The δ expansion for stochastic quantization

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Using a recently proposed perturbation expansion called the δ expansion, we show how to solve the Langevin equation associated with a $g\phi^4$ field theory. We illustrate the technique in zero- and one-dimensional space-time, and then generalize this approach to d dimensions.

I. INTRODUCTION

Recently, a new perturbative technique, called the δ expansion, was proposed to solve nonlinear problems in physics.[1] The technique involves replacing in a differential equation nonlinear terms such as ϕ^3 by $\phi^{1+2\delta}$ and expanding this term in powers of δ:

$$\phi^{1+2\delta}=\phi\sum_{n=0}^{\infty}\frac{\delta^n}{n!}(\ln\phi^2)^n . \tag{1.1}$$

We then obtain a solution to the differential equation as a perturbation series in powers of δ. The perturbation parameter δ is a measure of the nonlinearity of the theory. When $\delta=0$ the theory is linear and typically can be solved in closed form. As δ increases from zero, the effects of the nonlinearity turn on smoothly. Thus, one would expect and we have indeed found in our research that the δ-series representation of the solution has a finite radius of convergence. Furthermore, the δ expansion is nonperturbative in all physical parameters such as the coupling constant g.

As an example of a difficult nonlinear problem that we have successfully treated, consider the Blasius equation

$$y'''+yy''=0,\quad y(0)=y'(0)=0,\quad y'(\infty)=1 . \tag{1.2}$$

This problem cannot be solved analytically. However, we can introduce the perturbation parameter δ:

$$y'''+y^{\delta}y''=0 . \tag{1.3}$$

If we represent the solution to this equation as a series in powers of δ,

$$y=\sum_{n=0}^{\infty}y_n\delta^n , \tag{1.4}$$

we can easily calculate the coefficients y_n. Even a small number of terms in the δ series gives an accurate approximation[1] to the exact solution to (1.2).

Our success in solving classical nonlinear differential equations using the δ expansion suggests that one could apply these same methods to the Langevin equation, a nonlinear classical differential equation whose solution can be used to obtain the Green's functions of a quantum field theory. The Langevin equation for a quantum field theory is obtained by adding two terms to the classical equation of motion, a random source term η, and a derivative with respect to a fictitious time τ. For example, suppose we want to solve a $g\phi^4$ field theory in d-dimensional space-time. The Euclidean action for this theory is given by

$$S[\phi]=\int[\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}g\phi^4]d^dx . \tag{1.5}$$

We replace the classical field equation

$$\frac{\delta S}{\delta\phi}=(-\partial^2+m^2)\phi+g\phi^3=0 \tag{1.6}$$

by the Langevin equation

$$\frac{\partial\phi}{\partial\tau}+\frac{\delta S}{\delta\phi}=\eta , \tag{1.7a}$$

or for $S[\phi]$ given by (1.5) we obtain

$$\frac{\partial}{\partial\tau}\phi(x,\tau)+(-\partial^2+m^2)\phi(x,\tau)+g\phi^3(x,\tau)=\eta(x,\tau) . \tag{1.7b}$$

This diffusion equation must be solved subject to the initial condition

$$\phi(x,\tau_0)=0 , \tag{1.7c}$$

where τ_0 is the time at which the source term η first turns on. Thus, ϕ is regarded as quiescent before the source term begins to operate. The source term η represents white noise. This means that there is no correlation between the noise at two different points in (x,τ) space:

$$\langle\eta(x,\sigma)\eta(y,\tau)\rangle=2\delta(x-y)\delta(\sigma-\tau) . \tag{1.8}$$

We also assume that $\langle\eta(x,\sigma)\rangle=0$.

