Two-dimensional quantum cosmology: Directions of dynamical and thermodynamic arrows of time

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Two-dimensional quantum cosmology is studied to clarify the issue associated with the arrow of time. We introduce a density-matrix description for the minisuperspace by projecting out nonzero modes of the wave function of the whole Universe. Thermodynamic time is determined by the direction in which the quantum coherence of minisuperspace variables decreases. This arrow of time is compared with that of dynamical time which is determined by the semiclassical approximation for the density matrix. We reconfirm that the global time cannot properly be defined for the thermodynamic times as well as for the dynamical time in a closed universe. The arrows of dynamical and thermodynamic times coincide with the cosmological arrow of time.

I. INTRODUCTION

The singularity theorem[1] says that the classical concept of space-time cannot be unlimitedly applicable to the vicinity of the big bang, whereas the present Universe is fairly well described by the classical theory. At present there seems to be no quantum correlation which, if it exists, drastically destroys the consistency in the deterministic interpretation of cosmic observations. Thus, there must be a transition from an era which is fully quantum mechanical to an era which has no quantum coherence in the course of cosmological evolution. In this paper, we call this direction from the quantum to the classical era the thermodynamic arrow of time. Precisely, this thermodynamic time is measured by the quantum coherence width of the density matrix.[2,3] For the case of four-dimensional flat space-time, as is studied in Ref. 3, the width turns out to be inversely proportional to the cosmic scale factor. This shows that the gradual transition to a classical era occurs due to cosmic expansion. In this paper we examine gravitation in two space-time dimensions (2D) (Ref. 4). Using this model, we previously studied how to extract dynamical time variables which assure the positive definiteness of the probability density of the Wheeler-DeWitt (WD) wave function.[5] Here we do not treat the wave function but the density matrix. The latter seems more suitable than the former for considerations of the thermodynamic arrow of time. Corresponding to this change of strategy, dynamical time is defined also in a different form from Ref. 5. The merit of the 2D model is that it enables us to estimate the quantum coherence of minisuperspace in every case: open, flat, and closed universes. It is also possible to study the relationship between the thermodynamic arrow of time and the dynamical one.

The outline of this paper is as follows. In Sec. II, the density matrix of minisuperspace is constructed in the path-integral form from the influence functional method. Then the density matrix is evaluated by the steepest-descent method and the direction of the thermodynamic time is discussed. Subsequently in Sec. III, that direction is compared with that of dynamical time t which is determined from a semiclassical solution of the density matrix. The time t is written in terms of gravitational variables and no additional clock variable is needed. Section IV is devoted to discussions. The Appendix gives the derivation of the influence functional which is used in Sec. II.

II. DENSITY MATRIX OF MINISUPERSPACE

We consider a 2D space-time model universe. Beyond their own meaning, 2D models in particle physics are the guideposts to the corresponding theories in four dimensions. In the case of the minisuperspace model, its essential features seem to be involved in the 2D model. The gravitational action in 2D reads

$$S_g=\frac{1}{2}\int d^2x\sqrt{-g}\,b(R-2\Lambda)+\text{surface term}\ , \qquad (2.1)$$

where R is a scalar curvature and Λ is a cosmological constant. The variable b is a scalar field, which yields the nontrivial contribution of 2D pure gravity to the equation of motion. The gravitational coupling constant $1/(16\pi G)$ is included in b unless it is explicitly needed. The matter action is given by

$$S_M=\frac{1}{2}\int d^2x\sqrt{-g}\,[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-(m^2+\xi R)\phi^2]\ , \qquad (2.2)$$

where ξ is a coupling constant to the scalar curvature. In this paper we use the following parametrization for the metric:

$$g_{\mu\nu}=a^2\text{diag}(N^2,-1)\ , \qquad (2.3)$$

where N is the lapse function which represents the arbitrariness in the time parametrization. The shift function is already set to be zero. The above actions become, for this parametrization ($\chi \equiv \ln a$, an overdot denotes $\partial/\partial x^0$, a prime denotes $\partial/\partial x^1$),

$$S_g = \int d^2x \left[-\frac{\dot{b}}{N}\dot{\chi} + b'(N\chi' + N') - Na^2 b\Lambda \right], \quad (2.4)$$

$$S_m = \tfrac{1}{2}\int d^2x\, N(N^{-2}\dot{\phi}^2 - \phi'^2) \quad (2.5)$$

and terms which break the conformal invariance are treated as perturbations:

$$S_{\text{int}} = -\tfrac{1}{2}\int d^2x\, Nh\phi^2 , \quad (2.6)$$

where

$$h = m^2a^2 - 2\xi \left[\frac{\ddot{\chi}}{N^2} - \chi'' - \frac{N'}{N}\chi' - \frac{N''}{N} - \frac{\dot{N}}{N^3}\dot{\chi} \right]. \quad (2.7)$$

Our basic strategy is to construct the effective dynamics of minisuperspace variables (the zero modes of a, b, and φ) by projecting out all the inhomogeneous modes. Even if the total wave function of the Universe ideally exists, it will not directly relate to our actual observables. Thus, the inconsistency in the probabilistic interpretation of the wave function of the Universe does not worry us seriously. For an experimentally accessible argument, we switch to the subdynamics of minisuperspace as motivated in Sec. I. Since the subsystem cannot be described by a superposition of pure states, we have to introduce a density matrix description. Just for simplicity of calculation in this paper, we take into account only the nonzero modes of the scalar matter field ϕ. That is, we consider the back reaction of the spatial fluctuation of ϕ into homogeneous space-time. This reflects just what we actually observed in our Universe.

