Derivation of Ashtekar variables from tetrad gravity

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The new gravitational variables introduced recently by Ashtekar are derived from the standard tetrad gravity formalism with full local Lorentz invariance. This is done by a succession of canonical transformations, which involve as a first step the transition from arbitrary tetrads to three-dimensional triads and pure gauge boost variables. It is then shown that the weighted contravariant triads and mixed components of the extrinsic curvature are also canonically conjugate. Finally, the new variables are explicitly derived and proved to be canonical because of a remarkable identity obeyed by the spatial spin connection in three dimensions.

I. INTRODUCTION

Recently, Ashtekar has introduced new variables in general relativity, in terms of which the Hamiltonian constraints become homogeneous polynomials in the new momenta.[1] The interest of these variables is that they may simplify some questions of quantum gravity.

Now, although the phase space considered by Ashtekar is the same as that of standard gravity in the tetrad formalism, the explicit connection between the new variables and the canonical variables of known formulations of tetrad gravity[2,3] has never been fully explained.

It is the purpose of this paper to derive Ashtekar's variables starting from canonical tetrad gravity. We show how one can, by a succession of canonical transformations, go from the canonical pairs of Refs. 2 and 3 to the variables considered in Ref. 1. Our analysis therefore explicitly establishes the equivalence of Ashtekar's formulation with standard canonical tetrad gravity, and should make the formalism of Ref. 1 accessible to workers familiar with more conventional canonical approaches to general relativity.

The transition to the new variables is made in three steps. First, we show how one can go from arbitrary tetrads to triads along the spatial sections $x^0=$const and pure gauge boost variables (Sec. II). This step appears necessary to define Ashtekar variables and is achieved without fixing the time gauge. Next, we make a further canonical transformation to contravariant triad densities and mixed components of the extrinsic curvature (Sec. III). The fact that it is the contravariant triad vector densities which are conjugate to the extrinsic curvature is analogous to the well-known property of metric gravity, viz., gg^{km} is conjugate to $K_{km}g^{-1/2}$. We also relate this property to the covariant canonical formalism based on forms developed in Ref. 4, where a similar result was also found. Finally, we derive Ashtekar variables and prove that they are canonical because of a remarkable identity obeyed by the spatial spin connection in three dimensions (Sec. IV). We also rewrite the Hamiltonian constraints in terms of the new variables using an identity due to Witten.[5]

Although there is no need to introduce spinorial variables to perform the analysis, as it has already been pointed out previously,[6,7] the use of spinors suggests the consideration of the final variables in terms of which the constraints are polynomial. For this reason we rewrite the final variables, which do not refer to spinors at all, in the spinor notations used by Ashtekar. This is simply done by saturating triad indices with the Pauli matrices.

II. FROM ARBITRARY TETRADS TO TIME-GAUGE TETRADS

The canonical variables of tetrad gravity in which the full local Lorentz gauge freedom is retained are the spatial components e_{ak} of the tetrads and their momenta π^{ak} ($a=$tetrad index$=0,1,2,3$; $k=$spatial coordinate index$=1,2,3$), satisfying

$$[e_{ak}(x),\pi^{bl}(x')]=\delta_a{}^b\delta_k{}^l\delta(x,x') . \tag{1}$$

The Hamiltonian constraints read

$$\mathcal{H}_\perp=\tfrac{1}{2}g^{-1/2}(g_{ik}g_{jl}+g_{il}g_{jk}-g_{ij}g_{kl})\pi^{ij}\pi^{kl}-Rg^{1/2}\approx 0 , \tag{2a}$$

$$\mathcal{H}_i=-2\pi_i{}^j{}_{|j}\approx 0 , \tag{2b}$$

$$J^{ab}=(\pi^{ak}e^b{}_k-\pi^{bk}e^a{}_k)\approx 0 , \tag{2c}$$

where

$$g_{ij}=e_{ai}e^a{}_j , \tag{3a}$$

$$\pi^{ij}=\tfrac{1}{4}(e_a{}^i\pi^{aj}+e_a{}^j\pi^{ai}) . \tag{3b}$$

The constraints J^{ab} generate local Lorentz rotations and are present because the full Lorentz gauge freedom has been retained.