We can express the correlation function of a product of white-noise sources η in terms of the functional integral:

$$\langle \eta(x_1,\tau_1)\eta(x_2,\tau_2)\cdots\eta(x_n,\tau_n)\rangle = \frac{\int \mathcal{D}\eta \exp\left[-\frac{1}{4}\int d^dx \int_{\tau_0}^{\infty} d\tau\, \eta^2(x,\tau)\right]\eta(x_1,\tau_1)\eta(x_2,\tau_2)\cdots\eta(x_n,\tau_n)}{\int \mathcal{D}\eta \exp\left[-\frac{1}{4}\int d^dx \int_{\tau_0}^{\infty} d\tau\, \eta^2(x,\tau)\right]} . \quad (1.9)$$

Evaluating this functional integral, we find that if n is odd, the correlation function vanishes, and, if $n=2m$,

$$\langle \eta(x_1,\tau_1)\eta(x_2,\tau_2)\cdots\eta(x_{2m},\tau_{2m})\rangle$$
$$=2^m[\delta(x_1-x_2)\delta(\tau_1-\tau_2)\delta(x_3-x_4)\delta(\tau_3-\tau_4)\cdots\delta(x_{2m-1}-x_{2m})\delta(\tau_{2m-1}-\tau_{2m})+\text{permutations}] . \quad (1.10)$$

In all, there are $(2m-1)!!$ terms on the right-hand side of (1.10). To obtain the N-point Green's functions $G_N(x_1,\ldots,x_N)$ for the quantum field theory given by (1.5), which are conventionally expressed as a path integral,

$$G_N(x_1,\ldots,x_N)=\frac{\int \mathcal{D}\phi \exp(-S[\phi])\phi(x_1)\cdots\phi(x_N)}{\int \mathcal{D}\phi \exp(-S[\phi])} , \quad (1.11)$$

we first solve (1.7b) for $\phi(x,\tau)$. We regard the classical field ϕ as a functional of the noise source η. Second, we calculate the equal-τ stochastic average using (1.10) and compute the Green's functions from the prescription[2]

$$G_N(x_1,\ldots,x_N)=\lim_{\tau\to\infty}\langle \phi(x_1,\tau)\phi(x_2,\tau)\cdots\phi(x_N,\tau)\rangle . \quad (1.12)$$

For the special case of supersymmetric quantum mechanics, there is another, and even simpler, procedure for obtaining the Green's functions.[3] The rules are as follows. We consider the zero-dimensional Langevin equation

$$\dot{x}+W(x)=\eta(\tau) . \quad (1.13)$$

We compute the corresponding Green's function

$$G_2(\tau,\sigma)=\langle x(\tau)x(\sigma)\rangle \quad (1.14)$$

using (1.10). We then take the limit $\tau+\sigma\to\infty$, $|\tau-\sigma|=T$ fixed. The result agrees with the two-point function of supersymmetric quantum mechanics whose Euclidean Lagrangian is

$$L=\tfrac{1}{4}\dot{x}^2+\tfrac{1}{4}W^2(x)-\tfrac{1}{2}W'(x) . \quad (1.15)$$

Generalizing this procedure to calculate the $2n$-point Green's function is straightforward. The advantage of this technique is that it is only necessary to solve a Langevin equation in the single variable τ. This result is surprising because quantum mechanics is a quantum field theory in one-dimensional space-time, and, therefore, the Langevin equation corresponding to such a theory would ordinarily require two variables, the fictitious time τ and the real time t. Supersymmetric quantum mechanics is remarkable because it obviates the necessity of introducing the time t. Supersymmetry allows the one variable τ to play the role of the fictitious time as well as the real time.

We organize this paper as follows. In Sec. II we obtain the first two terms in the δ-series solution of the zero-dimensional Langevin equation. We show that the equal-τ correlation function at large τ corresponds to the two-point function of zero-dimensional field theory. In Sec. III we verify that the unequal-τ correlation function corresponds to the two-point function of supersymmetric quantum mechanics. In Sec. IV we show how to generalize our procedure to arbitrary dimensions.

