Brief Reports

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$\langle \phi^2 \rangle$ for massive fields in Schwarzschild spacetime

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The quantity $\langle \phi^2 \rangle$ is computed for massive scalar fields in a background Schwarzschild spacetime. For conformally coupled massive scalar fields the expectation value of the trace of the stress-energy tensor, $\langle T \rangle$, is also computed.

Since Hawking's discovery that black holes emit a thermal distribution of particles,[1] a great deal of effort has gone into the study of quantum effects in black-hole spacetimes. Two very useful tools in such studies are the quantities $\langle \phi^2 \rangle$, where ϕ is a quantum field, and $\langle T_{\mu\nu} \rangle$, where $T_{\mu\nu}$ is the stress-energy tensor operator for ϕ. $\langle \phi^2 \rangle$ is a useful quantity because it gives one qualitative information about $\langle T_{\mu\nu} \rangle$ and can often be computed with much less effort. $\langle \phi^2 \rangle$ also provides information about spontaneous symmetry breaking in a given background spacetime. $\langle T_{\mu\nu} \rangle$ gives direct insight into such quantum effects as vacuum polarization and particle production. Through use of the semiclassical back-reaction equations it also provides insight into the ways in which quantum fields can affect the spacetime geometry.

To date, most computations of $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ in black-hole spacetimes have focused on Schwarzschild spacetime. In a background Schwarzshild spacetime Fawcett and Whiting,[2] Candelas and Howard,[3] and Candelas and Jensen[4] have computed $\langle \phi^2 \rangle$ for a massless scalar field, while Fawcett[5] and Howard and Candelas[6] have computed $\langle T_{\mu\nu} \rangle$ for a massless scalar field. Most recently Jensen and Ottewill[7] have computed $\langle T_{\mu\nu} \rangle$ for the electromagnetic field.

For massive fields no exact computations have been made. However, an approximate computation of $\langle \phi^2 \rangle$ has been made by Frolov[8] and an approximate computation of $\langle T_{\mu\nu} \rangle$ has been made by Frolov and Zel'nikov[9,10] for massive scalar fields in Schwarzschild and Kerr spacetimes. The approximations used are expected to be valid when $\tilde{m} \equiv Mm \gg 1$, where M is the mass of the black hole and m is the mass of the scalar field. Throughout this paper, units are used such that $\hbar = G = c = k_B = 1$. The results of these calculations show that for $\tilde{m} \gg 1$ vacuum-polarization effects for massive fields are smaller than for massless ones and their magnitude tends to decrease with increasing mass by a factor of $\tilde{m}^{-2}$.

In this paper $\langle \phi^2 \rangle$ is computed for massive scalar fields in Schwarzschild spacetime using a generalization of the method used by Candelas and Howard[3] for massless fields. It is shown that for $\tilde{m} \gtrsim 2$, the approximate expressions obtained by Frolov and Zel'nikov are valid. Vacuum-polarization effects for small values of $\tilde{m}$ are also explored. It is found that even for $\tilde{m} = 0.1$ there is a significant decrease in the value of $\langle \phi^2 \rangle$ compared to the $\tilde{m} = 0$ case.

For massive scalar fields with conformal coupling to the scalar curvature, the expectation value of the trace of the stress-energy tensor $\langle T \rangle$ is also computed. For $\tilde{m} \gtrsim 2$, the approximate expressions of Frolov and Zel'nikov are shown to be valid. As is predicted by their approximation, the trace anomaly is effectively canceled by other terms in $\langle T \rangle$ for $\tilde{m} \gtrsim 2$.

The computation of $\langle \phi^2 \rangle$ proceeds in much the same way as in Ref. 3. One begins by noting that for the Euclideanized version of Schwarzschild spacetime the metric has the form

$$ds^2 = (1-2M/r)d\tau^2 + (1-2M/r)^{-1}dr^2 + r^2 d\Omega^2 . \quad (1)$$

The Euclidean Green's function $G_E(x,x)$ is equal to $\langle \phi^2 \rangle$ as can be seen from the relationships

$$\langle \phi^2 \rangle = G^{(1)}(x,x)/2 = iG_F(x,x) = G_E(x,x) . \quad (2)$$

As in Ref. 3, the renormalization scheme used to compute $\langle \phi^2 \rangle$ is point splitting. First $G_E(x,x')$ is computed for a scalar field in the background (1) and then the DeWitt-Schwinger expansion for $G_E(x,x')$ is subtracted from it. The result is an expression which is finite in the limit $x' \to x$.