The vectors $e_{a\mu}$ have no particular relation to the hypersurfaces $x^0=\text{const}$. For instance, $e_{0\mu}$ is not necessarily normal to these hypersurfaces, and the vectors $e_{a'\mu}$ $(a'=1,2,3)$ are not necessarily tangent. For the subsequent discussion it is necessary to use variables which are adapted to the hypersurfaces $x^0=\text{const}$ (see below). This can be done by performing a point canonical transformation as follows. Given $e_{a\mu}$ and the hypersurface $x^0=\text{const}$, one can, by a unique Lorentz rotation in the plane $(e_{0\mu},n_\mu)$ where n_μ is the normal to $x^0=\text{const}$, define a triad $h_{a'k}$ tangent to $x^0=\text{const}$. The components of the triad, and the parameters $\omega_{a'}$ characterizing the Lorentz boost, can be used as new variables instead of e_{ak}.

The formulas which connect these two sets are

$$e_{0k}=-h_{a'k}\omega^{a'} , \tag{4a}$$

$$e_{a'k}=h_{a'k}+\frac{1}{\gamma+1}h_{b'k}\omega^{b'}\omega_{a'} , \tag{4b}$$

where

$$\gamma=(1+\omega_{a'}\omega^{a'})^{1/2} . \tag{4c}$$

The inverse transformation reads

$$\omega_{a'}=n_{a'} , \tag{5a}$$

$$h_{a'k}=e_{a'k}-\frac{1}{\gamma(\gamma+1)}e_{b'k}n^{b'}n_{a'} , \tag{5b}$$

where n_a are the components of the normal in the tetrad frame $e_{a\mu}$ and only depend on e_{ak} (Ref. 2). One easily checks that $h_{a'k}$ are indeed triads on $x^0=\text{const}$, i.e., that the spatial metric (3a) is given by

$$g_{ij}=h_{a'i}h^{a'}{}_j . \tag{6}$$

The transformation (4) and (5) can be completed as a canonical transformation by means of the formula $p\,dq=p'dq'$. One finds

$$p^{a'k}=\pi^{a'k}-\pi^{0k}\omega^{a'}+\frac{1}{\gamma+1}\pi^{b'k}\omega_{b'}\omega^{a'}, \tag{7a}$$

$$\pi^{a'}=-\pi^{0k}h^{a'}{}_k+\frac{1}{\gamma+1}\omega_{b'}(h^{a'}{}_k\pi^{b'k}+\pi^{a'k}h^{b'}{}_k) , \tag{7b}$$

where $p^{a'k}$ and $\pi^{a'}$ are, respectively, the momenta conjugate to $h_{a'k}$ and $\omega_{a'}$.

Because the theory is invariant under local Lorentz transformations, and because the variables $\omega_{a'}$ are just Lorentz parameters, the canonical pairs $(\omega_a,\pi^{a'})$ should be pure gauge. That this is so can be seen by expressing the Lorentz generators J^{ab} in terms of the momenta $\pi^{a'}$ and the spatial rotation generators $j^{a'b'}$ of the triads $h_{a'k}$,

$$j^{a'b'}=(p^{a'k}h^{b'}{}_k-p^{b'k}h^{a'}{}_k) . \tag{8}$$

One finds, after elementary calculations,

$$J^{0b'}=-\pi^{b'}+\frac{1}{\gamma+1}\omega_{c'}j^{c'b'} , \tag{9a}$$

$$J^{a'b'}=j^{a'b'}-\omega^{a'}\pi^{b'}+\omega^{b'}\pi^{a'} , \tag{9b}$$

and

$$j^{a'b'}=J^{a'b'}-\omega^{a'}J^{0b'}+\omega^{b'}J^{0a'} +\frac{1}{\gamma+1}\omega_{c'}(\omega^{a'}J^{c'b'}-\omega^{b'}J^{c'a'}) , \tag{9c}$$

$$\pi^{a'}=\frac{1}{\gamma+1}\omega_{b'}J^{b'a'}+\left[-\delta^{a'}_{c'}+\frac{\omega_{c'}\omega^{a'}}{\gamma(\gamma+1)}\right]J^{0c'} . \tag{9d}$$

It follows that the constraints $J^{ab}\approx 0$ are entirely equivalent to

$$\pi^{a'}\approx 0,\quad j^{a'b'}\approx 0 \tag{10}$$

and, therefore, $\omega_{a'}$ is pure gauge, since it can be transformed at will by a gauge transformation. The imposition of the time-gauge condition is equivalent to $\omega_{a'}=0$.

