Regularization of the path-integral measure for anomalies

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As was shown for the first time by Fujikawa, the anomaly is fundamentally a variation of the functional integral measure under transformation. Fujikawa's original prescription of 1979 for the variation of the integral measure looks to be at first sight an artifact. In this paper we will show that it is not and that it is fully equivalent to the authentic field-theoretical treatment for a two-point function. To do this we first examine various ways of solving the factor $A(x)$ in Fujikawa's expression for the functional integral measure. We define the anomaly as $A(x)-A_f(x)$, where $A_f(x)$ is the Fujikawa factor for the free field. We propose a regulator which leads to a finite result for any anomaly. We show that $A(x)$ can be defined in terms of the proper time through a splitting procedure. The original Fujikawa prescription for $A(x)$ is shown to be closely related to the proper-time description of the anomaly, initiated by Schwinger. Its equivalence to the authentic field-theoretical treatment is proven as a consequence of these investigations. The ζ-functional regularization for $A(x)$ is also examined. We examine the way to deduce the anomaly from the effective potential by adopting the ϕ^4 model as an example. Comparison of the path-integral prescription with this procedure enables one to clarify the nature of divergence appearing in the original Fujikawa form of $A(x)$. The renormalization-group equation for the effective potential is solved exactly to obtain the precise form of the β function in terms of which we reexpress the result obtained earlier for $A(x)$. Finally we discuss the physical significance of the renormalization-group equation for the case of broken symmetry.

I. INTRODUCTION

As is well known, the phenomenon of the conservation of certain currents, valid in classical mechanics but not valid anymore in quantum mechanics, nowadays is called an anomaly. The evidence for the chiral anomaly was found for the first time by Fukuda and Miyamoto[1] in 1949, and soon after was confirmed by Steinberger.[2] The effective Lagrangian for the anomaly was constructed by Schwinger[3] already in 1951. However it was only since Adler, Bell, and Jackiw's pioneering works[4] that the anomaly became recognized as one of the fundamentally important subjects in field theory. Since then, all sorts of anomalies have been investigated in an immense number of publications. Several ways of handling infinities associated with the anomaly have been proposed, apart from a simple subtraction method:[5] namely, dimensional regularization,[6] ζ-functional regularization,[7] Schwinger's proper-time approach combined with certain regularization procedures,[8] etc. The chiral and the energy-momentum-tensor anomalies are most frequently adopted as examples. In the present field-theoretical language, the anomaly is presented as an anomalous form of the generalized Ward-Takahashi identity. Then the anomaly can most conveniently be dealt with by the path-integral method.

In the path-integral form of quantum field theory, any field transformation should be reduced to the change of measure of functional integration. Recently, in 1979, Fujikawa,[9] noticing this fact, proposed a new and simple method to handle the functional integral measure. This method is particularly suited to the chiral anomaly. When applied to other cases, this method gives rise to other infinities in addition to the wanted anomalies. A few procedures have been proposed to eliminate such infinities.[10] Fujikawa himself has adopted the ζ-functional regularization for the case of the gravitational anomaly.[11]

As Fujikawa's method has great practical advantage due to its simplicity, it is worth investigating the nature of these infinities and examining what sort of regularization procedure has to be added further. Also we wish to know whether or not his method is applicable to the case of broken symmetry. In this paper we will analyze the consequences of Fujikawa's treatment when applied to the ϕ^4 theory. Before doing this, we first briefly review Fujikawa's prescription[12] for the chiral anomaly and present a few alternative ways of formulating the transformation of the path-integral measure. By doing so, we can clarify the physical meaning of Fujikawa's method.

Fujikawa's prescription is almost incredibly simple, and yet totally successful. In due course of our investigation we will understand the reason for this. In general when one calculates a physical observable, an arbitrary measure such as a cutoff factor, arbitrary mass, etc., appears. For example, the effective potential for ϕ^4 theory contains an arbitrary mass measure. The anomaly is free from such arbitrary measure. In fact, Fujikawa isolates and picks only the physical quantity which is independent of such an arbitrary measure. This is the reason why his method works so well.

II. CHIRAL ANOMALY

A. Fujikawa prescription

The Lagrangian density

$$\mathcal{L}=\bar{\psi}(i\not{D}-m)\psi+\frac{1}{2g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} \tag{2.1}$$

undergoes an infinitesimal variation

$$\delta L=-\partial_\mu\alpha(x)\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)-2mi\alpha(x)\bar{\psi}(x)\gamma_5\psi(x) \tag{2.2}$$

for an infinitesimal chiral transformation

$$\psi\to e^{i\alpha(x)\gamma_5}\psi(x)\ . \tag{2.3}$$

From now on we use the Euclidean coordinates in this subsection following Fujikawa.[9,12] The gauge field part of the functional integral measure is invariant, but the particle part

$$(d\mu)_p=\prod_n d\bar{b}_n\prod_m da_m \tag{2.4}$$

is not, where

$$\psi(x)=\sum a_n\varphi_n(x) \tag{2.5}$$

with

$$\not{D}\varphi_n(x)=\lambda_n\varphi_n(x)\ . \tag{2.6}$$

For the chiral transformation expressed as

$$\begin{aligned}\psi(x)\to\psi'(x)&\equiv e^{i\alpha(x)\gamma_5}\psi(x)\\ &\equiv\sum_n a'_n\varphi_n(x)\end{aligned} \tag{2.7}$$

with

$$\begin{aligned}a'_m&=\sum\int d^4x\,\varphi_m(x)^\dagger e^{i\alpha(x)\gamma_5}\varphi_n(x)a_n\\ &=C_{m,n}a_n\ ,\end{aligned} \tag{2.8}$$

the measure $\prod_n da_n$ transforms into

$$\prod_m da'_m=(\det C_{m,n})^{-1}\prod_n da_n\ . \tag{2.9}$$

The factor $(\det C_{m,n})^{-1}$ diverges as one can see from the expression

$$\begin{aligned}(\det C_{m,n})^{-1}&=\det\left[\delta_{m,n}+i\int\alpha(x)\varphi_m(x)^\dagger\gamma_5\varphi_n(x)dx\right]^{-1}\\ &=\exp\left[-i\int dx\,\alpha(x)A(x)\right]\ ,\end{aligned} \tag{2.10}$$

where

$$A(x)\equiv\sum_n\varphi_n(x)^\dagger\gamma_5\varphi_n(x')\ . \tag{2.11}$$

The quantities such as $\sum_n\varphi_n(x)^\dagger\Gamma_{\mu\nu}\ldots\varphi_n(x')$ are all singular at $x=x'$, where $\Gamma_{\mu\nu}\ldots$ is a product of the γ_μ. The way of handling such infinities has been under investigation for about 40 years already and various ideas have been proposed. For the case of $A(x)$ for the chiral anomaly, Fujikawa[9] has found a very simple method to deal with the infinity. By correcting each contribution from φ_n by a factor $f(\lambda_n^2/M^2)$ satisfying