Candelas[11] has derived $G_F(x,x')$ for a scalar field in Schwarzschild spacetime in the Hartle-Hawking vacuum[12] which is a thermal state at the black-hole temperature $T = \kappa/(2\pi)$, where $\kappa = 1/(4M)$ is the surface gravity and M is the mass of the black hole. Using his results, one finds that when the points have the same values of r, θ, and ϕ, and different values of τ, such that $\tau - \tau' = \epsilon$,

$$G_E(x,x') = (64\pi^2 M^2)^{-1} \sum_{l=0}^{\infty} (2l+1) p_{0l}(s) q_{0l}(s) + (32\pi^2 M^2)^{-1} \sum_{n=1}^{\infty} n^{-1} \cos(n\kappa\epsilon) \times \sum_{l=0}^{\infty} (2l+1) p_{nl}(s) q_{nl}(s) . \quad (3)$$

A new radial coordinate s has been defined here such that $r=M(s+2)$. Note that $s=0$ on the event horizon. The modes p_{nl} and q_{nl} satisfy the equation

$$\left[\frac{d}{ds}(s^2+2s)\frac{d}{ds}-l(l+1)-\frac{n^2(s+2)^3}{16s}-\tilde{m}^2(s+2)^2\right]R_{nl}=0\ . \quad (4)$$

They are normalized so that in the limit $s\to 0$, $p_{0l}\sim 1, q_{0l}\sim -\ln s, p_{nl}\sim s^{n/2}$, and $q_{nl}\sim s^{-n/2}$.

Note that with the definition $\tilde{m}\equiv mM$, the mass M of the black hole scales out of the mode equation.

Candelas[11] pointed out that, as it stands, the left side of Eq. (3) is finite as long as x and x' are separated, but for the given separation the right-hand side is divergent. This can be remedied by subtracting the quantity $2/(s^2+2s)^{1/2}$ from $p_{0l}q_{0l}$ in Eq. (3) and adding it to $p_{nl}q_{nl}$ using the identity $\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)=-\frac{1}{2}$. The result is

$$G_E(x,x')=(64\pi^2M^2)^{-1}\sum_{l=0}^{\infty}[(2l+1)p_{0l}(s)q_{0l}(s)-2/(s^2+2s)^{1/2}]$$
$$+(32\pi^2M^2)^{-1}\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)\sum_{l=0}^{\infty}[(2l+1)n^{-1}p_{nl}(s)q_{nl}(s)-2/(s^2+2s)^{1/2}]\ . \quad (5)$$

To renormalize, one subtracts the DeWitt-Schwinger expression for $G_E(x,x')$ which is given by[13]

$$G_{\rm DS}(x,x')=(8\pi^2\sigma)^{-1}+\frac{m^2}{8\pi^2}\left[\gamma+\tfrac{1}{2}\ln\left|\frac{m^2\sigma}{2}\right|-\tfrac{1}{2}\right]\ , \quad (6a)$$

where σ is one-half the square of the geodesic distance separating x and x' and γ is Euler's constant. For the points chosen

$$\sigma=\frac{\epsilon^2 s}{2s+4}-\frac{\epsilon^4 s}{24M^2(s+2)^5}\ . \quad (6b)$$

To get $G_{\rm DS}$ into the same form as G_E, one can use the Plana sum formula[14] which says that, for a function $f(n)$,

$$\sum_{n=j}^{\infty}f(n)=\frac{f(j)}{2}+\int_j^{\infty}dx\,f(x)+i\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}[f(j+it)-f(j-it)]\ . \quad (7)$$

The result is[15]

$$G_{\rm DS}(x,x')=\frac{-(s+2)}{16\pi^2M^2s}\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)[n^2/16+\tilde{m}^2s/(s+2)]^{1/2}+\frac{\tilde{m}^2}{16\pi^2M^2}\ln[\tilde{m}^2s/(s+2)]$$
$$-\frac{\tilde{m}^2}{8\pi^2M^2}\ln\{\tfrac{1}{4}+[\tfrac{1}{16}+\tilde{m}^2s/(s+2)]^{1/2}\}$$
$$+\frac{i(s+2)}{16\pi^2M^2s}\int_0^{\infty}\frac{dt}{[\exp(2\pi t)-1]}\{[(1+it)^2/16+\tilde{m}^2s/(s+2)]^{1/2}-[(1-it)^2/16+\tilde{m}^2s/(s+2)]^{1/2}\}$$
$$+\frac{1}{48\pi^2M^2s(s+2)^3}+O(\epsilon)\ . \quad (8)$$