We will not impose the time-gauge condition in the remainder of this paper and rather keep the variables $\pi^{a'}$ and $\omega_{a'}$. However, because they are pure gauge, these variables decouple from the rest. In particular, one finds that π^{ij} is given by the expression

$$\pi^{ij}=\tfrac{1}{4}(p^{a'i}h_{a'}{}^j+p^{a'j}h_{a'}{}^i) \tag{11}$$

so that the constraints $\mathcal{H}_\perp$ and $\mathcal{H}_i$ given by (2a) and (2b) do not depend on the pure gauge variables $\omega_{a'}$ or $\pi^{a'}$.

III. THE EXTRINSIC CURVATURE AS A CANONICAL VARIABLE

The canonical variables $(h_{a'k},p^{a'k})$ obtained in Sec. II satisfy

$$[h_{a'k}(x),p^{b'l}(x')]=\delta^{b'}_{a'}\delta^l_k\,\delta(x,x') .$$

In this section we show how, by a further canonical transformation, the variables $(\tilde{h}^{a'k},2K_{a'k})$ arise as another canonical pair, where $\tilde{h}^{a'k}=g^{1/2}g^{ik}h^{a'}{}_i$ and

$$K_{a'k}=h_{a'}{}^iK_{ik}+\tfrac{1}{4}g^{-1/2}j_{a'b'}h^{b'}{}_k . \tag{12}$$

In (12), K_{ik} is the extrinsic curvature, given in terms of metric momenta π^{ij} by

$$K_{ik}=g^{-1/2}(\tfrac{1}{2}\pi_{jl}g^{jl}g_{ik}-\pi_{ik})$$

and, by (11), in terms of the triad momenta $p^{a'i}$, by

$$K_{ik}=\tfrac{1}{4}g^{-1/2}p^{a'j}(h_{a'j}g_{ik}-h_{a'i}g_{kj}-h_{a'k}g_{ij}) . \tag{13}$$

That this transformation is canonical can be checked either by directly evaluating the Poisson brackets $[\tilde{h}^{a'k}(x),2K_{b'i}(x')]$ and $[K_{b'i}(x),K_{a'j}(x')]$ or, as in Sec. II, by checking that

$$2K_{a'j}d\tilde{h}^{a'j}=p^{a'j}dh_{a'j} .$$

In either case one needs the definition of $j_{a'b'}$ [Eq. (8)] and the relationship

$$\frac{\delta\tilde{h}^{a'j}}{\delta h_{b'i}}=g^{1/2}(h^{a'j}h^{b'i}-h^{a'i}h^{b'j}) . \tag{14}$$

This result is analogous to the well-known result of metric gravity, that is, if

$$[g_{ij}(x),\pi^{kl}(x')]=\tfrac{1}{2}(\delta_i{}^k\delta_j{}^l+\delta_j{}^k\delta_i{}^l)\delta(x,x') ,$$

then also

$$[gg^{ij}(x),g^{-1/2}K_{kl}(x')]=\tfrac{1}{2}(\delta^i{}_k\delta^j{}_l+\delta^i_l\delta^j{}_k)\delta(x,x') .$$

We point out that if the full Lorentz gauge freedom was maintained, it would be necessary to add to K_{aj} a term proportional to the normal n^a to the hypersurface $x^0=$const; that is, in an arbitrary gauge, the fields conjugate to the tetrad densities $g^{1/2}g^{ij}e_{aj}$ $(a=0,1,2,3)$, are given by

$$K_{ak}=e_a{}^iK_{ik}+\tfrac{1}{4}g^{-1/2}J_{ab}e^b{}_k-\tfrac{1}{4}g^{-1/2}n_an^be^c{}_kJ_{bc} .$$

That the $(\tilde{h}^{a'k},2K_{a'k})$ form a canonical pair can alternatively be seen considering the identity

$$\omega_j{}^{ab}=\Omega_j{}^{ab}+(n^be^{ai}-n^ae^{bi})K_{ij} \tag{15}$$

which relates the spatial components of the four-dimensional spin connection ω to those of the connection on $x^0=$const, denoted Ω, via the extrinsic curvature. In the time gauge, this identity (15) has components

$$\omega_j^{a'0}=h^{a'i}K_{ij} . \tag{16}$$

In (15) and (16), time derivatives of the $e^a{}_i$ appear on both sides. Since these are given uniquely in terms of the π^{ai} only on the constraint surface $J_{ab}=0$, one can always add terms proportional to J_{ab} (or $j_{a'b'}$) to the right-hand side of (15) or (16).