$$f(\infty)=f'(\infty)=f''(\infty)=\cdots=0\ \text{ and }\ f(0)=1\ , \tag{2.12}$$

one obtains a finite result for $A(x)$ as follows:

$$\begin{aligned}A(x)&=\lim_{M\to\infty}\left[\sum_n\varphi_n(x)^\dagger\gamma_5 f(\lambda_n^2/M^2)\varphi_n(x)\right]\\ &=\lim_{M\to\infty}\left[\sum_n\varphi_n(x)^\dagger\gamma_5 f(\not{D}^2/M^2)\varphi_n(x)\right]\\ &=\lim_{M\to\infty}\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_5 e^{-ikx}f(\not{D}^2/M^2)e^{ikx}\\ &=\lim_{M\to\infty}\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_5 f\left[\frac{1}{M^2}\{[k_\mu+D_\mu(x)]^2+\tfrac{1}{4}\{\gamma_\mu,\gamma_\nu\}F^{\mu\nu}\}\right]\\ &=\lim_{M\to\infty}\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_5\left[f(k^2/M^2)+f'(k^2/M^2)\frac{B}{M^2}+\frac{1}{2!}f''(k^2/M^2)\frac{B^2}{M^4}\right]\\ &=\lim_{M\to\infty}\mathrm{Tr}\gamma_5(\{\gamma^\mu,\gamma^\nu\}F_{\mu\nu})^2\frac{1}{4M^4}\frac{1}{2!}\int\frac{d^4k}{(2\pi)^4}f''(k^2/M^2)\\ &=\lim_{M\to\infty}=\mathrm{Tr}({}^*F^{\mu\nu}F_{\mu\nu})\int\frac{d^4k}{(2\pi)^4}f''(k^2)\\ &=\frac{-1}{16\pi^2}\mathrm{Tr}{}^*FF\ ,\end{aligned} \tag{2.13}$$

where

$$B=[k_\mu+D_\mu(x)]^2+\tfrac{1}{4}\{\gamma_\mu,\gamma_\nu\}F^{\mu\nu}-k_\mu^2 \ . \quad (2.14)$$

There are several important choices for $f(\lambda_n^2/M^2)$.

(a) The form

$$f(\lambda_n^2/M^2)=e^{-(\lambda_n/M)^2} \quad (2.15)$$

has been used extensively by Fujikawa in order to demonstrate his method. This form has a very important relationship to Schwinger's proper-time approach,[3] as will be seen later.

Here the sign in front of $(\lambda_n/M)^2$ has no importance. One can as well adopt $e^{(\lambda_n/M)^2}$ for $f[(\lambda_n/M)^2]$ (see Sec. VI A).

(b) (Ref. 13)

$$f(\lambda_n^2/M^5)=\frac{1}{1+(\lambda_n/M)^2} \quad (2.16)$$

corresponds to the most common regularization[14]

$$\frac{1}{D\!\!\!/^2+m^2}-\frac{1}{D\!\!\!/^2+M^2}\doteq\frac{1}{(D\!\!\!/^2+M^2)(1+D\!\!\!/^2/M^2)} \quad (2.17)$$

satisfying only the condition I of Pauli and Villars.[15]

For the chiral anomaly, these two factors give rise to the same finite result as seen in (2.13). This is not the case for other anomalies. When $f(\lambda_n^2/M^2)$ is expanded around k^2/M^2, the first and second terms containing $f(k^2/M^2)$ and $f'(k^2/M^2)$ remain infinite. There exists however a particular form of f, which leads to a finite result for any anomaly, except for an infinite constant having no physical significance (we will return to this in Sec. II B. It is

(c)

$$f(\lambda_n^2/M^2)=\frac{1}{1-(\lambda_n/M)^4} \ , \quad (2.18)$$

which is a limiting case of the regulator given by Coleman and Jackiw[16] with $C_1^2=C_2^2=-\frac{1}{2}$ and $M_1^2=M_2^2\to\infty$: namely,

$$\frac{1}{\mu^2+p^2}+\frac{C_1^2}{M_1^2+p^2}+\frac{C_2^2}{M_2^2+p^2}\to\frac{1}{(\mu^2+p^2)(1-p^4/M^4)} \quad (2.19)$$

[for the notations, see (III.9) and (III.10) of Ref. 16].

With this f, the strong singularity, namely, the second term in the fifth expression of $A(x)$ in (2.13) [the term proportional to $f'(k^2/M^2)$] vanishes already before the trace is taken. This f satisfies Pauli and Villar's first condition I, and further satisfies second condition I_a at the limit of infinite M.

We will discuss the correct usage of this regulator (2.18) in Sec. VI A.

B. Proper-time description

The proper-time approach of Schwinger[3] is closely related to Fujikawa's method as will be seen in Sec. II C. In fact, as will be seen shortly, the square M^2 of Fujikawa's mass measure can be interpreted as the inverse of the Euclidean proper-time s, or equivalently as an arbitrary measure μ where $\mu s=\hat{s}$ and $\hat{s}$ is a dimensionless proper time.

1. Proper-time splitting method—linear expression

In Ref. 17, Dirac regarded a relativistic operator

$$P_5=(\gamma^\mu p_\mu-m) \quad (2.20)$$

as the Hamiltonian which is the conjugate variable to the proper time s, and introduced a covariant equation

$$P_5\psi=\frac{1}{i}\frac{d}{ds}\psi \ . \quad (2.21)$$

In terms of $\varphi_\mu(x)$ defined in (2.6), $\psi(x,s)$ is decomposed as

$$\psi=\sum_n\psi_n(x,s)=\sum_n e^{i\lambda_n s}a_n\varphi_n(x) \ , \quad (2.22)$$

and reduces to $\psi(x)$ in (2.5) at $s=0$. In this subsection we define the Fujikawa factor $A(x)$ as

$$A(x)=\lim_{s\to0}\sum_n\bar{\psi}_n(x,s)\gamma_5\psi_n(x,0)$$
$$=\lim_{s\to0}\sum_n\varphi_n^\dagger(x)e^{-i\lambda_n s}\gamma_5\varphi_n(x) \ . \quad (2.23)$$

In Euclidean proper time, this is just Fujikawa's $A(x)$ corrected by $e^{-\lambda_n s}$ instead of $e^{-\lambda_n^2/M^2}$ [see (2.15) of Ref. 12].