II. ZERO-DIMENSIONAL LANGEVIN EQUATION

Consider the massless quantum field theory in zero-dimensional space-time defined by

$$S(\phi)=g\frac{\phi^4}{4} . \quad (2.1)$$

The vacuum-persistence amplitude for this field theory reduces to an ordinary Riemann integral:

$$Z[J]=\frac{\int_{-\infty}^{\infty} d\phi \exp(-\frac{1}{4}g\phi^4+J\phi)}{\int_{-\infty}^{\infty} d\phi \exp(-\frac{1}{4}g\phi^4)} . \quad (2.2)$$

To obtain the δ expansion we first replace 4 everywhere in (2.2) by $2+2\delta$, and instead, we study

$$Z_\delta[J]=\frac{\int_{-\infty}^{\infty} d\phi \exp[-g(\phi^2)^{1+\delta}/(2+2\delta)+J\phi]}{\int_{-\infty}^{\infty} d\phi \exp[-g(\phi^2)^{1+\delta}/(2+2\delta)]} . \quad (2.3)$$

The formula for the two-point function in this theory can be obtained in closed form by evaluating the integral exactly:

$$G_2=\frac{\int_{-\infty}^{\infty} d\phi \exp[-g(\phi^2)^{1+\delta}/(2+2\delta)]\phi^2}{\int_{-\infty}^{\infty} d\phi \exp[-g(\phi^2)^{1+\delta}/(2+2\delta)]}$$
$$=\left[\frac{2+2\delta}{g}\right]^{1/(1+\delta)}\frac{\Gamma(3/(2+2\delta))}{\Gamma(1/(2+2\delta))} . \quad (2.4)$$

It is straightforward to Taylor expand (2.4) as a series in powers of δ:

$$G_2=\frac{1}{g}\{1-\delta L+\delta^2[-1+L+\tfrac{1}{2}L^2+\psi'(\tfrac{3}{2})]+\cdots\} , \quad (2.5)$$

where

$$L=\psi(\tfrac{3}{2})+\ln\frac{2}{g} . \quad (2.6)$$

We will now show how to obtain this same expansion from the Langevin equation. The Langevin equation for the original ϕ^4 theory is

$$\frac{\partial\phi}{\partial\tau}+g\phi^3=\eta \ . \tag{2.7}$$

We replace (2.7) by

$$\frac{\partial\phi}{\partial\tau}+g\phi^{1+2\delta}=\eta \ . \tag{2.8}$$

Next, we assume that ϕ has the expansion

$$\phi=\phi_0+\delta\phi_1+\delta^2\phi_2+\cdots \ . \tag{2.9}$$

Inserting (2.9) into (2.8), we obtain a sequence of linear equations:

$$\frac{\partial\phi_0}{\partial\tau}+g\phi_0=\eta \ , \tag{2.10a}$$

$$\frac{\partial\phi_1}{\partial\tau}+g\phi_1=-g\phi_0\ln\phi_0^2 \ , \tag{2.10b}$$

$$\frac{\partial\phi_2}{\partial\tau}+g\phi_2=-2g\phi_1-g\phi_1\ln\phi_0^2-\frac{g}{2}\phi_0(\ln\phi_0^2)^2 \ , \tag{2.10c}$$

and so on. From (1.7c) the initial conditions are

$$\phi_0(\tau_0)=\phi_1(\tau_0)=\phi_2(\tau_0)=\cdots=0 \ . \tag{2.11}$$

The solutions to (2.10) and (2.11) are

$$\phi_0(\tau)=e^{-g\tau}\int_{\tau_0}^{\tau}dt\, e^{gt}\eta(t) \ , \tag{2.12a}$$

$$\phi_1(\tau)=-ge^{-g\tau}\int_{\tau_0}^{\tau}dt\, e^{gt}\phi_0(t)\ln\phi_0^2(t) \ , \tag{2.12b}$$

$$\phi_2(\tau)=-ge^{-g\tau}\int_{\tau_0}^{\tau}dt\, e^{gt}\{2\phi_1(t)+\phi_1(t)\ln\phi_0^2(t) + \tfrac{1}{2}\phi_0(t)[\ln\phi_0^2(t)]^2\} \ . \tag{2.12c}$$

Note that to each order n in δ, the equation in (2.10) for ϕ_n has the same homogeneous part and an inhomogeneous part depending on all previous orders. Therefore, we can, in principle, compute to any required order in perturbation theory.