Now, in the first-order formalism of tetrad gravity constructed covariantly using differential forms,[4] the two-form primary constraint

$$\phi_{ab}=\pi_{ab}-e^c\wedge e^d\epsilon_{abcd} \tag{17}$$

with $\epsilon_{0123}=-1=-\epsilon^{0123}$, relates the (two-form) momenta conjugate to the (one-form) spin connection ω^{ab} to the square of the (one-form) tetrad e^a. When the time gauge is implemented,[8] this constraint (17) has components on $x^0=$const given by

$$\phi_{ab}=\tfrac{1}{2}\phi_{abij}dx^i\wedge dx^j ,$$

etc., with (17′)

$$\phi_{0a'ij}=(\pi_{0a'}{}^k+2g^{1/2}h_{a'}{}^k)\epsilon_{ijk} ,$$

where $\pi_{0a'}{}^k=\tfrac{1}{2}\pi_{0a'ij}\epsilon^{ijk}$ and $\epsilon_{123}=1$. In (17′) we have used the three-dimensional identity

$$h^{a'}{}_ih^{b'}{}_j\epsilon_{a'b'c'}=g^{1/2}\epsilon_{ijk}h_{c'}{}^k .$$

When the constraints (17′) ar eliminated, one sees immediately that the momenta $\pi_{0a'}{}^k$ conjugate to the components $\omega^{0a'}{}_k=-h^{a'i}K_{ik}(+j_{a'b'}\text{terms})$ from (16) are precisely

$$\pi_{0a'}{}^k=-2\tilde{h}_{a'}{}^k . \tag{18}$$

That is, the pairs $(\omega_k{}^{0a'},-2\tilde{h}_{a'}{}^k)$ are canonical; therefore, the pairs $(2K_{a'k},\tilde{h}^{a'k})$ are canonical, which is exactly our previous result.

IV. THE "NEW VARIABLES"

The variables considered by Ashtekar[1] are the $\tilde{h}^{a'k}$ themselves and the self-dual combination of the four-dimensional spin connection,

$$A_{a'k}=2K_{a'k}+\frac{i}{2}\epsilon_{a'b'c'}\omega_k^{b'c'} , \tag{19}$$

where $\omega_k{}^{b'c'}$ is given in terms of the variables $h^{a'}{}_i$ in the standard way:

$$\omega_k{}^{a'b'}=\tfrac{1}{2}[g_{kj,i}h^{a'j}h^{b'i}+h^{a'i}h^{b'}{}_{i,k}-(a'\leftrightarrow b')] . \tag{20}$$

We now show that these variables also form a canonical pair. This property follows because, in three dimensions, the spin connection (20) obeys a remarkable identity. That is, the second term on the right-hand side of (19) is a functional derivative with respect to $\tilde{h}^{a'i}$ of the functional $G[\tilde{h}^{a'i}]$ with

$$G=\int d^3x\, \epsilon^{ijk}h^{a'}{}_kh_{a'j,i} , \tag{21a}$$

$$\omega_k{}^{a'b'}\epsilon_{a'b'c'}=\frac{\delta G}{\delta\tilde{h}^{c'k}} . \tag{21b}$$

The replacement of $2K_{a'k}$ by $A_{a'k}$ in the canonical momenta conjugate to $\tilde{h}^{a'k}$ just corresponds to a canonical phase transformation and one has indeed[9]

$$\int d^3x\, 2K_{a'k}d\tilde{h}^{a'k}=\int d^3x\, A_{a'k}d\tilde{h}^{a'k}-\frac{i}{2}dG . \tag{22}$$

One could of course have also considered the self-dual combination of the $\tilde{h}^{a'k}$,

$$\tilde{h}^{a'k}+\frac{i}{4}\epsilon^{ijk}h^{a'}{}_ih^0{}_j , \tag{23}$$

but, in the time gauge, this second term is identically zero. Therefore, it is only in the time gauge that the triad densities $\tilde{h}^{a'k}$ are real.