One can easily verify that the $A(x)$ in (2.23) gives rise to the correct value (2.13) for the anomaly. To see this clearly, we first replace λ_n by $\sqrt{D\!\!\!/^2}$, since only the fourth-order term in the series

$$e^{-\lambda_n s}=\sum\frac{1}{n!}(-\lambda_n s)^n$$

remains when the trace is taken. Then in our $A(x)$, the integral

$$\int d^4k\frac{1}{4!}f''''(k)dk \quad (2.24)$$

replaces the integral

$$\int d^4k\frac{1}{2!}f''(k^2)dk \quad (2.25)$$

in Fujikawa's $A(x)$ in (2.15) of Ref. 12, both of which have an identical value $\pi^2/4$.

2. Proper-time splitting method—bilinear expression

There is another expression for $\psi(x,s)$ which leads us directly to Fujikawa's $A(x)$ as a limit of vanishing s. Adopting the second-order Hamiltonian $\mathcal{H}=D\!\!\!/^2-m^2$, Schwinger described the particle motion in terms of x_μ in the Heisenberg picture, as will be referred to in the following subsection. In a similar manner, we start from the second-order wave equation

$$(D\!\!\!/^2-m^2)\psi=\frac{d}{ids}\psi \ , \quad (2.26)$$

and define $A(x)$ as

$$A(x)=\lim_{s\to 0} A(x,s)$$

$$=\lim_{s\to 0} \bar{\psi}(x,s)\gamma_5 \frac{\not{D}-m}{2m}\psi(x,0) \tag{2.27}$$

$$=\lim_{s\to 0} \bar{\psi}(x)e^{-i\mathcal{H}s}\gamma_5 \frac{\not{D}-m}{2m}\psi(x)$$

$$=\lim_{s\to 0} \sum_n \varphi_n^{-1}(x)e^{-i(\not{D}^2-m^2)s}\gamma_5\varphi_n(x)$$

$$=\lim_{s\to 0} \sum_n \varphi_n^{-1}(x)e^{-i\not{D}^2 s}\gamma_5\varphi_n(x)$$

$$=\lim_{s\to 0} \sum_n \varphi_n^{-1}(x)e^{-i\lambda_n^2 s}\gamma_5\varphi_m(x) , \tag{2.28}$$

which is exactly identical to Fujikawa's $A(x)$ in (2.15) of Ref. 12, if one replaces Euclidean time s by $1/M^2$. In deriving (2.28), the double degeneracy on λ_n for $\psi(x)$ is taken into account.

C. Proper-time integral form for the anomaly

1. Schwinger's form of coördinate representation

We mention that Schwinger's proper-time description in the coordinate representation[3] can be applied easily to the Fujikawa factor $A(x)$. For notation one should refer to Ref. 3.

In the Fourier integral form of the Green's function

$$\frac{1}{\not{D}-m}=\frac{\not{D}-m}{\not{D}^2+m^2}=-i\int ds(\not{D}-m)e^{-i(\not{D}^2+m^2)s} , \tag{2.29}$$

the s is simply an arbitrary parameter in most cases.[15] Schwinger considered it as the proper time and made full use of the physical content of this expression. Then the Green's functions as a matrix operator in the coordinate space is

$$\langle x|G|x'\rangle=-i\int_0^\infty ds\langle x|(\not{D}-m)e^{-i(\not{D}^2+m^2)s}|x'\rangle$$

$$=-i\int_0^\infty ds\, e^{-im^2 s}\langle x|(\not{D}-m)e^{-iD^2 s}|x'\rangle$$

$$=-i\int_0^\infty ds\, e^{-im^2 s}\langle x(s)|(\not{D}-m)|x'\rangle . \tag{2.30}$$

Here $x(s)$ is the coordinate variable in the Heisenberg picture and obeys

$$\frac{d}{ids}x=[-i\not{D}^2,x] . \tag{2.31}$$

The difference between $A(x)$ and G is principally γ_5. Then $A(x)$ can be represented in coordinate space as

$$A(x)=\lim_{x'\to x} -i\int_0^\infty ds\, \mathrm{Tr}\langle x|\gamma_5(\not{D}-m)e^{-i(\not{D}^2+m^2)s}|x'\rangle$$

$$=\lim_{x'\to x}\int_0^\infty ds\, e^{-im^2 s}m\, \mathrm{Tr}\langle x|e^{-\not{D}^2 s}\gamma_5|x'\rangle$$

$$=\lim_{x'\to x}\int_0^\infty ds\, me^{-im^2 s}\mathrm{Tr}\langle x(s)|\gamma_5|x'(0)\rangle$$

$$=\lim_{x'\to x}\int_0^\infty ds\, me^{-im^2 s}\mathrm{Tr}\gamma_5\langle x(s)|x'(0)\rangle \tag{2.32}$$

[note that the first part of Eq. (2.4) of Ref. 3 is valid not only for the constant field $F_{\mu\nu}=0$ but also for any $F_{\mu\nu}$. Also the term proportional to D disappears in our case when the trace is taken]. Now $A(x)$ is reduced to the transformation function $\langle x(s)|x'(0)\rangle$ which is calculated by Schwinger for the case of the constant field. The result for $A(x)$ is still infinite. A finite result can be obtained either by replacing $e^{im^2 s}$ by a regular,[3] or by taking into account an ordinary renormalization procedure for mass and charge, etc.[3] Finally we obtain

$$A(x)=\frac{1}{4\pi^2}E\cdot H , \tag{2.33}$$

which is identical to Fujikawa's result.

Schwinger's proper-time description in this subsection is in fact identical to the proper-time splitting method given in Sec. II B 1 and consequently is equivalent to Fujikawa's method mentioned in Sec. II A. To see this, the coordinate representation (2.32) is not convenient, because the propagation of the particle during the interval of time s is described in terms of the coordinate variable $x(s)$ in the Heisenberg representation, instead of $\psi(x)$ itself. We will return to the momentum representation in the next subsection.