A. Zeroth-order calculation

The next step in calculating the two-point Green's function is to evaluate the white-noise average of the product of two fields:

$$\begin{aligned}G_2(\sigma,\tau)=\langle\phi(\sigma)\phi(\tau)\rangle&=\langle[\phi_0(\sigma)+\delta\phi_1(\sigma)+\delta^2\phi_2(\sigma)+\cdots][\phi_0(\tau)+\delta\phi_1(\tau)+\delta^2\phi_2(\tau)+\cdots]\rangle\\&=\langle\phi_0(\sigma)\phi_0(\tau)\rangle+\delta[\langle\phi_0(\sigma)\phi_1(\tau)\rangle+\langle\phi_1(\sigma)\phi_0(\tau)\rangle]\\&\quad+\delta^2[\langle\phi_0(\sigma)\phi_2(\tau)\rangle+\langle\phi_2(\sigma)\phi_0(\tau)\rangle+\langle\phi_1(\sigma)\phi_1(\tau)\rangle]+\cdots \ .\end{aligned} \tag{2.13}$$

In this subsection we calculate the first term in this series:

$$\begin{aligned}\langle\phi_0(\sigma)\phi_0(\tau)\rangle&=e^{-g(\sigma+\tau)}\int_{\tau_0}^{\tau}dt\int_{\tau_0}^{\sigma}ds\, e^{g(t+s)}\langle\eta(t)\eta(s)\rangle\\&=\frac{e^{-gT}}{g}-\frac{1}{g}e^{g(2\tau_0-\tau-\sigma)} \ ,\end{aligned} \tag{2.14}$$

where $T=|\tau-\sigma|$. In the limit as τ and σ approach infinity, with the time difference T held fixed, the result in (2.14) approaches

$$\langle\phi_0(\sigma)\phi_0(\tau)\rangle\to\frac{e^{-gT}}{g} \ . \tag{2.15}$$

This is the form of a free Green's function in one-dimensional space-time, where g plays the role of the mass. Note that, even though the theory described by (2.1) has no bare mass term, to zeroth order in δ, a mass has been generated. We have already seen this effect in previous studies of the δ expansion.[4] It is surprising that this one-dimensional result arises from the zero-dimensional Langevin equation. We return to this point in Sec. III. Here, we simply set $T=0$ and obtain the result

$$\langle\phi_0^2\rangle=\frac{1}{g} \ , \tag{2.16}$$

which agrees with the first term in (2.5).

B. First-order calculation

To calculate the contribution to the two-point Green's function to first order in δ, we must evaluate the correlation functions $\langle\phi_0(\sigma)\phi_1(\tau)\rangle$ and $\langle\phi_1(\sigma)\phi_0(\tau)\rangle$. This calculation is nontrivial because the source η appears in the argument of a logarithm:

$$\langle\phi_0(\sigma)\phi_1(\tau)\rangle+\langle\phi_1(\sigma)\phi_0(\tau)\rangle=-ge^{-g(\sigma+\tau)}\left\langle\int_{\tau_0}^{\sigma}dt\, e^{gt}\eta(t)\int_{\tau_0}^{\tau}dr\int_{\tau_0}^{r}ds\, e^{gs}\eta(s)\ln\left[e^{-gr}\int_{\tau_0}^{r}du\, e^{gu}\eta(u)\right]^2\right\rangle+(\sigma\leftrightarrow\tau) \ . \tag{2.17}$$

