)]J_0(\omega t)\}\\ &\quad - \sum d_i \ln\{z+z_i+[(z+z_i)^2-t^2]^{1/2}\} .\end{aligned} \tag{16}$$

The function f can then be expressed as

$$f(t,z) = \bar{p}(t,z) + \frac{1}{2}\int_0^t ds\, s\left[\left(\frac{\partial \bar{p}(s,z)}{\partial s}\right)^2 + \left(\frac{\partial \bar{p}(s,z)}{\partial z}\right)^2\right], \quad t>0 . \tag{17}$$

When condition (14) holds the functions f and $\bar{p}$ satisfy

$$\lim_{t\to 0^+}[f(t,z)-\bar{p}(t,z)] = \lim_{t\to 0^+}[\dot{f}(t,z)-\dot{\bar{p}}(t,z)] = 0 . \tag{18}$$

Introducing a pair of new coordinates

$$t' = t^2, \quad x' = x - \ln(t) \tag{19a}$$

or

$$t'' = t^2, \quad x'' = x + \ln(t) \tag{19b}$$

one obtains, similarly to Moncrief,[10] two inequivalent extensions of the metric (15) defined now for all values of (t',x') and $(t'',x'') \in (-1,1)\times(-\infty,\infty)$. Here we have supposed $\min|z_i| = 1$.

III. BOUNDARY CONDITIONS

We will now show that among the solutions given by Eqs. (13), (14), and (9) there is an infinite family of metrics which describe scattering of plane gravitational waves provided there are at least two solitonic terms in Eq. (13).

The solution of the Einstein's vacuum field equations, which describes the spacetime with two spacelike commuting Killing vectors, can be interpreted as the interaction region produced after the collision of two gravitational plane waves only if certain boundary conditions on the null hypersurfaces $u=0$ and $v=0$ are satisfied.

Without any loss of generality we consider the following form for the solution of Eq. (8):

$$p = d_1 \operatorname{arccosh}\frac{z+1}{t} + d_2 \operatorname{arccosh}\frac{1-z}{t} . \tag{20}$$

It can be shown that other terms of the solution given by Eq. (13) such as the homogeneous term $k\ln(t)$ and the linear combination of the terms $A_\omega \cos(\omega z)J_0(\omega t)$ are redundant for the discussion of the boundary conditions on the null hypersurfaces $u=0$ and $v=0$.

One can also show that inclusion of more solitary terms, unless their poles z_i (origins on the t axis) are within the interaction region bounded by $t=-z+1$ and $t=z+1$ (see Fig. 1), does not change the behavior of gravitational field relevant for the boundary conditions on the $u=0$ and $v=0$ hypersurfaces. If, however, the solution for the function p contains another pair of solitons with poles $(\alpha,-\alpha)$ such that $\alpha<1$, one should define the interaction region to be bounded by $t=-z+\alpha$ and $t=z+\alpha$ hypersurfaces and rescale accordingly Eqs. (3) and (4).

The appropriate boundary conditions for the colliding wave problem were formulated by O'Brien and Synge[18] (see also Ref. 4). In fact, with the chosen coordinate gauge given by Eqs. (3) and (4) one has to verify only the continuity of the function $F(u,v)$ of Eq. (1). The relation between this function and the function $f(u,v)$ is given by

$$e^{F(u,v)} = -4t_u t_v e^{f(u,v)} . \tag{21}$$

All other boundary conditions are automatically verified.

Since t_u and t_v of Eq. (21) tend to zero as $O(u^{n-1})$ and $O(v^{m-1})$ on the null boundaries $u=0$ and $v=0$, respectively, the function $e^{f(u,v)}$ must diverge as u^{1-n} and v^{1-m} on these hypersurfaces to ensure the smooth matching between the interaction and the extended precollision regions. This divergence in the function $e^{f(u,v)}$ comes "surprisingly" from the solitonic terms.

The first soliton

$$d_1 \operatorname{arccosh}\frac{z+1}{t}$$

contributes to the function $f(u,v)$ via Eqs. (9) the following term on $v=0$:

$$-\tfrac{1}{2}d_1^2 \ln[(z+1)^2-t^2] = -\tfrac{1}{2}d_1^2 \ln(v^m) + \text{bounded terms} , \tag{22}$$

which stabilizes the behavior of the gravitational field as $v \to 0$ provided

$$d_1^2 = 2 - 2/m \ . \tag{23}$$

By the same token the second soliton

$$d_2 \operatorname{arccosh} \frac{z-1}{t}$$

contributes, as $u \to 0$,

$$-\tfrac{1}{2} d_2^2 \ln(u^n) \ , \tag{24}$$

which smoothes the behavior of the field on $u=0$ hypersurface, provided

$$d_2^2 = 2 - 2/n \ . \tag{25}$$

As we have already mentioned, we have checked all other possible contributions of p_r [Eq. (13)] to the function $f(u,v)$ and found them to be bounded on the boundaries $u=0$ and $v=0$. Consequently, we conclude that the presence of at least two solitonic terms in the transversal degree of freedom p are necessary to ensure the verification of the boundary conditions relevant to the colliding wave problem.

It is important to note in this context that each soliton acts on a *different* null boundary thus providing a balancing mechanism for ensuring the proper boundary conditions to be satisfied. The understanding of this mechanism combined with our previous conclusions related to the evolution of curvature singularity allows one to construct regular solutions describing collisions between plane gravitational waves which do not preserve the usual $(u\text{-}v)$ symmetry. The simplest such regular solution is given by Eqs. (20), (23), and (25) along with

$$e^{f(z,t)} = (\mu_1+\mu_2)^{2d_1d_2}(\mu_1)^{2d_1^2-qd_1}(\mu_2)^{2d_2^2-qd_2} \times (t^2-\mu_1^2)^{-d_1^2}(t^2-\mu_2^2)^{-d_2^2} \ , \tag{26}$$

where

$$q = d_1 + d_2 = \pm 1 \ ,$$
$$\mu_1 = 1 - z + [(1-z)^2 - t^2]^{1/2} \ , \tag{27}$$
$$\mu_2 = z + 1 + [(1+z)^2 - t^2]^{1/2} \ .$$

The results of the study of this solution and the relation between the development of the curvature singularity and the energy content of the incoming plane waves will be presented elsewhere.

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