2. Momentum-space description

The following pseudo-two-point function, denoted as $-iG_5(x,x')$,

$$-iG_5(x,x')=\sum_n \varphi_n^\dagger(x')\gamma_5\varphi_n(x) , \tag{2.34}$$

satisfies

$$(\not{D}-m)\gamma_5(-i)G_5(x,x')=-i\delta^4(x-x') . \tag{2.35}$$

The $G_5(x,x')$ can be expressed as

$$G_5(x,x')=\gamma_5\frac{\not{D}+m}{\not{D}^2-m^2}\delta^4(x-x')$$

$$=i\int ds\,\gamma_5(\not{D}+m)e^{-i(\not{D}^2-m^2)s}\delta^4(x-x')$$

$$=i\int ds\, d^4k\frac{1}{(2\pi)^4}e^{-ikx'}\gamma_5(\not{D}+m)e^{-i(\not{D}^2-m^2)s}e^{ikx}$$

$$=i\int ds\, d^4k\frac{1}{(2\pi)^4}e^{-im^2 s}e^{-ikx's}\gamma_5(\not{D}+m)e^{-i\not{D}^2 s}e^{ikx} . \tag{2.36}$$

The trace of G_5 is

$$\mathrm{Tr}G_5(x,x')=\frac{i}{(2\pi)^4}\int ds\, d^4k\, m e^{im^2s}\mathrm{Tr}e^{-ikx'}\gamma_5 e^{-i\not{D}^2 s}e^{ikx}. \tag{2.37}$$

In the above we have followed the most commonly used approach for the Green's function.[15] For our purpose one can adopt the linear expression

$$\begin{aligned} G_5(x,x')&=\gamma_5\frac{1}{\not{D}-m}\delta^4(x-x')\\ &=i\int ds\,\gamma_5 e^{i(\not{D}-m)s}\delta^4(x-x')\\ &=i\int ds\,\gamma_5 e^{ims}e^{i\not{D}s}\delta^4(x-x')\\ &=\frac{1}{(2\pi)^4}\int ds\, d^4k\, e^{-ims}e^{-ikx'}e^{i\not{D}s}e^{ikx}\,. \end{aligned} \tag{2.38}$$

The Fujikawa factor $A(x)$ is a kind of pseudo Green's function in five-dimensional space of x and s at $x=x'$ and $s=s'$. The pseudo Green's function, denoted as $G_5(x',x,s',s)$,

$$G_5(x',x,s',s)=\sum_n \bar{\psi}_n(x',s')\gamma_5\psi(x,s) \tag{2.39}$$

obeys

$$\left[D-\frac{\partial}{i\partial s}\right]\gamma_5 G_5(x',x,s',s)=\delta^4(x-x')\delta(s-s') \tag{2.40}$$

and can be expressed as

$$G_5(x',x,s',s)=i\int d\alpha\,\gamma_5\exp\left[i\left[\not{D}-\frac{\partial}{i\partial s}\right]\alpha\right]\delta^4(x-x')\delta(s-s')\,. \tag{2.41}$$

Expressing $\delta(s-s')$ as

$$\delta(s-s')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\kappa\, e^{i\kappa(s-s')} \tag{2.42}$$

one can rewrite $G_5(x',x,s',s)$ as

$$\begin{aligned} G_5(x',x,s',s)&=\frac{i}{2\pi}\int d\kappa\, d\alpha\,\gamma_5 e^{i(\not{D}-\kappa)\alpha}\delta^4(x-x')e^{i\kappa(s-s')}\\ &=\frac{i}{2\pi}\int d\kappa\, d\alpha\, e^{-i\kappa(\alpha-s+s')}\gamma_5 e^{i\not{D}\alpha}\delta^4(x-x')\\ &=i\int d\alpha\,\delta(\alpha-s+s')\gamma_5 e^{i\not{D}\alpha}\delta^4(x-x')\\ &=i\gamma_5 e^{i\not{D}(s-s')}\delta^4(x-x')\\ &=i\frac{1}{(2\pi)^4}\int d^4k\,\gamma_5 e^{-ikx'}e^{-i\not{D}(s-s')}d^{ikx}\,. \end{aligned} \tag{2.43}$$

Then the Fujikawa factor is

$$\begin{aligned} A(x)&=\lim_{\substack{x'\to x\\ s\to 0}}\mathrm{Tr}G_5(x,x',s,0)\\ &=\lim_{\substack{x'\to x\\ s\to 0}}\frac{1}{(2\pi)^4}\int d^4k\,\mathrm{Tr}e^{-ikx}e^{-i\not{D}s}e^{ikx'}\,, \end{aligned} \tag{2.44}$$

which is exactly identical to the proper-time splitting prescription (2.23), and consequently is equivalent to Fujikawa's (2.15) in Ref. 12.

As we have seen, Fujikawa's prescription is not simply an artifact, but is a totally orthodox field-theorical treatment, presented in a concise manner. Its identity with Schwinger's proper-time approach suffices to understand the nature of the divergences appearing in Fujikawa's result for anomalies other than chiral one.

The third line in the expression (2.43) for $G_5(x',x,s',s)$ is very similar to the third line of $G_5(x,x')$ in (2.38). The former can be obtained from the latter by replacing s by α, and then further replacing $e^{-im\alpha}$ by $\int dm\, e^{-im(\alpha-s+s')}$. This is because the set of $\psi_n(x,s)$ can be thought of as the spectrum of the $\psi_n(x)$ with continuous mass from zero to infinity, since $(\not{D}-\lambda_n)\psi_n=0$. The phase shift $\exp[i\kappa(s-s')]$ is due to a further relation $(\partial/i\partial_s)\psi_n=\lambda_n\psi_n$.

D. Generalized ζ-function approach

Since the ζ-functional regularization, invented by Salam and Strathdee,[18] was shown to be very useful for the effective Lagrangian by Hawking,[19] it has been applied on various occasions to eliminate infinities. It has been used also for chiral anomalies.[20] Fujikawa himself has adopted this approach of the gravitational anomaly.[11] The way of applying this method is not unique.[21] In this subsection we choose the simplest procedure for achieving our purpose.

If one adopts the imaginary proper time $\tau=is$, instead of s, the function

$$\begin{aligned} F(x,x',\tau)&=\sum_n \bar{\psi}_n(x,\tau)\psi_n(x',0)\\ &=\sum e^{-\lambda_n\tau}\varphi_n(x)\varphi_n(x') \end{aligned} \tag{2.45}$$

is called a heat function, which represents how a unit quantity of heat, in our case a particle, initially placed at the point x', diffuses with time τ (Ref. 19). The function

$$\zeta(x,s)=\frac{1}{\Gamma(s)}\int_0^{\infty}d\tau\,\tau^{s-1}F(x,x',\tau)d\tau$$

is called a generalized ζ function. It is a means of regularizing the value of $\int F(x,x',\tau)$ at $x=x'$. Analogously, we split the Fujikawa factor $A(x)$ as $A(x,x',\tau)$:

$$\begin{aligned} A(x,x',\tau)&=\sum_n \bar{\psi}(x,\tau)\gamma_5\psi_n(x',0)\\ &=\sum_n e^{-\lambda_n\tau}\varphi_n^{\dagger}(x)\gamma_5\varphi(x')\\ &=\gamma_5^{\sigma\rho}\sum_n e^{-\lambda_n\tau}\varphi_{n\sigma}(x)\varphi_{n\rho}(x')\\ &=\gamma_5^{\sigma\rho}B_{\sigma\rho}(x,x',\tau)\,. \end{aligned} \tag{2.46}$$