Subtracting Eq. (8) from Eq. (5) and taking the limit $\epsilon\to 0$ gives the following renormalized expression for $\langle\phi^2\rangle$:

$$\langle\phi^2\rangle=(64\pi^2M^2)^{-1}\sum_{l=0}^{\infty}[(2l+1)p_{0l}(s)q_{0l}(s)-2/(s^2+2s)^{1/2}]$$
$$+(32\pi^2M^2)^{-1}\sum_{n=1}^{\infty}\left[\sum_{l=0}^{\infty}[(2l+1)n^{-1}p_{nl}(s)q_{nl}(s)-2/(s^2+2s)^{1/2}]+\frac{2(s+2)}{s}[n^2/16+\tilde{m}^2s/(s+2)]^{1/2}\right]$$
$$-\frac{\tilde{m}^2}{16\pi^2M^2}\ln[\tilde{m}^2s/(s+2)]+\frac{\tilde{m}^2}{8\pi^2M^2}\ln\{\tfrac{1}{4}+[\tfrac{1}{16}+\tilde{m}^2s/(s+2)]^{1/2}\}$$
$$-\frac{i(s+2)}{16\pi^2M^2s}\int_0^{\infty}\frac{dt}{\exp(2\pi t)-1}\{[(1+it)^2/16+\tilde{m}^2s/(s+2)]^{1/2}-[(1-it)^2/16+\tilde{m}^2s/(s+2)]^{1/2}\}$$
$$-\frac{1}{48\pi^2M^2s(s+2)^3}\ . \quad (9)$$

To compute $\langle\phi^2\rangle$, one solves the mode equations numerically with the appropriate boundary conditions at the event horizon, substitutes the results into Eq. (9), and uses appropriate cutoffs for the sums over l and n. The details of the numerical calculations will be presented elsewhere.

The results for $\tilde{m}=0$, 0.1, 0.2, 0.5, 1, 2, and 3 are shown in Figs. 1 and 2. From these figures it is seen that

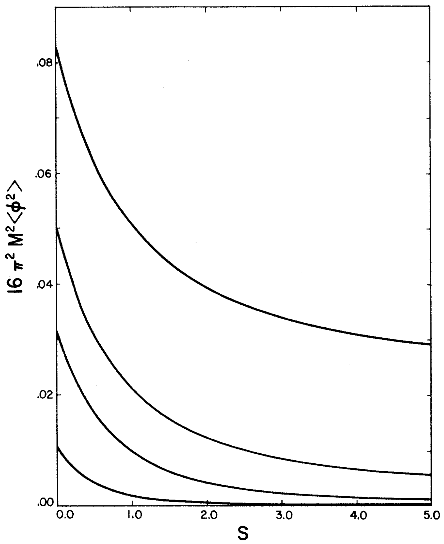

FIG. 1. From top to bottom the curves in this figure correspond to the cases $\tilde{m}=0$, 0.1, 0.2, and 0.5, respectively. Note that in each case $\langle \phi^2 \rangle$ has its maximum value on the event horizon $s=0$, and decreases monotonically to its flat-space value at $s=\infty$.

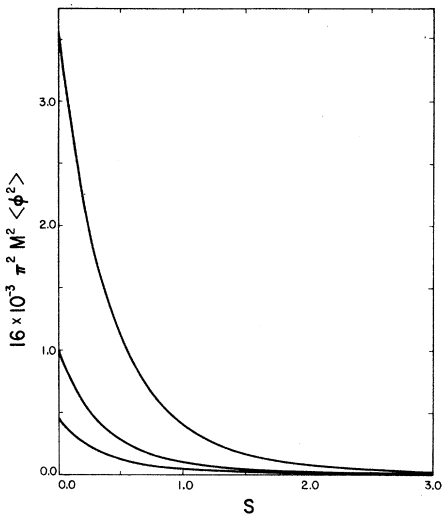

FIG. 2. The curves from top to bottom in this figure correspond to the cases $\tilde{m}=1$, 2, and 3. In each case $\langle \phi^2 \rangle$ has its maximum value on the event horizon $s=0$, and decreases monotonically to its flat-space value at $s=\infty$.