Using the WD wave function Ψ, the total density matrix of the whole Universe is given by

$$\bar{\rho}[a_+,b_+,\phi_+;a_-,b_-,\phi_-] \equiv \Psi(a_+,b_+,\phi_+)\Psi^*(a_-,b_-,\phi_-) . \quad (2.8)$$

The above-mentioned situation of realistic observation implies that we should treat the reduced density matrix ρ defined by

$$\rho[a_+,b_+;a_-,b_-] \equiv \int d\phi_+ \int d\phi_- \bar{\rho}[a_+,b_+\phi_+;a_-,b_-,\phi_-] \times \delta(\phi_+ - \phi_-) . \quad (2.9)$$

Equation (2.9) is evaluated in the Appendix using the in-in formalism of quantum field theory. The result becomes

$$\rho[a_+,b_+;a_-,b_-] = \int da'_+ \int da'_- \int db'_+ \int db'_- \int_{a'_+}^{a_+} \mathcal{D}a_+ \int_{a'_-}^{a_-} \mathcal{D}a_- \int_{b'_+}^{b_+} \mathcal{D}b_+ \int_{b'_-}^{b_-} \mathcal{D}b_- \rho[a'_+,b'_+;a'_-,b'_-] \times \exp(i\tilde{S}[a_+,b_+;a_-,b_-]) . \quad (2.10)$$

Here

$$\exp(i\tilde{S}[a_+,b_+;a_-,b_-]) = \mathcal{F}[a_+,b_+;a_-,b_-]\exp\{i(S_g[a_+,b_+] - S_g[a_-,b_-])\} \quad (2.11)$$

with

$$\mathcal{F}[a_+,b_+;a_-b_-] = \exp\left[-\frac{i}{4\pi}\int dx^0 \int dx^{0'} \theta(x^0 - x^{0'}) H_\Delta(x^0)(2\pi)^{-2} \times \int dp\, e^{-ip\Delta x^0} \ln p^2 H_c(x^{0'}) - \frac{V}{32}\int dx^0 H_\Delta^2(x^0) \right], \quad (2.12)$$

where V is a coordinate length which is an arbitrary constant. We have to introduce it in order to avoid divergence due to the infinite spatial volume. $H(x^0)$ is the time integral of the perturbation: $H_\pm(x^0) = \int^{x^0} dx^0 h_\pm(x^0)$.

Now we evaluate the path-integral expression for the density matrix by the steepest-descent method; the path integral is approximated by the integrand with the extremum path (classical solution). As far as the lowest order of the perturbation is concerned, perturbation-free classical solutions are sufficient for the evaluation of the integrand. The perturbation-free action for the density matrix is given by

$$\tilde{S}_0 = V\int dx^0 \left[-\frac{\dot{b}_+}{N_+}\dot{\chi}_+ - N_+a_+^2b_+\Lambda + \frac{\dot{b}_-}{N_-}\dot{\chi}_- + N_-a_-^2b_-\Lambda \right]. \quad (2.13)$$

The classical solutions must satisfy the extremum condition

$$\frac{\delta\tilde{S}_0}{\delta a_\pm}=0 \Longrightarrow \left[\frac{\ddot{b}_\pm}{a_\pm}-2\Lambda a_\pm b_\pm\right]=0 ,$$
$$\frac{\delta\tilde{S}_0}{\delta b_\pm}=0 \Longrightarrow (\ddot{\chi}_\pm - a_\pm^2\Lambda)=0 , \tag{2.14}$$

and the constraint equation which comes from the lapse function independence of the density matrix:

$$\frac{\delta\tilde{S}_0}{\delta N_\pm}=0 \Longrightarrow \dot{b}_\pm\dot{\chi}_\pm - a_\pm^2 b_\pm\Lambda=0 . \tag{2.15}$$

In Eqs. (2.14) and (2.15), we have set $N_\pm=1$ after the Euler variations. In the remainder of this section, we study the steepest-descent approximation for the density matrix in the following three cases: (A) the case $0<\Lambda\equiv\alpha^2$, (B) the case $0=\Lambda$, and (C) the case $0>\Lambda\equiv-\alpha^2$.