Then $B(x,x',\tau)$ satisfies

$$\not{D}^2B(x,x',\tau)+\frac{\partial}{\partial\tau}B=0 \tag{2.47}$$

or equivalently

$$(\partial^\mu\partial_\mu+\tfrac{1}{2}\sigma_{\mu\nu}F^{\mu\nu})B(x,x',\tau)+\frac{\partial}{\partial\tau}B(x,x',\tau)=0 \ . \tag{2.48}$$

For the external constant field $F_{\mu\nu}$, the solution of this equation can be written as

$$B(x,x',\tau)=B^0(x,x',\tau)e^{-\sigma_{\mu\nu}F^{\mu\nu}\tau/2} \ ,$$

where

$$B^0(x,x',\tau)=\frac{1}{16\pi^2\tau^2}e^{(x-x')^2/4\tau} \tag{2.49}$$

satisfies the free equation

$$\partial^\mu\partial_\mu B^0(x,x',\tau)+\frac{\partial}{\partial\tau}B^0(x,x',\tau)=0 \ . \tag{2.50}$$

Introducing a generalized pseudofunction, denoted as ζ_5,

$$\zeta_5(x,x',s)=\frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}A(x,x',\tau) \ , \tag{2.51}$$

we obtain $A(x)$ as the regularized value of $A(x,x',\tau)$ at $x=x'$: namely,[22]

$$\begin{aligned}\zeta_5(x,x',s)&=\frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}\frac{1}{16\pi^2\tau^2}\\&\quad\times\gamma_5 e^{-\sigma_{\mu\nu}F^{\mu\nu}/2}\\&=\frac{1}{16\pi^2}\gamma_5(\tfrac{1}{2}\sigma_{\mu\nu}F^{\mu\nu})^{2-s}\frac{\Gamma(s-2)}{\Gamma(s)} \ ,\end{aligned} \tag{2.52}$$

the trace of which at $s=0$, is identical to Fujikawa's result (2.15) of Ref. 12.

The merit of this method lies in the fact that the result is already finite before the trace is taken. However this method is still an artifact.

III. SCALING ANOMALY IN THE ϕ^4 MODEL

For the chiral transformation, the Fujikawa factor $A(x)$ is finite and gives rise to the anomaly alone, irrespective of the choice of $f(\lambda_n^2/M^2)$. This is not the case for the other anomalies. In this section we will examine the nature of the divergences appearing in $A(x)$.

Among various choices for $f(\lambda_n^2/M^2)$, Fujikawa's original choice $e^{-\lambda_n^2/M^2}$ has an important physical meaning. It represents how the particle proceeds, or more precisely, how the particle phase develops within the interval $\Delta^s=1/M^2$ of the Euclidean proper time. Fujikawa's treatment of $A(x)$ with this $e^{-\lambda_n^2/M^2}$ is equivalent to the orthodox field-theoretical treatment for the two-point function. Then we expect that the divergences in this $A(x)$ can be interpreted in terms of conventional renormalization theory. To demonstrate this, we adopt the ϕ^4 theory as an example. In the first part of this section, we will closely examine one by one the infinite terms appearing in this $A(x)$.

The anomaly for the conformal transformation has a long history. Very often it has been dealt with in connection with the energy-momentum-tensor trace anomaly.[23] Recently Buchmüller and Dragon[24] have derived such anomaly in the ϕ^4 theory from the effective potential as $\mu(\partial/\partial\mu)V_{\text{eff}}$, where μ is an arbitrary mass measure which always accompanies the dilation invariant theory. In the latter part of this section, we will compare Fujikawa's procedure with the effective potential approach.

A. Path-integral measure with $e^{-(\lambda/M)^2}$

For the real field ϕ with the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{\lambda}{4!}\phi^4 \ , \tag{3.1}$$

the quantum fluctuation $\hat{\phi}=\phi-\langle\phi\rangle=\phi-\phi_c$ around the saddle point ϕ_c obeys

$$\left[\partial^\mu\partial_\mu+\frac{\lambda}{2}\phi_c^2\right]\hat{\phi}=0 \ . \tag{3.2}$$

In terms of a complete set $\hat{\phi}_n$ satisfying

$$\left[\partial^\mu\partial_\mu+\frac{\lambda}{2}\phi_c^2\right]\hat{\phi}_n=\lambda_n\hat{\phi}_n \ , \tag{3.3}$$

the Fujikawa factor $A(x)$ corrected by

$$f(\lambda_n^2/M^2)=e^{-\lambda_n^2/M^2}$$

is

$$\begin{aligned}A(x)&=\sum_n\hat{\phi}_n(x)(1+x^\mu\partial_\mu)e^{-(\lambda_\mu/M)^2}\hat{\phi}_n(x)\\&=\lim_{M\to\infty}\frac{1}{(2\pi)^4}\int e^{-ikx}(1+x^\mu\partial_\mu)\exp\left[-\frac{\partial^2+\frac{\lambda}{2}\phi_c^2}{M^2}\right]e^{ikx}d^4k\\&=\lim_{M\to\infty}\frac{1}{(2\pi)^4}\int d^4k\,e^{-ikx}(1+ix^\mu k_\mu)\exp\left[\frac{-\frac{\lambda}{2}\phi_c^2}{M^2}\right]\exp\left[\frac{-k^2}{M^2}\right]e^{ikx}\\&=\lim_{M\to\infty}\frac{1}{(2\pi)^4}\int d^4k\,\exp\left[\frac{-\frac{\lambda}{2}\phi_c^2}{M^2}\right]\exp\left[\frac{-k^2}{M^2}\right]\end{aligned} \tag{3.4}$$

in the Euclidean space-time. Unlike the case of the chiral anomaly, the above $A(x)$ is not finite and includes two further infinite terms originating from f and f', as will be seen below:

$$A(z)=\lim_{M\to\infty}\frac{1}{(2\pi)^4}\left[\int\exp\left[\frac{-k^2}{M^2}\right]d^4k+\frac{\lambda}{2}\frac{\phi_c^2}{M^2}\int\exp\left[\frac{-k^2}{M^2}\right]d^4k+\frac{1}{2}\left[\frac{\lambda}{2}\frac{\phi_c^2}{M^2}\right]^2\int\exp\left[\frac{-k^2}{M^2}\right]\right]d^4k$$

$$=\lim_{M\to\infty}\frac{1}{16\pi^2}M^4+\lim_{M\to\infty}\frac{1}{16\pi^2}\frac{\lambda}{2}M^2\phi_c^2+\frac{1}{16\pi^2}\frac{1}{2}\left[\frac{\lambda}{2}\phi_c^2\right]^2 . \tag{3.5}$$

The anomaly is independent of any arbitrary measure and the third term must be the anomaly. To see this clearly, we derive the anomaly from the effective potential (3.10) of Coleman and Weinberg.[25] The result is

$$-M\frac{\partial}{\partial M}V_{\rm eff}=\frac{1}{32\pi^2}\left[\frac{\lambda}{2}\phi_c^2\right]^2 , \tag{3.6}$$

which coincides with the third term in the above $A(x)$.