$\langle \phi^2 \rangle$ is smaller than in the massless case. However, its overall behavior is much the same in that it attains its maximum value on the event horizon and decreases monotonically to its flat-space value at $s=\infty$. The reason this asymptotic value of $\langle \phi^2 \rangle$ is nonzero is that the field is at the black-hole temperature $T=(8\pi M)^{-1}$. Table I gives, to three significant figures, the value of $\langle \phi^2 \rangle$ at $s=0$ and $s=\infty$ for $\tilde{m}=0$, 0.1, 0.2, 0.5, 1, 2, and 3.

The approximate expression for $\langle \phi^2 \rangle$ obtained by Frolov[8] comes from higher-order terms in the DeWitt-Schwinger expansion than are needed for renormalization. He kept terms to $O(\tilde{m}^{-2})$. To $O(\tilde{m}^{-4})$ one finds[8,13,16,17]

$$\langle \phi^2 \rangle \approx (16\pi^2 M^2)^{-1}\{\tfrac{4}{15}\tilde{m}^{-2}(s+2)^{-6} + \tilde{m}^{-4}[\tfrac{9}{7}(s+2)^{-8} - \tfrac{194}{63}(s+2)^{-9}] + O(\tilde{m}^{-6})\} \; . \tag{10}$$

For $\tilde{m} \gtrsim 2$ there is good agreement between (10) and the numerical results for $s \lesssim 5$.

Once $\langle \phi^2 \rangle$ is known, it is easy to compute $\langle T \rangle$ for a conformally coupled massive scalar field. Since $T=-m^2\phi^2$ for this field, the renormalized value of $\langle T \rangle$ is just the trace anomaly plus $\langle \phi^2 \rangle$ (Ref. 13): i.e.,

$$\langle T \rangle = \frac{1}{60\pi^2 M^4 (s+2)^6} - m^2 \langle \phi^2 \rangle \; . \tag{11}$$

From Eqs. (9) and (11) it can be seen that if $\langle T \rangle$ is written in terms of $\tilde{m}$ and s, then the black-hole mass only appears as an overall factor of M^{-4}.

The results of numerical computations for $\tilde{m}=0$, 0.1, 0.2, 0.5, 1, 2, and 3 are shown in Figs. 3 and 4. From the figures it is seen that the magnitude of $\langle T \rangle$ decreases with increasing $\tilde{m}$ and M. In the massless case, $\langle T \rangle$ decreases monotonically from its maximum value on the event horizon to zero at $s=\infty$. For the massive scalar field, $\langle T \rangle$ also attains its maximum value on the event horizon. It then decreases to a minimum negative value and asymptotically approaches its flat-space value, which is also negative, at $s=\infty$. The values of $\langle T \rangle$ at $s=0$ and $s=\infty$ for $\tilde{m}=0$, 0.1, 0.2, 0.5, 1, 2, and 3 are given in Table I to three significant figures.

An approximation for $\langle T \rangle$ which reproduces these features can be obtained by substituting Eq. (10) into Eq.

TABLE I. $\langle \phi^2 \rangle$ and $\langle T \rangle$ at $s=0$ and $s=\infty$.

$\tilde{m}$	$16\pi^2 M^2 \langle \phi^2 \rangle_{s=0}$	$16\pi^2 M^2 \langle \phi^2 \rangle_{s=\infty}$	$16\pi^2 M^4 \langle T \rangle_{s=0}$	$16\pi^2 M^4 \langle T \rangle_{s=\infty}$
0	8.33×10^{-2}	2.08×10^{-2}	4.17×10^{-3}	0.00
0.1	4.99×10^{-2}	2.38×10^{-3}	3.67×10^{-3}	-2.38×10^{-5}
0.2	3.17×10^{-2}	2.51×10^{-4}	2.90×10^{-3}	-1.00×10^{-5}
0.5	1.07×10^{-2}	2.02×10^{-7}	1.48×10^{-3}	-5.05×10^{-8}
1.0	3.51×10^{-3}	9.82×10^{-13}	6.54×10^{-4}	-9.82×10^{-13}
2.0	9.88×10^{-4}	1.68×10^{-23}	2.16×10^{-4}	-6.71×10^{-23}
3.0	4.51×10^{-4}	2.49×10^{-34}	1.03×10^{-4}	-2.24×10^{-33}