(A) For the case $0<\Lambda\equiv\alpha^2$, the classical solutions become

$$a_\pm=-\frac{A_\pm}{\sinh[\alpha A_\pm(x^0-x_\pm^0)]} ,$$
$$b_\pm=-\frac{B_\pm}{\tanh[\alpha A_\pm(x^0-x_\pm^0)]} . \tag{2.16}$$

In the above, $A_\pm$, $B_\pm$, and $x_\pm^0$ are integral constants. Among them, $A_\pm$ represents a scale of the scale factor $a(x^0)$ which has no meaning by itself. Thus, we restrict our consideration on the solutions with $A_\pm=1$. In the gauge $N_\pm=1$, the perturbation becomes

$$h_\pm(\eta)=(m^2a_\pm^2-2\xi\ddot{\chi}_\pm)=(m^2+2\xi\Lambda)a_\pm^2\equiv M^2a_\pm^2 . \tag{2.17}$$

We put the classical solutions Eq. (2.16) into the action Eq. (2.11) and expand it as a series in a_Δ^2 ($a_\Delta=a_+-a_-$) assuming the smallness of a_Δ: $|a_\triangle/a_c|\ll1$ ($2a_c\equiv a_++a_-$). Then the approximate density matrix becomes

$$\rho[a_+,a_-]\approx\mathcal{N}\exp\left[-i\alpha V(b_+\sqrt{1+a_+^2}-b_-\sqrt{1+a_-^2}) - \frac{V}{64}\frac{M^4}{\alpha^3}a_c^{-2}(1+a_c^2)^{-1}\{\tfrac{4}{3}+[a_c^2-1-\tfrac{1}{3}(1+a_c^2)]\sqrt{1+a_c^2}\}a_\triangle^2+O(a_\triangle^4)\right] . \tag{2.18}$$

From the above expression, the dispersion (the width of the quantum coherence) σ^2 of this density matrix becomes [see Fig. 1(a)]

$$\sigma^2=\frac{32}{V}\frac{\alpha^3}{M^4}a_c^2(1+a_c^2)\{\tfrac{4}{3}+[a_c^2-1-\tfrac{1}{3}(1+a_c^2)]\times\sqrt{1+a_c^2}\}^{-1} . \tag{2.19}$$

This expression has the following asymptotic forms:

$$\sigma^2\propto\begin{cases}a_c^{-2} & \text{for } a_c\to0,\\ a_c & \text{for } a_c\to\infty .\end{cases} \tag{2.20}$$

We now proceed to the second case.

(B) For the case $0=\Lambda$, the classical solutions become

$$a_\pm=A_\pm\exp[A_\pm(x^0-x_\pm^0)],\quad b_\pm=B_\pm . \tag{2.21}$$

Then the approximate density matrix becomes

$$\rho[a_+,a_-]\approx\mathcal{N}\exp\left[-\frac{VM^4}{128}a_c^2a_\triangle^2\right] . \tag{2.22}$$

Thus, the dispersion is given by [see Fig. 1(b)]

$$\sigma^2=\frac{64}{VM^4}a_c^{-2} . \tag{2.23}$$

In this case, the phase part of the density matrix completely vanishes for this perturbation-free evaluation. Thus the semiclassical approximation may seem implausible. However, we expect that the nonlocal imaginary exponent in $\mathcal{F}[a_+,a_-]$ of Eq. (A9) gives rise to a nonvanishing contribution to the action. Whether or not this actually happens is not yet clear.

(C) For the case $0>\Lambda\equiv-\alpha^2$, the classical solutions become

$$a_\pm=\frac{A_\pm}{\cosh[\alpha A_\pm(x^0-x_\pm^0)]} ,$$
$$b_\pm=-B_\pm\tanh[\alpha A_\pm(x^0-x_\pm^0)] . \tag{2.24}$$

The approximate density matrix becomes

$$\rho[a_+,a_-]\approx\mathcal{N}\exp\left[-i\alpha V(b_+\sqrt{1-a_+^2}-b_-\sqrt{1-a_-^2}) - \frac{V}{32}\frac{M^4}{\alpha^3}a_c^{-2}(1-a_c^2)^{-1}\{\tfrac{4}{3}\pm[a_c^2+1+\tfrac{1}{3}(1-a_c^2)]\sqrt{1-a_c^2}\}a_\triangle^2+O(a_\triangle^4)\right] \tag{2.25}$$