To investigate the nature of the infinites in $A(x)$, we derive the anomaly from the bare effective Lagrangian $V_{\rm eff}^{\rm NR}$

$$V_{\rm eff}^{\rm NR}=\frac{\lambda}{4!}\phi_c^4+\frac{\lambda\Lambda^2}{64\pi^2}\phi_c^2+\frac{\lambda^2\phi_c^4}{256\pi^2}\left[\ln\frac{\lambda\phi_c^2}{2\Lambda^2}-\frac{1}{2}\right] . \tag{3.7}$$

Then we obtain

$$-\Lambda\frac{\partial}{\partial\Lambda}V_{\rm eff}^{\rm NR}=\frac{\lambda\Lambda^2}{32\pi^2}\phi_c^2+\frac{\lambda^2\phi_c^4}{128\pi^2} . \tag{3.8}$$

The second term is the anomaly. The first term is identical to the second term in $A(x)$, if one replaces M in $A(x)$ by Λ. Now we have confirmed that Fujikawa's measure M is in fact the cutoff factor Λ in conventional field theory. (One should not confuse Fujikawa's M with Coleman and Weinberg's M.)

This implies that the second infinite term in $A(x)$ would not have appeared if we had started from the renormalized Lagrangian. The original Lagrangian is conformal invariant. However, the counterterms $-\frac{1}{2}B\phi^2$ in (3.1) of Ref. 25, is not, and gives rise to an additional variation

$$-\delta(\tfrac{1}{2}B\phi^2)=-B\phi^2 , \tag{3.9}$$

the vacuum value of which just cancels the second infinity in $A(x)$.

Coleman and Weinberg's renormalized Lagrangian has one more counterterm. We will return to this in Sec. VI B.

The rest of infinities $A(x)$, namely, the first term in (3.5), is equal to the Fujikawa factor for the free field ϕ_f, denoted as $A_f(x)$,

$$A_f(x)=\lim_{M\to\infty}\hat{\phi}^\dagger_{f,n}(x)(1+x\delta)e^{-\partial^2/M^2}\hat{\phi}_{f,n}(x)$$

$$=\lim_{M\to\infty}\frac{1}{4\pi^2}M^4 . \tag{3.10}$$

What should be observed is the difference between $A(x)$ and $A_f(x)$,

$$A(x)-A_f(x) , \tag{3.11}$$

which finally is reduced to a pure anomaly

$$[A(x)-A_f(x)]_{\rm renormalized}=\frac{1}{16\pi^2}\frac{1}{2}\left[\frac{\lambda}{2}\right]^2\phi_c^4 , \tag{3.12}$$

after the renormalization of the quandractic divergence by the counterterm (3.9).

Such was the case for the massless theory. For the massive ϕ^4 theory with

$$\mathcal{L}=\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi-\tfrac{1}{2}m^2\phi^2\frac{\lambda}{4!}\phi^4 , \tag{3.13}$$

the anomaly is obtained as

$$[A(x)-A_f(x)]_{\rm renormalized}$$

$$=\frac{1}{16\pi^2}\left[\frac{1}{2}\left[m^2+\frac{\lambda}{2}\phi_c^2\right]^2-\frac{1}{2}m^2\right]$$

$$=\frac{1}{16\pi^2}\left[\frac{\lambda}{2}m^2\phi_c^2+\frac{1}{2}\left[\frac{\lambda}{2}\right]^2\phi_c^4\right] , \tag{3.14}$$

which one may compare with Buchmüller and Dragon's result for the complex field.[24]

B. Path-integral measure with the Pauli-Villar's regulator

With the use of the Pauli-Villar's regulator (2.18), Fujikawa's procedure leads us immediately to the finite result, namely, the anomaly:

$$A(x)=\lim_{M\to\infty}\sum_n\hat{\phi}_n(x)(1+x\partial)f(\lambda_n^2/M^2)\hat{\phi}_n(x)$$

$$=\lim_{M\to\infty}\sum_n\hat{\phi}_n(x)(1+x\partial)f\left[\frac{\partial^\mu\partial_\mu+\frac{\lambda}{2}\phi_c^2}{M^2}\right]\hat{\phi}_n(x)$$

$$=\lim_{M\to\infty}\frac{1}{(2\pi)^4}\int d^4k\left[f\left[\frac{k^2}{M^2}\right]+f'\left[\frac{k^2}{M^2}\right]\frac{\frac{\lambda}{2}\phi_c^2}{M^2}+\frac{1}{2!}f''\left[\frac{k^2}{M^2}\right]\frac{\left[\frac{\lambda}{2}\phi_c^2\right]^2}{M^2}\right] . \tag{3.15}$$

The second integral vanishes for our regulator and consequently

$$
\begin{aligned}
A(x)-A_f(x) &= \lim_{M\to\infty}\frac{1}{(2\pi)^4}\int d^4k\frac{1}{2!}f''\left[\frac{k^2}{M^2}\right]\frac{\left[\frac{\lambda}{2}\phi_c^2\right]^2}{M^4}\\
&= -\frac{1}{16\pi^2}\frac{1}{2}[f(k)]_0^\infty\left[\frac{\lambda}{2}\phi_c^2\right]^2\\
&= \frac{1}{32\pi^2}\left[\frac{\lambda}{2}\phi_c^2\right]^2 , \qquad (3.16)
\end{aligned}
$$

which is the anomaly. From a practical point of view, our $f(\lambda_\mu^2/M^2)$ in (2.18) is the best. The correct use of this regulator, justifiable from its physical significance, will be discussed in Sec. VI A.