We encountered the analogous problem in our treatment of field theory in Ref. 4, and our approach to this problem here is similar; to wit, we use the identity

$$\left.\frac{d}{d\alpha}x^{\alpha}\right|_{\alpha=0}=\ln x \tag{2.18}$$

to replace the logarithm in (2.17) by a power α. In the subsequent calculation we regard α as an arbitrary *integer*, which allows us to use the identity (1.10). Specifically we have

$$\langle\phi_0(\sigma)\phi_1(\tau)\rangle+\langle\phi_1(\sigma)\phi_0(\tau)\rangle$$
$$=-g\frac{d}{d\alpha}e^{-g(\sigma+\tau)}2^{\alpha+1}(2\alpha+1)!!\int_{\tau_0}^{\sigma}dt\,e^{gt}\int_{\tau_0}^{\tau}dr\,e^{-2\alpha gr}\prod_{n=1}^{2\alpha+1}\int_{\tau_0}^{r}dz_n e^{gz_n}\delta(t-z_1)\delta(z_2-z_3)$$
$$\times\cdots\delta(z_{2\alpha}-z_{2\alpha+1})\bigg|_{\alpha=0}+(\sigma\leftrightarrow\tau)\ , \quad (2.19)$$

where the factor $(2\alpha+1)!!$ occurs because all permutations of (1.10) contribute equally. Performing the trivial integrations over the delta functions by integrating on $z_1,z_3,\ldots,z_{2\alpha+1}$, we obtain

$$\langle\phi_0(\sigma)\phi_1(\tau)\rangle+\langle\phi_1(\sigma)\phi_0(\tau)\rangle$$
$$=-g\frac{d}{d\alpha}2^{\alpha+1}(2\alpha+1)!!e^{-g(\sigma+\tau)}\int_{\tau_0}^{\sigma}dt\,e^{2gt}\int_{\tau_0}^{\tau}dr\,e^{-2\alpha gr}\theta(r-t)\prod_{m=1}^{\alpha}\int_{\tau_0}^{r}dz_{2m}e^{2gz_{2m}}\bigg|_{\alpha=0}+(\sigma\leftrightarrow\tau)\ . \quad (2.20)$$

The integrals over z_{2m}, $m=1,2,\ldots,\alpha$, are elementary:

$$\langle\phi_0(\sigma)\phi_1(\tau)\rangle+\langle\phi_1(\sigma)\phi_0(\tau)\rangle=-g\frac{d}{d\alpha}\frac{2^{\alpha+1}\Gamma(\alpha+\frac{3}{2})}{g^{\alpha}\Gamma(\frac{3}{2})}e^{-g(\sigma+\tau)}\int_{\tau_0}^{\tau}dr(1-e^{2g(\tau_0-r)})^{\alpha}\int_{\tau_0}^{\sigma}dt\,e^{2gt}\theta(r-t)\bigg|_{\alpha=0}+(\sigma\leftrightarrow\tau)\ . \quad (2.21)$$

At this point, we set $\sigma=\tau$. (We return to the case where $\sigma\neq\tau$ in the next section.) Doing the t integration we obtain

$$\langle\phi_0(\sigma)\phi_1(\sigma)\rangle+\langle\phi_1(\sigma)\phi_0(\sigma)\rangle=-\frac{d}{d\alpha}\frac{2^{\alpha+1}\Gamma(\alpha+\frac{3}{2})}{g^{\alpha}\Gamma(\frac{3}{2})}e^{-2g\sigma}\int_{\tau_0}^{\sigma}dr(e^{2gr}-e^{2g\tau_0})(1-e^{2g(\tau_0-r)})^{\alpha}\bigg|_{\alpha=0}\ . \quad (2.22)$$

Now we take the limit $\sigma\to\infty$:

$$\langle\phi_0(\sigma)\phi_1(\sigma)\rangle+\langle\phi_1(\sigma)\phi_0(\sigma)\rangle$$
$$\to-\frac{1}{g\Gamma(\frac{3}{2})}\frac{d}{d\alpha}\left[\left[\frac{2}{g}\right]^{\alpha}\Gamma(\alpha+\tfrac{3}{2})\right]\bigg|_{\alpha=0}$$
$$=-\frac{1}{g}L\ . \quad (2.23)$$

This agrees precisely with the order-δ term in (2.5) and establishes the validity of our computational method.