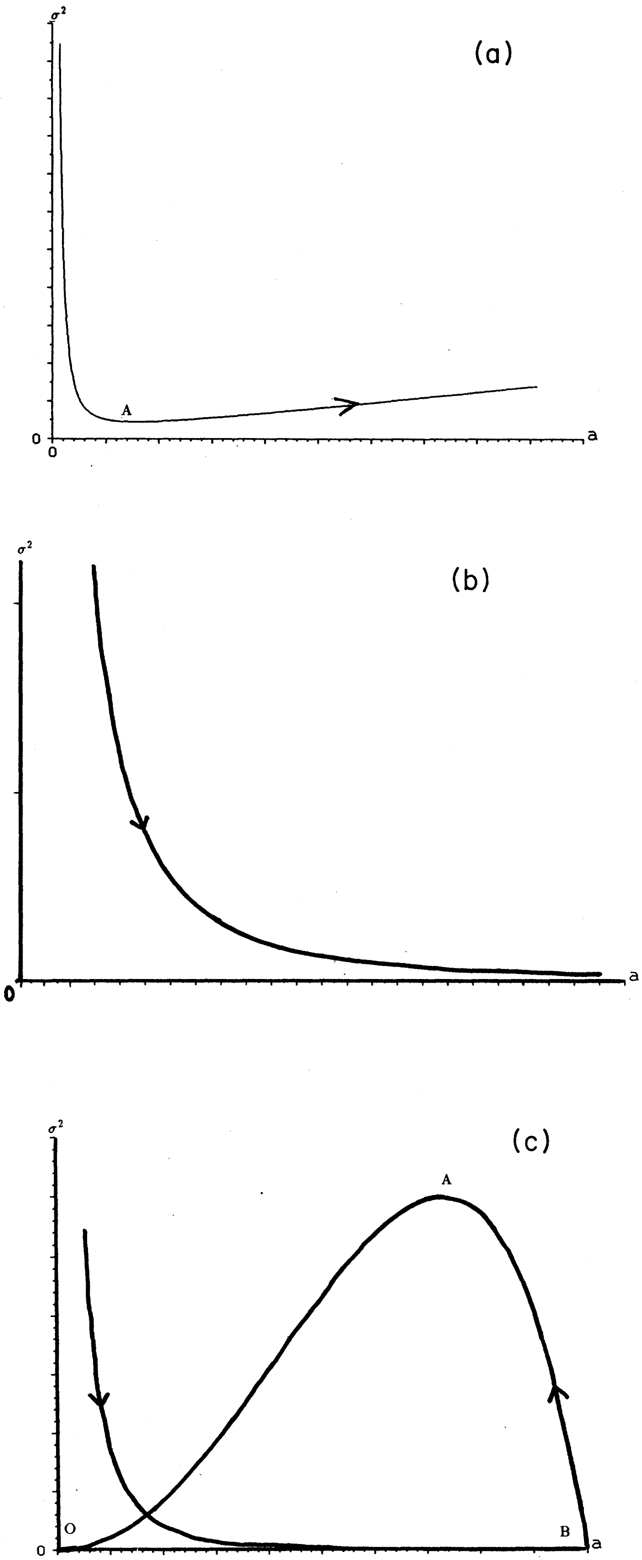

FIG. 1. Quantum coherence width σ^2 is shown in the vertical axis against the scale factor a is the horizontal axis. (a) the case $0<\Lambda\equiv\alpha^2$ (case A); (b) the case $0=\Lambda$ (case B), (c) the case $0>\Lambda\equiv-\alpha^2$ (case C).

Then the dispersion is given by [see Fig. 1(c)]

$$\sigma^2=\frac{16}{V}\frac{\alpha^3}{M^4}a_c^2(1-a_c^2)$$
$$\times\{\tfrac{4}{3}\pm[a_c^2+1+\tfrac{1}{3}(1-a_c^2)]\sqrt{1-a_c^2}\}^{-1}\,. \tag{2.26}$$

On the right-hand side of Eqs. (2.25) and (2.26), the signature $-$ $(+)$ corresponds to the expanding (contracting) phase of the Universe. These expressions have the following asymptotic forms:

$$\sigma^2\propto\begin{cases}a_c^{-2} & \text{for } a_c\rightarrow 0 \text{ in the expanding phase,}\\ a_c^2 & \text{for } a_c\rightarrow 0 \text{ in the contracting phase,}\\ a_{\max}^2-a_c^2 & \text{for } a_c\approx\text{maximum}\,.\end{cases} \tag{2.27}$$

The behavior of σ^2 for the cases (A), (B), and (C) is depicted in Fig. 1. According to Fig. 1(a) the quantum coherence reduces at the first stage of the expansion and then begins to increase at point A. Naively interpreting this figure, the thermodynamic arrow of time coincides with the cosmological arrow (the direction in which the Universe expands) until the point A. Beyond the point A, however, it seems to reverse its direction. However, this is not the case. We have treated the mass term and the coupling to the scalar curvature as perturbations. [See Eq. (2.17).] Thus the perturbations approximation simply breaks down for a large value of the scale factor; beyond the point A in Fig. 1.

The physical implication of Fig. 1(b) is obvious. We see that the quantum coherence monotonically decreases in the course of the cosmic expansion. This behavior fully agrees with that in the four-dimensional calculation.[3]

The behavior of σ^2 in case (C) is interesting. Figure 1(c) shows that the thermodynamic arrow changes its direction at the maximum expansion $a=a_{\max}$. This is the same conclusion as Hawking and Ref. 5 but differs from that of Penrose.[6] The arrow, however, changes its direction again at point A in the middle of the contracting phase. The complete understanding of this curious behavior is now not at our hand. However, the following remarks should be mentioned. In the framework of the WD equation, the probability density of the WD wave function becomes negative at just point A (Ref. 5). The unique time variable cannot be defined over the whole region, unlike a_c in cases (A) and (B). It cannot be compatible with the conserved positive-definite probability density due to the fact that the WD equation is a second-order hyperbolic differential equation.[7] The problem of dynamical time variables in the present approach will be discussed in the next section.

III. THE ARROW OF THE DYNAMICAL TIME

In the previous section, we studied the behavior of the quantum coherence width as a function of the cosmic scale factor. There we have derived the result that the thermodynamic arrow of time coincides with the cosmological arrow of time except for the AO stage in case (C). That is, the density matrix diagonalizes as a_c increases. Using this fact we will introduce a dynamical time variable which describes an evolution of the density matrix.