V. CONSTRAINTS

The variables $A_{a'k}$ are connections and one can thus define their field strengths $F_{a'ij}$. In terms of the Riemann tensor and the extrinsic curvature, $F_{a'ij}$ is given by

$$F_{a'ij}=\frac{i}{4}\epsilon^{klm}({}^3R_{klij}+2\bar{K}_{ki}\bar{K}_{lj})h_{a'm}+\tfrac{1}{2}(\bar{K}_{kj|i}-\bar{K}_{ki|j})h_{a'}{}^k \tag{24}$$

with

$$\bar{K}_{kj}=K_{kj}+\tfrac{1}{4}g^{-1/2}h^{a'}{}_kh^{b'}{}_jj_{a'b'}=K_{a'j}h^{a'}{}_k$$

as can be seen by a straightforward calculation [(24) is actually a reexpression of the Gauss-Codazzi equations].

Now, if one replaces, in $F_{a'ij}$, $\bar{K}_{ij}$ by K_{ij}, which does not change anything on the constraint surface $j_{a'b'}=0$, and if one computes $\tilde{h}^{a'i}F_{a'ij}$ and $\tilde{h}_{a'}{}^i\tilde{h}_{b'}{}^j\epsilon^{a'b'c'}F_{c'ij}$, one finds that the first expression yields $\mathcal{H}_j$ [Eq. (2b)] while the second one reduces to $\mathcal{H}_\perp g^{1/2}$ [Eq. (2a)]. This result is nothing but the transcription in the present formalism of an identity found by Witten.[5]

Therefore, the constraints

$$\tilde{h}^{a'i}F_{a'ij}=0 \ , \quad (25a)$$

$$\tilde{h}^{\,i}_{a'}\tilde{h}^{\,j}_{b'}F_{c'ij}=0 \ , \quad (25b)$$

and

$$j^{a'b'}=0 \quad (25c)$$

are equivalent to the original constraints (2a)–(2c). The advantage of (25a) and (25b) over the metric formulation is that, if one regards $\tilde{h}^{a'i}$ as momenta and $A_{a'i}$ as configuration variables, Eq. (25a) is linear in the momenta while (25b) is quadratic. Furthermore, $F_{a'ij}$ is a polynomial function of $A_{a'i}$ and of its derivatives.

Note that the constraint (25c) can be rewritten in the Yang-Mills form

$$\nabla_k \tilde{h}^{a'k} \equiv \tilde{h}^{a'k}{}_{|k} + i A_{c'k}\tilde{h}_{b'}{}^{k}\epsilon^{a'b'c'}=0 \ . \quad (26)$$

Indeed, the spatial connection term in $A_{c'k}$ combines with $\tilde{h}^{a'k}{}_{|k}$ to yield the full spatial derivative $\mathcal{D}_k h^{a'k}$ of the triads, which is zero since ${}^3\omega_{a'b'i}$ is the metric connection. At the same time, the extrinsic curvature $\bar{K}_{ck}$ piece in $A_{c'k}$ combines with $\tilde{h}_{b'}{}^{k}$ to yield $j_{c'b'}$.

VI. CONCLUSION

We have written the variables proposed by Ashtekar in terms of the well-known variables of triad gravity, which can in turn be reformulated in terms of the better known tetrad variables by means of (5b).

We have avoided introducing σ matrices to make it clear that spinors are not necessary to arrive at (25). However, for the sake of completeness, and because they suggested the appropriate change of variables, we point out that the spinor variables $(A^{kMN}, \tilde{\sigma}^{iMN})$ of Ref. 1 can be obtained from (19) and $\tilde{h}^{a'k}$ by contraction with the Pauli matrices $(\sigma_{a'})^{M}{}_{N}$ which satisfy $\sigma_{a'}\sigma_{b'}=\delta_{a'b'}+i\epsilon_{a'b'c'}\sigma^{c'}$ by the following relationships:

$$(A^{k})^{MN}=A_{a'}{}^{k}(\sigma^{a'})^{MN}, \quad (\tilde{\sigma}^{i})^{MN}=-\frac{i}{\sqrt{2}}\tilde{h}_{a'}{}^{i}(\sigma^{a'})^{MN} \ ,$$

$$(\sigma^{a'})^{MN}=(\sigma^{a'})^{M}{}_{L}\epsilon^{LN}, \quad \epsilon^{LN}=-\epsilon^{NL} \ .$$

Note added. After completion of this work, we became aware of recent papers[10] in which some aspects of the reformulation of Ashtekar variables in terms of time-gauge triads are discussed along different lines.

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