C. Fujikawa factor for complex scalar field

To compare with Buchmüller and Dragon's result,[24] and also to illustrate once again the utility of the $f(\lambda_\mu^2/M^2)$ in (2.18), we rapidly demonstrate how to calculate $A(x)$ for the complex scalar field ϕ with

$$\mathcal{L}=-\partial^\mu\phi^\dagger\partial_\mu\phi+m^2\phi^\dagger\phi+\frac{\lambda}{2}(\phi^\dagger\phi)^2 .$$

With the use of

$$f(\lambda_\mu^2/M^2) \text{ in } (2.18) , \qquad (3.17)$$

$$
\begin{aligned}
A(x)-A_f(x) &= \sum_n \phi_n^\dagger(x)(1+x\partial)f(\lambda_\mu^2/M^2)\phi_n(x)-\sum\phi_n^\dagger(x)(1+x\partial)f(0)\phi_n(x)\\
&= \frac{1}{(2\pi)^4}\int d^4k\frac{1}{2!}f''\left[\frac{k^2}{M^2}\right]\left[\left[\frac{-m^2-\lambda\langle\phi^\dagger\phi\rangle}{M^2}\right]^2-\left[\frac{-m^2}{M^2}\right]^2\right]\\
&= \frac{1}{32\pi^2}[2m^2\lambda\langle\phi^\dagger\phi\rangle+(\lambda\langle\phi^\dagger\phi\rangle)^2] , \qquad (3.18)
\end{aligned}
$$

which can be compared with the result

$$A(x)-A_f(x)=\frac{1}{32\pi^2}[m^4+2m^2\lambda\langle\phi^\dagger\phi\rangle+(\lambda\langle\phi^\dagger\phi\rangle)^2] , \qquad (3.19)$$

which Buchmüller and Dragon[24] would have obtained if they had adopted the U(1) model instead of the SU(2) $\times$U(1) model.

The term $(1/32\pi^2)m^4$ in (3.19) is related to the arbitrariness in the choice of zero-point energy.

IV. RENORMALIZATION-GROUP APPROACH

Among the numerous works on the anomalies by conventional methods, some authors, Collins[23] in particular, have dealt with the trace anomaly by taking into account the renormalization-group equation, based on the dimensional regularization. Then the anomaly is expressed in terms of the β function multiplied by ϕ_c^4. In this paper we adopt Weinberg's approach[26] for the renormalization group and deduce the β function. Finally our anomaly will also be expressed in terms of this β function.

With the Coleman-Weinberg potential[25]

$$V_{\text{eff}}=\frac{\lambda}{4!}\phi_c^4+\frac{\lambda^2\phi_c^4}{256\pi^2}\left[\ln\frac{\phi_c^2}{M^2}-\frac{25}{6}\right] \qquad (4.1)$$

the renormalization-group equation

$$\left[M\frac{\partial}{\frac{\partial}{M}}+\beta\frac{\partial}{\partial\lambda}\right]V_{\text{eff}}=0 , \qquad (4.2)$$

leads to the β function

$$\beta=M\frac{\partial}{\partial M}\lambda=\frac{\frac{2\lambda^2}{256\pi^2}}{\frac{1}{4!}+\frac{2\lambda}{256\pi^2}(2\ln\phi_c-2\ln M-\frac{26}{6})} . \qquad (4.3)$$

In the above V_{eff} there appear only two parameters λ and M, and consequently λ and M are mutually functions of each other. Then the β can be exactly solved as follows.

First, the β is rewritten as

$$\beta=\frac{\frac{2\lambda^2}{256\pi^2}}{\frac{1}{4!}+\frac{2\lambda}{256\pi^2}(-2\ln aM)} , \qquad (4.4)$$

where $a=e^{25/12}/\phi_c$. If one expresses $\ln aM$ as $\eta(\lambda)/\lambda$, the β is further simplified as

$$\beta=\frac{\frac{2\lambda^2}{256\pi^2}}{\frac{1}{4!}+\frac{-4\eta(\lambda)}{256\pi^2}} . \qquad (4.5)$$

Here $\eta(\lambda)$ means that η is a function of λ. Now the derivative of $\ln aM$ leads to

$$M\frac{\partial\lambda}{\partial M}=\frac{1}{\frac{1}{M}\frac{\partial M}{\partial\lambda}}=\frac{1}{\frac{\eta'\lambda-\eta}{\lambda^2}} , \qquad (4.6)$$

where $\eta'=(\partial/\partial\lambda)\eta$. Then Eq. (4.5) becomes a differential equation for $\eta(\lambda)$, which can be easily solved as

$$\eta(\lambda)=\frac{c}{\lambda}+\tfrac{16}{3}\pi^2\ , \tag{4.7}$$

where c is a constant. Finally $\ln aM$ and β are obtained as

$$\ln aM=\frac{16\pi^2}{3\lambda}+\frac{c}{\lambda^2}\ , \tag{4.8}$$

$$\beta=\frac{3\lambda^2}{-16\pi^2-6\dfrac{c}{\lambda}}\ . \tag{4.9}$$

We are in the Hilbert space where $\langle\phi\rangle=\phi_c$ and therefore ϕ_c is kept constant. c is independent of λ but may depend on ϕ_c. To see how c depends on ϕ_c we insert expression (4.8) into $V_{\rm eff}$ (4.1), and find

$$V_{\rm eff}=-\frac{2\phi_c^4 c}{256\pi^2}\ . \tag{4.10}$$

At the saddle point $\partial V_{\rm eff}/\partial\phi_c$ vanishes, and consequently

$$\frac{\partial}{\partial\phi_c}\phi_c^4 c=0\ , \tag{4.11}$$

which leads to

$$c=\frac{\hat{c}}{\phi_c^4}\ , \tag{4.12}$$

where $\hat{c}$ is a constant independent of ϕ_c. In terms of $\hat{c}$, the $V_{\rm eff}$ is

$$V_{\rm eff}=-\frac{2\hat{c}}{256\pi^2}\ , \tag{4.13}$$

which is simply another constant. Therefore the freedom in the expression of β corresponds to an arbitrary choice for the constant vacuum energy. In the following we normalize $V_{\rm eff}$ to vanish at $\phi_c=0$: namely, $\hat{c}=c=0$. Correspondingly,

$$M=a^{-1}e^{16\pi^2/3\lambda}=\frac{\exp\left[\dfrac{16\pi^2}{3\lambda}-\dfrac{25}{12}\right]}{\phi_c}\ , \tag{4.14}$$

$$\beta=-\frac{3\lambda^2}{16\pi^2}\ . \tag{4.15}$$

The anomaly, given in terms of β (Ref. 23) is

$$\partial^\mu D_\mu=\partial^\mu x_\alpha\theta^{\mu\alpha}=\theta^\mu_\mu=-\frac{1}{4!}\beta\phi_c^4\ , \tag{4.16}$$

which is exactly identical to our results obtained previously. In the above D_μ and $\theta^{\mu\nu}$ are the dilatation current and the energy-momentum tensor.[16]