This time variable is closely connected with that introduced in the case of the wave function of the Universe in the same model of Eq. (2.1) (Ref. 5). The method of this section is the extension of the WKB approximation in the WD wave function to that in the density matrix.[8]

Using the variables p_χ and p_b canonically conjugate, respectively, to χ and b, S_g of Eq. (2.4) is rewritten as

$$S_g=\int d^2x(p_\chi\dot{\chi}+p_b\dot{b}-N\mathcal{H}) \tag{3.1}$$

with

$$\mathcal{H}=p_\chi p_b+\chi'b'-b''-be^{2\chi}\Lambda\ . \tag{3.2}$$

Changing variables χ and b into

$$\alpha\equiv\frac{1}{\sqrt{2}}(\chi+b),\quad \beta\equiv\frac{1}{\sqrt{2}}(\chi-b)\ , \tag{3.3}$$

and therefore

$$p_\alpha=\frac{1}{\sqrt{2}}(p_\chi+p_b),\quad p_\beta=\frac{1}{\sqrt{2}}(p_\chi-p_b)\ , \tag{3.4}$$

the Hamiltonian of Eq. (3.2) for minisuperspace is diagonalized as

$$\mathcal{H}=\tfrac{1}{2}(p_\alpha^2-p_\beta^2)-\kappa U(\alpha,\beta) \tag{3.5}$$

with

$$U(\alpha,\beta)\equiv\frac{\Lambda}{\sqrt{2}}(\alpha-\beta)\exp[\sqrt{2}(\alpha+\beta)]\ . \tag{3.6}$$

Here we write out explicitly the gravitational coupling constant κ which has so far been included in b. In the same way as for the case of the wave function of the Universe, we get the energy constraint equation for the density matrix:

$$\left[\mathcal{H}[\alpha_+,\beta_+]-\frac{iV}{32}H_\triangle^2\right]\rho[\alpha_+,\beta_+,\alpha_-,\beta_-]=0 \tag{3.7}$$

and

$$\left[\mathcal{H}[\alpha_-,\beta_-]+\frac{iV}{32}H_\triangle^2\right]\rho[\alpha_+,\beta_+,\alpha_-,\beta_-]=0\ , \tag{3.8}$$

where the last terms in Eqs. (3.7) and (3.8) are the contributions from the influence functional in Eq. (2.12). From Eqs. (3.7) and (3.8), we get the Wheeler-DeWitt-type constraint equation for the density matrix:

$$\left[\mathcal{H}[\alpha_+,\beta_+]-\mathcal{H}[\alpha_-,\beta_-]-\frac{iV}{16}H_\triangle^2\right]\rho[\alpha_+,\beta_+,\alpha_-,\beta_-]=0\ . \tag{3.9}$$

Now we derive an evolution equation for the density matrix by applying the WKB approximation for this equation. Assuming a large magnitude of the variable $|\beta|$ compared to $|\alpha|$: $|\beta|\gg|\alpha|$. (This condition breaks down in the neighborhood of A in the contracting phase [see Fig. 1(c)]. This problem will be discussed shortly.) We parametrize the density matrix as

$$\rho[\alpha_\pm,\beta_\pm]=A[\alpha_\pm,\beta_\pm]\exp(iB[\beta_\pm])\ , \tag{3.10}$$

where the Hermiticity condition should be satisfied:

$$A[\alpha_\pm,\beta_\pm]^*=A[\alpha_\mp,\beta_\mp]\ ,\qquad B[\beta_\pm]=-B[\beta_\mp],\quad B\text{:real}\ . \tag{3.11}$$

Putting this form of the density matrix into Eq. (3.9), we get

$$\tilde{\mathcal{H}}A+\kappa^{-1}\left[+iA\frac{\partial^2B}{\partial\beta_+^2}+2i\frac{\partial B}{\partial\beta_+}\frac{\partial A}{\partial\beta_+}-A\left[\frac{\partial B}{\partial\beta_+}\right]^2-(+\leftrightarrow-)\right]-\frac{1}{16\pi}[U(\alpha_+,\beta_+)-U(\alpha_-,\beta_-)]A=0\ , \tag{3.12}$$

where

$$\tilde{\mathcal{H}}\equiv-\tfrac{1}{2}\left[\frac{\partial^2}{\partial\alpha_+^2}-\frac{\partial^2}{\partial\alpha_-^2}\right]-\frac{iV}{16}H_\triangle^2\ . \tag{3.13}$$