It is clear now how to proceed to any order in δ. In nth order we introduce n exponential parameters, $\alpha_1,\ldots,\alpha_n$, and use the identity (2.18) to replace logarithms by powers. We then regard the parameters α_k as integers so that we can apply the identity in (1.10). Finally, using Γ functions to analytically continue the combinatorial factors, we differentiate with respect to the parameters, and evaluate the resulting expressions at $\alpha_k=0$.

III. SUPERSYMMETRIC QUANTUM MECHANICS

To obtain the two-point Green's function in supersymmetric quantum mechanics, it is sufficient to consider $G_2(\sigma,\tau)$ in (2.13) at $|\sigma-\tau|=T\neq0$ in the limit $\sigma,\tau\to\infty$. We reevaluate the integral in (2.21) in this slightly more general case. We find that

$$\langle\phi_0(\sigma)\phi_1(\tau)\rangle=\begin{cases}-(L/2g)e^{-gT}, & \sigma>\tau\ ,\\ -(L/2g)e^{-2T}(1+2gT), & \sigma<\tau\ .\end{cases} \quad (3.1)$$

Combining this result with (2.15) we find, to first order in δ, that

$$G_2(\sigma,\tau)=\frac{e^{-gT}}{g}[1-\delta L(1+gT)+\cdots]\ . \quad (3.2)$$

We can verify this result using the techniques described in Ref. 4. The supersymmetric quantum-mechanical theory corresponding to (1.13) is defined by the vacuum-persistence functional[3]

$$Z[J]=\frac{\int\mathcal{D}\phi\exp\left[-S[\phi]+\int J\phi\,dt\right]}{\int\mathcal{D}\phi\exp(-S[\phi])}\ , \quad (3.3)$$

where

$$S[\phi]=\int dt[\tfrac{1}{4}\dot\phi^2+\tfrac{1}{4}W(\phi)^2-\tfrac{1}{2}W'(\phi)]\ . \quad (3.4)$$

The Langevin equation in (2.8) corresponds to the choice

$$W(\phi)=g\phi^{1+2\delta}\ . \quad (3.5)$$

If we substitute (3.5) into (3.4) and keep terms of order δ, we find the approximate action

$$S[\phi]=\int dt\left[\tfrac{1}{4}\dot\phi^2+\tfrac{1}{4}g^2\phi^2+\frac{\delta}{4}g^2\phi^2\ln\phi^4-\frac{\delta}{2}g\ln\phi^2+O(\delta^2)+\text{const}\right]\ . \quad (3.6)$$

This is a nonpolynomial action. Following the procedure in Ref. 4, to this order we replace (3.6) by a *provisional*

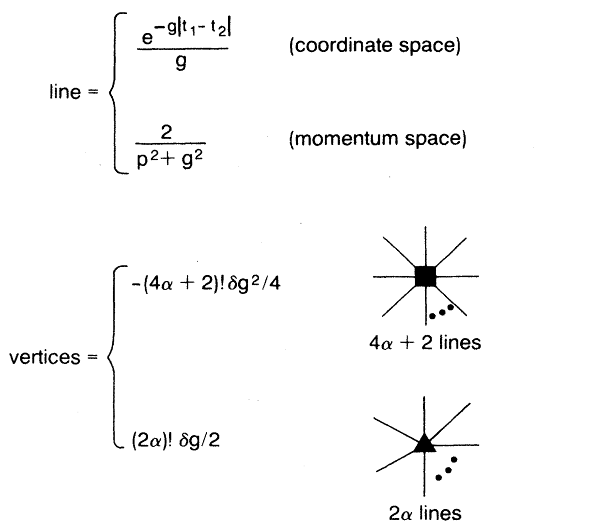

FIG. 1. The Feynman rules for the provisional action $\tilde{S}$ in (3.7).