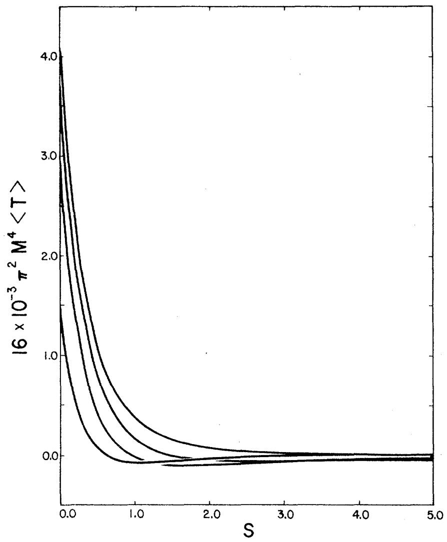

FIG. 3. The curves from top to to bottom on the event horizon $s=0$ in this figure correspond to the cases $\tilde{m}=0$, 0.1, 0.2, and 0.5. The magnitude of $\langle T\rangle$ decreases with increasing $\tilde{m}$. $\langle T\rangle$ approaches its flat-space value at $s=\infty$.

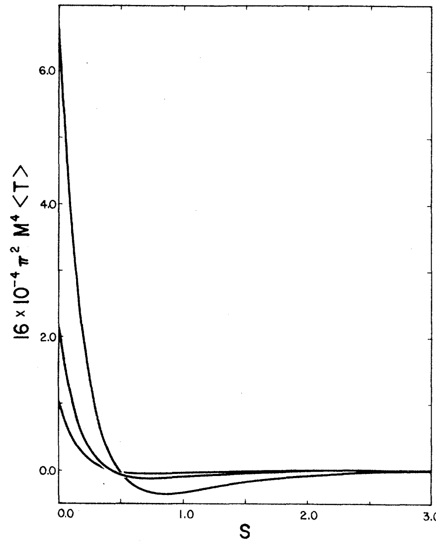

FIG. 4. The curves from top to bottom on the event horizon $s=0$ in this figure correspond to the cases $\tilde{m}=1$, 2, and 3. The magnitude of $\langle T\rangle$ decreases with increasing $\tilde{m}$. $\langle T\rangle$ approaches its flat-space value at $s=\infty$.

(11). The result is

$$\langle T\rangle \approx (16\pi^2 M^4 \tilde{m}^{\,2})^{-1}[-\tfrac{9}{7}(s+2)^{-8}+\tfrac{194}{63}(s+2)^{-9}] . \quad (12)$$

This is identical to the result that one finds by taking the trace of the approximate expressions for $\langle T_{\mu\nu}\rangle$ given by Frolov and Zel'nikov.[9,10] Equation (12) gives good quantitative agreement with the numerical results for $\tilde{m}\gtrsim 2$ and $s\lesssim 3$.

From the above results it is clear that vacuum-polarization effects in Schwarzschild spacetime due to massive scalar fields are always smaller in magnitude than those due to massless scalar fields. Further, as predicted by the calculations in Refs. 8 and 9, these vacuum-polarization effects are negligible compared to those due to massless scalar fields when $\tilde{m}\gtrsim 2$.

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[15]This was obtained by using the Plana sum formula on the quantity $\sum_{n=1}^{\infty}\cos(n\kappa\epsilon)[n^2/16+\tilde{m}^{\,2}s/(s+2)]^{1/2}$. An alternative formulation may be obtained using the Plana sum formula on the quantity $\sum_{n=0}^{\infty}\{\cos(n\kappa\epsilon)[n^2/16+\tilde{m}^{\,2}s/(s+2)]^{1/2}\}-\tilde{m}[s/(s+2)]^{1/2}$ with the result that the second, third, and fourth terms in Eq. (8) are replaced by the terms

$$-(32\pi^2M^2)^{-1}\tilde{m}[(s+2)/s]^{1/2}-(8\pi^2M^2s)^{-1}(s+2)\times\int_{4\tilde{m}}^{\infty}dt[t^2/16-\tilde{m}^{\,2}s/(s+2)]^{1/2}/[\exp(2\pi t)-1] .$$

The first formulation is easier to use near the event horizon and in general for $\tilde{m}^{\,2}s/(s+2)<1$, while the second is easier to use for $\tilde{m}^{\,2}s/(s+2)>1$.

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