Putting the expansion

$$B=\kappa B_1+B_0+\kappa^{-1}B_{-1}+\cdots \tag{3.14}$$

into this equation, we get

$$\frac{1}{2V}\left[\frac{\partial B_1}{\partial\beta_\pm}\right]^2+U(\alpha_\pm,\beta_\pm)=0 \tag{3.15}$$

to order $O(\kappa^1)$ with the assumption $|\beta|\gg|\alpha|$ and

$$\tilde{\mathcal{H}}A-\frac{i}{2}\left[\frac{\partial^2B_1}{\partial\beta_+^2}-\frac{\partial^2B_1}{\partial\beta_-^2}\right]-i\left[\frac{\partial B_1}{\partial\beta_+}\frac{\partial A}{\partial\beta_+}-\frac{\partial B_1}{\partial\beta_-}\frac{\partial A}{\partial\beta_-}\right]+\left[\frac{\partial B_1}{\partial\beta_+}\frac{\partial B_0}{\partial\beta_+}-\frac{\partial B_1}{\partial\beta_-}\frac{\partial B_0}{\partial\beta_-}\right]A=0 \tag{3.16}$$

to order $O(\kappa^0)$, and so on. Now we introduce a dynamical time t based on a particular solution of Eq. (3.15) $\bar{\beta}_\pm$ by the condition

$$\kappa\frac{d\bar{\beta}_\pm}{dt}=p_{\beta_\pm}=\pm\kappa\frac{\partial B_1}{\partial\bar{\beta}_\pm}\ . \tag{3.17}$$

According to the conclusions of the previous section, we can expect the classical property of the variable β for the region $|\beta|\gg|\alpha|$. We may put $\bar{\beta}_+=\bar{\beta}_-$ (i.e., $\bar{\beta}_\pm$ is almost classical). Equation (3.16) is reduced to

$$\tilde{\mathcal{H}}A-i\frac{\partial A}{\partial t}+\frac{\partial B_0}{\partial t}A=0\ . \tag{3.18}$$

Taking the trace of both right- and left-hand sides of Eq. (3.18), we obtain

$$\mathrm{Tr}(\tilde{\mathcal{H}}A)+\frac{\partial B_0}{\partial t}=0\ , \tag{3.19}$$

where use has been made of

$$\mathrm{Tr} A = 1 \ . \tag{3.20}$$

However, since

$$\mathrm{Tr}(\tilde{\mathcal{H}} A) = \mathrm{Tr}\left[[\mathcal{H}, A] - \frac{iV}{32}[H,[H,a]] \right] = 0 \ , \tag{3.21}$$

we get

$$\frac{\partial B_0}{\partial t} = 0 \ . \tag{3.22}$$

Consequently, we obtain the time-dependent Schrödinger equation for the density matrix:

$$i\frac{\partial A}{\partial t} = \tilde{\mathcal{H}} A \ . \tag{3.23}$$

The corresponding operator form of this equation becomes

$$i\frac{\partial A}{\partial t} = [\mathcal{H}_0, A] - \frac{iV}{32}[H,[H,A]] \tag{3.24}$$

with

$$\mathcal{H}_0 \equiv \tfrac{1}{2} p_\alpha^2 \ . \tag{3.25}$$

It should be mentioned here that $p_{\beta\pm}$, and then t, cannot be real unless $U(\alpha_\pm, \beta_\pm)$ is negative. This condition is satisfied only in the expanding phase. The treatment in the contracting phase is discussed in the last part of this section.

From Eq. (3.24), we can calculate t dependence of quantum coherence and entropy. The quantity $\mathrm{Tr} A^2$ per volume ($\equiv F$) indicates the degree of quantum coherence. Actually $F = 1$ for a pure quantum state since for this state, $A^2 = A$. On the other hand, for general mixed states, $0 \le F < 1$. Furthermore, for more dispersedly distributed mixed states the value of F is smaller. We get

$$\frac{\partial F}{\partial t} = -\frac{1}{16\pi}\mathrm{Tr} A [H,[H,A]] \ , \tag{3.26}$$

which is negative semidefinite. This is because in the matrix representation that A is diagonal:

$$\mathrm{Tr} A [H,[H,A]] = \sum_{ij} |H_{ij}|^2 (A_i - A_j)^2 \ge 0 \ . \tag{3.27}$$

Equation (3.26) shows that the system gradually becomes classical in the sense that the value of F decreases monotonically. For the ordinarily defined entropy $-\mathrm{Tr} A \ln A$ per volume ($\equiv S$), we get

$$\frac{\partial S}{\partial t} = \frac{1}{16\pi}\mathrm{Tr}(\ln A)[H,[H,A]] \ , \tag{3.28}$$

which is positive semidefinite. This is because, in the same representation,

$$\mathrm{Tr}(\ln A)[H,[H,A]] = -\sum_{ij} |H_{ij}|^2 (A_j - A_i) \ln\frac{A_i}{A_j} \ge 0 \ . \tag{3.29}$$

The existence of monotonically changing quantities suggests the intrinsic irreversibility of the system (second law of thermodynamics for the minisuperspace universe).

