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Cosmological solution of Einstein's equations with uniform density and nonuniform pressure

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An exact solution of the equations of general relativity is given that is most appropriately interpreted cosmologically and has some unusual properties. The solution is a perfect-fluid, spherically symmetric, nonstatic one in which the density is uniform but the pressure is nonuniform. A brief investigation, both algebraic and numerical, is made of the solution's main physical attributes. It may be used to model a localized inhomogeneity such as a bubble in the early Universe.

I. INTRODUCTION

In recent years it has become popular to look for solutions of Einstein's field equations that are significantly different from the known ones,[1] as a means of studying how effects of particle physics and quantum field theory can modify classical cosmology.[2–4] Not long ago, a perfect-fluid, spherically symmetric, nonstatic solution was found of a particularly interesting type.[5] In it, the density is uniform (homogeneous and isotropic) as in the Friedmann-Robertson-Walker solutions of standard cosmology, but the pressure is nonuniform. Solutions of this type are interesting because they depart modestly enough from the standard ones that they can be meaningfully interpreted, while also showing significant new properties. With this motivation in mind, in the present work another solution of the noted type will be presented and investigated. The present account is kept brief, since it is not known how useful the solution will be in astrophysics or cosmology. But it appears the solution may be used to model a localized inhomogeneity such as a bubble in the early Universe.

II. THE FIELD EQUATIONS AND A SOLUTION

Consider a spherically symmetric metric of the form

$$ds^2=e^{\sigma}c^2dt^2-e^{\omega}dr^2-R^2(d\theta^2+\sin^2\theta\,d\phi^2)\ . \qquad (1)$$

The coordinates are t, r, θ, and ϕ where r is comoving and σ, ω, and R depend in general on t and r. The speed of light is c, and conventional units will be used for ease of physical interpretation both here and below, where the constant of gravity G is introduced. Einstein's field equations for metric (1) and a perfect fluid are fairly well known in the form due to Misner and Sharp[6] and Podurets:[7] namely,

$$m\equiv\frac{c^2R}{2G}\left[1+\frac{e^{-\sigma}\dot{R}^2}{c^2}-e^{-\omega}R'^2\right]\ , \qquad (2a)$$

$$\dot{m}=\frac{-4\pi pR^2\dot{R}}{c^2}\ , \qquad (2b)$$

$$m'=4\pi\rho R^2R'\ , \qquad (2c)$$

$$\sigma'=\frac{-2p'}{\rho c^2+p}\ , \qquad (2d)$$

$$\dot{\omega}=\frac{-2\dot{\rho}c^2}{\rho c^2+p}-\frac{4\dot{R}}{R}\ . \qquad (2e)$$

The density and pressure of the matter are ρ and p, respectively, and the mass within radius r from the origin of coordinates is m. Derivatives with respect to t and r are denoted by an overdot and a prime, respectively. Equations (2) have been used quite widely (see, for example, Refs. 5 and 8–11). And while they are ill defined for problems involving static ($\dot{R}=0$) or Kantowski-Sachs-type ($R'=0$) metrics, they are eminently suited to most problems in astrophysics and cosmology because of their ready physical interpretation (see Ref. 4). Equations (2) admit the standard Friedmann-Robertson-Walker solutions where both the density and the pressure are uniform, but also solutions where the density is uniform and the pressure nonuniform. One of these will now be presented.

It may be verified by direct substitution and several hours of boring algebra that an exact solution of (2) is given by

$$e^{\sigma}=\frac{1}{g^2},\quad g\equiv(1-\alpha\beta^2t^{2/3}r^2)\ , \qquad (3a)$$

$$e^{\omega}=\left[\frac{\beta ct^{2/3}}{g}\right]^2\ , \qquad (3b)$$

$$R=\frac{\beta ct^{2/3}r}{g}\ , \qquad (3c)$$

$$\rho=\frac{1}{2\pi G}\left[\frac{1}{3t^2}-\frac{3\alpha}{t^{2/3}}\right]\ , \qquad (3d)$$

$$p=\frac{c^2}{2\pi G}\left[\frac{2\alpha}{t^{2/3}}-\frac{\alpha\beta^2r^2}{3t^{4/3}}+\alpha^2\beta^2r^2\right]\ , \qquad (3e)$$

$$m=\frac{4\pi R^3\rho}{3}=\frac{2\beta^3c^3r^3(1-9\alpha t^{4/3})}{9Gg^3}\ . \qquad (3f)$$

Two constants appear here, with dimensions $\alpha=T^{-4/3}$ and $\beta=L^{-1}T^{1/3}$. Of these, the second is not very important because it may if so desired be absorbed by a rescaling of r, but it is kept here for dimensional consistency.