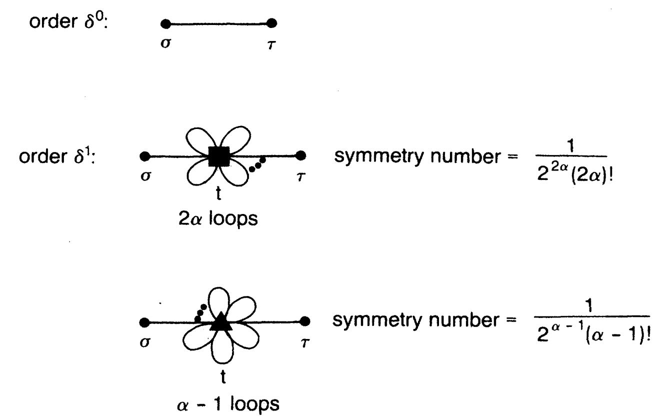

FIG. 2. The Feynman graphs in coordinate space that contribute to $G_2(\sigma,\tau)$ to order δ.

Euclidean action $\tilde{S}[\phi]$ having polynomial interaction terms:

$$\tilde{S}[\phi]=\int dt\left[\tfrac{1}{4}\dot{\phi}^2+\tfrac{1}{4}g^2\phi^2+\frac{\delta g^2}{4}\phi^{4\alpha+2}-\frac{\delta g}{2}\phi^{2\alpha}\right] . \qquad (3.7)$$

Note that we recover the theory described by S from the theory described by $\tilde{S}$ by taking one derivative with respect to the parameter α and setting $\alpha=0$.

Again, we treat α as an integer and read off the Feynman rules for $\tilde{S}[\phi]$. The free propagator is $e^{-g|t_1-t_2|}/g$ in coordinate space and $2(p^2+g^2)^{-1}$ in momentum space. There are two vertices, a $(4\alpha+2)$-point vertex, whose amplitude is $-(4\alpha+2)!\delta g^2/4$, and a 2α-point vertex, whose amplitude is $(2\alpha)!\delta g/2$. These rules are illustrated in Fig. 1.

The three graphs contributing to $G_2(\sigma,\tau)$ to order δ are shown in Fig. 2. To order δ^0 we have $e^{-g|\tau-\sigma|}/g$, which agrees with the first term in (3.2). To order δ^1 there are two terms corresponding to two graphs shown in Fig. 2:

$$\left[-\frac{\delta g^2}{4}(4\alpha+2)!\left[\frac{1}{g}\right]^{2\alpha}\frac{1}{2^{2\alpha}(2\alpha)!}+\frac{\delta g}{2}(2\alpha)!\left[\frac{1}{g}\right]^{\alpha-1}\frac{1}{2^{\alpha-1}(\alpha-1)!}\right]\int_{-\infty}^{\infty}dt\frac{e^{-g|\sigma-t|}}{g}\frac{e^{-g|t-\tau|}}{g} . \qquad (3.8)$$

In (3.8) we have included the symmetry numbers shown in Fig. 2 for each graph. We have also included the amplitude for the loops; each loop has the value $1/g$. Evaluating the integral in (3.8), taking the derivative with respect to α, and setting $\alpha=0$, we obtain

$$-\frac{e^{-gT}}{g}\delta L(1+gT) ,$$

which agrees exactly with the order δ contribution in (3.2). We have thus verified that the δ-expansion techniques when applied to the purely classical Langevin equation give, simply and directly, the correct field-theoretic Green's functions.