Thus, we have shown that the direction of t coincides with that of entropy increase. Then, does the latter change its direction in the contracting phase? Our answer is affirmative. In the contracting phase, an overall factor b of S_g changes its sign at $a_{\max}$. And quite the same argument as that developed in Eqs. (3.10)–(3.29) is valid also in the contracting phase provided that α and β are exchanged. We denote the dynamical time variable in the expanding (contracting) phase as $t_\alpha (t_\beta)$. That is,

$$\frac{d\bar{\beta}_\pm}{dt_\beta} = \pm\frac{\partial B_1(\bar{\beta}_\pm)}{\partial \bar{\beta}_\pm} = \pm\frac{\partial B_1(\bar{\alpha}_\pm)}{\partial \bar{\alpha}_\pm} = \frac{d\bar{\alpha}_\pm}{dt_\alpha} \ , \tag{3.30}$$

where the middle equality holds since both right- and left-hand sides satisfy the same Hamilton-Jacobi equation (3.15). If $|\beta| \gg |\alpha|$ in the expanding phase and $|\alpha| \gg |\beta|$ in the contracting phase are satisfied, then $\bar{\beta}_\pm = \beta_{\pm(\mathrm{cl})}$ and $\bar{\alpha}_\pm = \alpha_{\pm(\mathrm{cl})}$, with $\beta_{\pm,(\mathrm{cl})}$ and $\alpha_{\pm(\mathrm{cl})}$ given by substituting Eq. (2.24) into Eq. (3.3). The classical trajectory in the $\alpha_{\pm(\mathrm{cl})} - \beta_{\pm(\mathrm{cl})}$ plane is shown in Fig. 2. From the inequality

$$\frac{d\alpha_{(\mathrm{cl})}}{dx^0}\frac{d\beta_{(\mathrm{cl})}}{dx^0} < 0 \ , \tag{3.31}$$

together with Eq. (3.30), it follows that

$$\frac{dt_\alpha}{dx^0}\frac{dt_\beta}{dx^0} < 0 \ . \tag{3.32}$$

This means that the dynamical time reverses its direction at $a_{\max}$ as well as entropy.

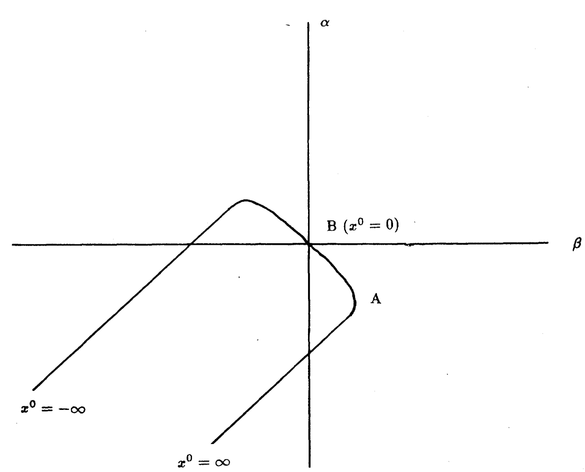

FIG. 2. The classical solution Eq. (2.24) for the case (C): $0 > \Lambda$ in terms of new variables α, β introduced in Eq. (3.3). Points A and B, respectively, correspond to those of Fig. 1(c).

IV. CONCLUSIONS AND DISCUSSIONS

We have studied arrows of time in 2D quantum cosmology. First in Sec. II, we introduced a thermodynamic time which is measured by the quantum coherence width. The width associates with the density matrix of minisuperspace. We found that this thermodynamic time coincides with a cosmological time which is defined by the increase of scale factor. Second in Sec. III, the quantum coherence measured by the linear entropy and the ordinary entropy was studied based on the semiclassical property of the clock variable argued in Sec. II. We were forced to define *local* time variables α and β in the closed universe. Using these time variables, we showed that entropy increases (decreases) in the expanding (contracting) phase as x^0 increases. This is consistent with the results of Sec. II.

In this paper, the density matrix on the minisuperspace has been naturally introduced by projecting out the other inhomogeneous modes. This subdynamics description seems to become essential whenever we make any prediction of the results of realistic cosmic observation. This subdynamics description is also used in Refs. 9 and 10, which study the loss of quantum coherence in the theory of quantum gravity. Their method essentially relies on projecting out the topologically excited baby universes of Planck size and on the concentrating upon the subdynamics of nearly flat global universe (mother universe). Their standpoint is that topologically excited other universes are causally unobservable while our standpoint is that the globally distributed physical quantities on a spacelike hypersurface are causally unobservable. In either case the intrinsic limitation of the cosmic observation introduces the loss of quantum coherence and the reduction of the quantum system to the classical evolution. In order to proceed further, including the problem of which of the above projections is more realistic, we have to consider the specific properties of the actual observational processes of the Universe. What we can say, at least, is that the naive application of quantum theory to the whole Universe, neglecting the specific structure of the detection, does not give any realistic prediction; the prediction is detector dependent. Furthermore, if we introduce the concrete structure of the detection process, any nonstandard interpretation on the wave function will become unnecessary.

APPENDIX

In this appendix, we derive the expression for the reduced density matrix Eqs. (2.9)–(2.12) based on the influence functional method developed in Ref. 11.