By comparison, the first constant α is important. It determines the density and pressure, as well as the geometry. It should be noted that for $\alpha=0$, ρ and p reduce to those of the Einstein–de Sitter solution of conventional cosmology, as does the metric. This correspondence, together with the fact that the density is in any case uniform, suggests that (3) be interpreted cosmologically.

For $\alpha\neq 0$, p is finite and nonuniform. However, if $\rho>0$ is required for all t, then $\alpha<0$ is necessary. This also ensures that the function g and the geometry are always well behaved. A consequence of $\alpha<0$ is that there is a region near the origin of coordinates where $p<0$, though the more usual behavior holds away from the origin where $p>0$. It should be recalled that in recent years $p<0$ has come to be regarded as physically acceptable, since a negative pressure corresponds to an attractive force between the particles of the fluid, and this can arise naturally under a variety of circumstances.[5,12–15] But while this unusual property of (3) can be understood physically without much effort, other properties of the solution are harder to grasp. This comment refers especially to the mass m and the dynamics of the model as specified by the azimuthal distance measure R. The mass (3f) has the strange property that $m\sim 0$ for $r\sim 0$ *and*

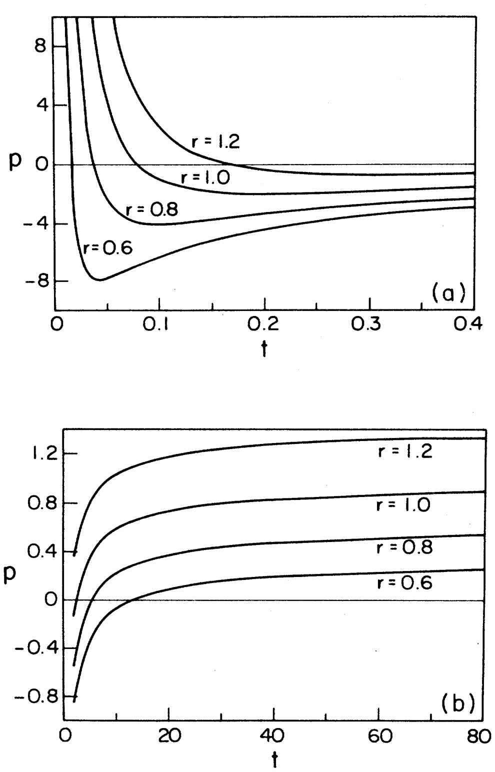

FIG. 1. (a) The pressure p against (small) t for various r. From (3e) with $\alpha=-1$, $\beta=+1$, and units satisfying $c^2/2\pi G=1$. (b) The pressure p against (large) t, with quantities as specified in (a).

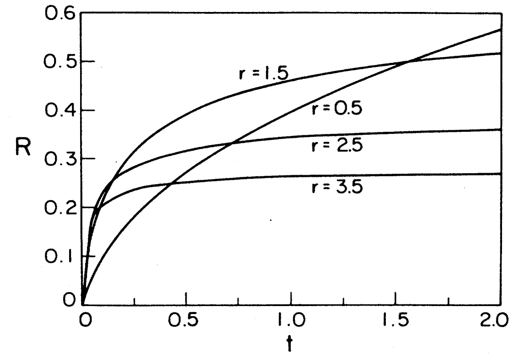

FIG. 2. The azimuthal distance measure R against t for various r. From (3c) with $\alpha=-1$, $\beta=+1$, and units satisfying $c=1$.