IV. THE LANGEVIN EQUATION IN d-DIMENSIONAL FIELD THEORY

In higher dimensions we must include the d'Alembertian in the Langevin equation (1.7). Again, we utilize the δ expansion by replacing (1.7b) by

$$\frac{\partial\phi}{\partial\tau}+(-\partial^2+m^2)\phi+g\phi^{1+2\delta}=\eta . \qquad (4.1)$$

As before, we expand in powers of δ and assume the form (2.9). Thus, in place of the system (2.10) we have

$$\frac{\partial\phi_0}{\partial\tau}+(-\partial^2+m^2+g)\phi_0=\eta , \qquad (4.2a)$$

$$\frac{\partial\phi_1}{\partial\tau}+(-\partial^2+m^2+g)\phi_1=-g\phi_0\ln\phi_0^2 , \qquad (4.2b)$$

$$\frac{\partial\phi_2}{\partial\tau}+(-\partial^2+m^2+g)\phi_2=-2g\phi_1-g\phi_1\ln\phi_0^2-\frac{g}{2}\phi_0(\ln\phi_0^2)^2 , \qquad (4.2c)$$

and so on. To solve the first of these equations, (4.2a), we Fourier transform in all variables except in the artificial time variable τ. The solution of the transformed equation is

$$\tilde{\phi}_0(k,\tau)=e^{-(k^2+m^2+g)\tau}\int_{\tau_0}^{\tau}ds\, e^{(k^2+m^2+g)s}\tilde{\eta}(k,s) . \qquad (4.3)$$

From (1.8) the transformed sources satisfy

$$\langle\tilde{\eta}(k,\sigma)\tilde{\eta}(p,\tau)\rangle=2(2\pi)^d\delta(k+p)\delta(\sigma-\tau) . \qquad (4.4)$$

The zeroth-order two-point function is then obtained as

$$\lim_{\sigma=\tau\to\infty} \langle \tilde{\phi}_0(k,\sigma)\tilde{\phi}_0(p,\tau)\rangle = \frac{1}{k^2+m^2+g}(2\pi)^d\delta(p+k) \ . \tag{4.5}$$

Again, we note a shift, to zeroth order in δ, of the square of the bare mass by g.

To proceed further we use the Langevin Green's function

$$\tilde{D}(k,\tau,s)=e^{-(k^2+m^2+g)(\tau-s)}\theta(\tau-s)$$

used in (4.3) to solve (4.2b) for ϕ_1 in terms of ϕ_0. As usual we introduce a parameter α and use the identity (2.18) in order to replace the logarithm in that equation by a power of ϕ_0. This allows us to compute the average over the noise using (1.10). We are left with integrals over the Langevin Green's function $\tilde{D}$. These integrals will diverge unless we introduce a regularization scheme. Thus, following Ref. 5, we modify the Langevin equation (1.7a) to read

$$\frac{\partial\phi}{\partial\tau}(x,\tau)+\frac{\delta S}{\delta\phi}(x,\tau)=\int d^dy\, R_{xy}(\partial^2)\eta(y,\tau) \ , \tag{4.6}$$

where, for example, the regulator R may be taken to be

$$R=\left[1-\frac{\partial^2}{\Lambda^2}\right]^{-j} , \tag{4.7}$$

where j is chosen large enough to make all integrals that occur finite. Introducing such a regulator modifies the zeroth-order two-point function (4.5):

$$G_2^{(0)}(k,p)=\frac{\tilde{R}^{\,2}(k^2)}{k^2+m^2+g}(2\pi)^4\delta(p+k) \ . \tag{4.8}$$

We will not pursue this calculation any further because it is not the purpose of this paper to conduct a regulated Langevin-equation calculation. Rather, our purpose here was to establish that the δ expansion, which we have applied successfully to a wide variety of classical differential equations, is equally effective in solving the Langevin equation corresponding to a quantum field theory.

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