From the total density matrix $\bar{\rho}[a_+,b_+,\phi_+;a_-,b_-,\phi_-]$, which describes the entire Universe, a reduced density matrix ρ for the minisuperspace variables is defined:

$$\rho[a_+,b_+;a_-b_-]\equiv\int d\phi_+\int d\phi_-\bar{\rho}[a_+,b_+\phi_+;a_-,b_-,\phi_-]\delta(\phi_+-\phi_-)\ . \tag{A1}$$

We assume the matter unperturbed state is the conformal vacuum. Then the reduced density matrix is expressed as

$$\rho[a_+,b_+;a_-,b_-]=\int da'_+\int da'_-\int db'_+\int db'_-\int_{a'_+}^{a_+}\mathcal{D}a_+\int_{a'_-}^{a_-}\mathcal{D}a_-\int_{b'_+}^{b_+}\mathcal{D}b_+\int_{b'_-}^{b_-}\mathcal{D}b_-\rho[a'_+,b'_+;a'_-,b'_-]$$
$$\times\exp(i\tilde{S}[a_+,b_+;a_-,b_-])\ , \tag{A2}$$

where

$$\exp(i\tilde{S}[a_+,b_+;a_-,b_-])=\mathcal{F}[a_+,b_+;a_-,b_-]\exp\{i(S_g[a_+,b_+]-S_g[a_-,b_-])\} \tag{A3}$$

and

$$\mathcal{F}[a_+,b_+;a_-,b_-]=\int\mathcal{D}\phi_p\exp\{i(S_m[\phi_+]+S_{\rm int}[\phi_+,a_+]-S_m[\phi_-]-S_{\rm int}[\phi_-,a_-])\}\ . \tag{A4}$$

In this equation, ϕ_p means a pair of scalar fields ϕ_+ and ϕ_- and its path integration is performed on the conformal vacuum. This is the in-in formalism of quantum field theory which is natural and useful for calculating density matrices.[12]. It is straightforward to evaluate Eq. (A4) in perturbation series in the interaction. In two-by-two matrix representation, it becomes

$$\mathcal{F}=\exp\left\{-\tfrac{1}{2}{\rm Tr}\ln\left[-\Box_p+\begin{bmatrix}-h_+ & 0\\ 0 & h_-\end{bmatrix}\right]\right\}$$
$$\approx-\tfrac{1}{4}\int d^2x\int d^2x'[h_\triangle(x)\Pi_1(x,x')h_\triangle(x')+h_\triangle(x)\Pi_2(x,x')h_c(x')]+O(h^3)\ , \tag{A5}$$

where

$$h_{\pm}=N_{\pm}\left[m^2a_{\pm}^2-2\xi\left[\frac{\ddot{\chi}}{N^2}-\chi''-\frac{N'}{N}\chi'-\frac{N''}{N}-\frac{\dot{N}}{N^3}\dot{\chi}\right]\right]. \tag{A6}$$

$$h_{\triangle}\equiv h_{+}-h_{-},\quad 2h_c\equiv h_{+}+h_{-}\ ,$$

and

$$\frac{1}{\Box_p}\equiv i\begin{vmatrix}\langle T\phi(x)\phi(x')\rangle & \langle \phi(x')\phi(x)\rangle\\ \langle \phi(x)\phi(x')\rangle & \langle \bar{T}\phi(x)\phi(x')\rangle\end{vmatrix}\ , \tag{A7}$$

$$\begin{aligned}\Pi_1(x,x')&\equiv \mathrm{Re}\langle T\phi(x)\phi(x')\rangle^2\ ,\\ \Pi_2(x,x')&\equiv 2i\theta(x^0-x^{0\prime})\mathrm{Im}\langle T\phi(x)\phi(x')\rangle^2\ .\end{aligned} \tag{A8}$$

In deriving Eq. (A5), we have assumed the regularization: $\langle \phi^2(x)\rangle=0$. It is a standard calculation to evaluate Eq. (A5):

$$\begin{aligned}\mathcal{F}[a_{+}a_{-}]=\exp\Bigg[&-\frac{i}{4\pi}\int d^2x\int d^2x'h_{\triangle}(x)(2\pi)^{-2}\int d^2p\, e^{-ip\triangle x}p^{-2}\ln p^2 h_c(x')\\ &-\tfrac{1}{32}\int d^2x\int d^2x'h_{\triangle}(x)(2\pi)^{-2}\int d^2p\, e^{-ip\triangle x}p^{-2}\theta(p^2)h_{\triangle}(x')\Bigg]\ .\end{aligned} \tag{A9}$$

The second term of the exponent in Eq. (A9) arises as a back reaction of ϕ-particle production and will cause diffusion of quantum coherence of the scale factor. We concentrate upon the effect of this term in Sec. III. The first term of the exponent in Eq. (A9) is very similar to one which appeared in Ref. 13. It represents a dispersive back reaction of the ϕ field and is needed for the concrete evaluation of the diagonal elements of the density matrix. However, it does not have any relation to a reduction of classical properties of the density matrix.

As we have remarked in the Introduction, what we want to describe is the homogeneous minisuperspace variables. Thus, we consider the homogeneous perturbation $h_{\pm}(t)$ though general perturbations are also tractable. Equation (A9) reduces to

$$\exp\left[-\frac{V}{32}\int dx^0 H_{\triangle}^2(x^0)\right]\ , \tag{A10}$$

where V is a constant coordinate length over which we study the homogeneous universe and $H(x^0)$ is the time integral of the perturbation: $H_{\pm}(x^0)=\int^{x^0}dx^0 h_{\pm}(x^0)$.

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