$r\sim\infty$. This kind of behavior has been noted before, though, in other situations.[16,17] Here, it arises because the azimuthal distance measure (3c) also has the property that $R\sim 0$ for $r\sim 0$ and $r\sim\infty$. This in turn arises because of the influence of the function g in (3a). That is, the unusual properties of m and R arise from the geometry.

The unusual behavior of the solution (3) for large radii suggests that it be truncated at some appropriate distance from the origin of coordinates and joined to another solution. This is the most reasonable way to make use of solutions such as the one being considered here.[5] The present solution in the form (3) itself suggests truncation, because the pressure (3e) increases away from the origin, a behavior which might arise from the operation of certain processes of particle physics but which cannot extend indefinitely far. Thus the most logical way to consider (3) is that it represents a localized inhomogeneity, such as a bubble,[3] embedded in a global background, such as that of the early Universe.

It can be remarked in passing that if (3) is to be embedded in a background in this way then it may be con-

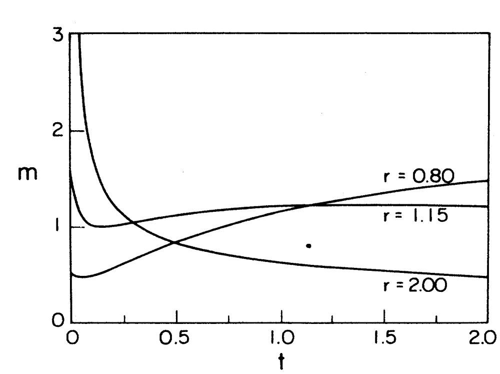

FIG. 3. The mass m against t for various r. From (3f) with $\alpha=-1$, $\beta=+1$, and units satisfying $2c^3/9G=1$.

venient to express the solution in other coordinates, a procedure which also suggests other applications of the solution but which will not be pursued here.[18]

Returning to the simple form (3), more insight into it may be gained from numerical work. The density (3d) is a simple decreasing function of time and so not very interesting, but the pressure p of (3e), the azimuthal distance measure R of (3c), and the mass m of (3f) are worth plotting. Choosing $\alpha=-1$ and $\beta=+1$ for reasons given above, the behavior of these three parameters is illustrated in Figs. 1–3.

Finally, leaving the solution's physical properties, it can be pointed out that it also has some interesting mathematical properties. Most notably, it admits a one-parameter group of conformal motions. In the usual notation, it may be shown that

$$\mathcal{L}_{\xi} g_{ij} = \left[\frac{6k + 2\alpha\beta^2 k t^{2/3} r^2}{1-\alpha\beta^2 t^{2/3} r^2} \right] g_{ij} \,, \tag{4a}$$

$$\xi^i = 3kct\delta_0^i + kr\delta_1^i \,, \tag{4b}$$

with k a constant.

III. CONCLUSION

A solution of Einstein's equations has been given that is spherically symmetric and nonstatic, with matter that consists of a perfect fluid with uniform density but nonuniform pressure. The solution is most appropriately interpreted as representing a localized inhomogeneity such as a bubble in the early Universe. It has unusual properties that can be studied further.

ACKNOWLEDGMENTS

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[18]At the risk of being mathematically trivial, it may be physically useful to note that a more general form of (3), which also satisfies (2), is given by $e^{\sigma}=(\dot{T}/g)^2$, $g\equiv(1-\alpha T^{2/3}X^2)$, $e^{\omega}=(cT^{2/3}X'/g)^2$, $R=cT^{2/3}X/g$, $\rho=(1/2\pi G)(1/3T^2-3\alpha/T^{2/3})$, $p=(c^2/2\pi G)(2\alpha/T^{2/3}-\alpha X^2/3T^{4/3}+\alpha^2X^2)$, $m=4\pi R^3\rho/3=(2c^3X^3/9Gg^3)(1-9\alpha T^{4/3})$. Here the two functions $T(t)$ and $X(r)$ can be chosen as desired. The choices $T=t$, $X=\beta r$ give back (3) of the main text, but other choices (e.g., $X=\text{const}/r$) give forms that suggest other physical applications.