Until now, we have dealt with the massless ϕ^4 theory. In the case of the finite mass, we further have to introduce a mass scaling parameter γ_m. With two parameters, one cannot easily solve the renormalization-group equation exactly. Still it is instructive to express the result (3.14) in terms of γ_m and β, as

$$A(x)=+\tfrac{1}{2}\gamma_m m^2\phi_c^2-\frac{1}{4!}\beta\phi_c^4\ . \tag{4.17}$$

Then we find that

$$\gamma_m=\frac{\lambda}{16\pi^2},\quad \beta=-\frac{3\lambda^2}{16\pi^2}\ . \tag{4.18}$$

One may also try for the result (3.19) for the complex field as

$$A(x)=+\gamma_m m^2\langle\phi^\dagger\phi\rangle-(\tfrac{1}{2})^2 3\langle\phi^\dagger\phi\rangle^2\ . \tag{4.19}$$

Then the γ_m and β are

$$\gamma_m=+\frac{\lambda}{16\pi^2},\quad \beta=\frac{-\lambda^2}{32\pi^2}\ , \tag{4.20}$$

which are the values Buchmüller and Dragon would have obtained, if they had solved the U(1) model exactly.

V. DEFORMATION— A NEW INVARIANCE IN CONDENSED SPACE

As particle condenses, certain original symmetries are no longer valid. Instead there appear new symmetries. In analogy to the group-theoretical terminology, this phenomenon is called the "deformation."[27] Our $V_{\rm eff}$ has also a new symmetry.

The value of the coupling parameter λ depends on where the measurement is made. In our formalism, λ depends on the mass scale M on which the measurement is made. This is expressed explicitly in the β function (4.3), the solution of which is the relations (4.8) and (4.9).

On another mass scale M', the coupling parameter λ' will satisfy

$$\ln aM'=\frac{16\pi^2}{3\lambda'}+\frac{c}{\lambda'^2}\ . \tag{5.1}$$

With such λ' and M',

$$V'_{\rm eff}=\frac{\lambda'}{4!}\phi_c^4+\frac{\lambda'^2\phi_c^4}{256\pi^2}\left[\ln\frac{\phi_c^2}{M'^2}-\frac{25}{6}\right] \tag{5.2}$$

must be equal to $V_{\rm eff}$ (4.1). To confirm this most quickly, one inserts (5.1) into (5.2) and finds the result (4.13), which is independent of λ' and M'. The transformation from (λ,M) to (λ',M') can be presented as

$$\ln\frac{M'}{M}=\left[\frac{c}{\lambda'^2}+\frac{16}{3}\frac{16\pi^2}{\lambda'}\right]-\left[\frac{c}{\lambda^2}+\frac{16}{3}\frac{\pi^2}{\lambda}\right]\ . \tag{5.3}$$

For an infinitesimal transformation this reduces to

$$\lambda'-\lambda=\frac{-1}{\dfrac{2c}{\lambda^3}+\dfrac{16\pi^2}{3\lambda^2}}\ln\frac{M'}{M}\ . \tag{5.4}$$

At $c=0$

$$\lambda'-\lambda=\frac{3\lambda^2}{16\pi^2}\ln\frac{M'}{M}\ . \tag{5.5}$$

The last result (5.5) differs from Coleman and Weinberg's (Ref. 25) in sign. While their relation is considered to be approximate, ours is exact. In fact the meaning of "approximate" is vague because of the presence of $\ln M$ in

$V_{\rm eff}$. The $\ln M$ cannot be expanded in terms of λ. The problem is of nonperturbative nature.

A similar situation occurs when one solves a renormalization equation. If one expands the β function in terms of λ, hoping the M to be a series function of λ, one gets the wrong sign.

Throughout this paper the anomalous dimension for the field is not considered, as the $V_{\rm eff}$ in (4.1) is of the order of one loop.[25] For the finite mass, we further have to introduce the γ_m function and the exact symmetry is hard to find.

VI. DISCUSSION

A. Regularization of the Fujikawa factor

As is shown in Sec. II C 2 Fujikawa's prescription $A(x)$ with $f(\lambda^2/M^2)=e^{-\lambda^2/M^2}$ is fully equivalent to the orthodox field-theoretical treatment extended to (x_μ,s) space. The infinity appearing in $[A(x)-A_f(x)]$, is identical to the one in one-loop calculation, as the higher-order loops do not concern the anomaly. We may just discard this infinity, since it would not appear if we start from the renormalization Lagrangian (see Sec. III A). However, one can also regularize it.

As seen in the second line of (2.43), $A(x)$ appears as a sum of contribution from a continuous mass spectrum. The integration on the mass parameter κ reminds us of Pauli and Villar's regularization. However the insertion of a factor $\rho(\kappa)$ or $\rho(\kappa^2)$ does not help to eliminate the infinity at all. It rather deteriorates the equivalence between the proper-time description and Fujikawa's method. In fact, to apply a Pauli-Villars regulator is to multiply a factor $\rho(\lambda/M)$ or $\rho(\lambda^2/M^2)$, and not $\rho(\kappa)$ or $\rho(\kappa^2)$. For example, if one inserts $\rho(\lambda^2/M^2)=1/1-\lambda^2/M^2$, the λ^2/M^2 term in $A(x)$ drops, since

$$\frac{1}{1-\lambda^2/M^2}e^{-\lambda^2/M^2}=1+(\lambda/M)^4+\cdots\ . \tag{6.1}$$

As a matter of fact one can adopt e^{λ^2/M^2} instead of Fujikawa's choice $e^{-\lambda^2/M^2}$. It is equivalent to a change of sign of i in iD^2s in the proper-time description, having no physical significance. Then the corresponding regulator is $1/(1+\lambda^2/M^2)$ given in (2.16). Finally $A(x)-A_f(x)$ becomes finite.[27]

For the ϕ^4 theory, Coleman-Weinger's regulator (2.18) has to be applied as

$$\frac{1}{1-(\lambda^2/M^2)^2}\left[\tfrac{1}{2}(e^{-\lambda^2/M^2}+e^{\lambda^2/M^2})\right]\ . \tag{6.2}$$

One can find easily a factor

$$\tfrac{1}{2}(e^{-i\lambda^2 s}+e^{i\lambda^2 s})\ , \tag{6.3}$$

if one splits $A(x)$ in the proper-time axis in a suitable manner. In fact however, $A(x)-A_f(x)$ corrected by the factor (6.3), is already finite by itself, and the regulator (2.18) becomes redundant.

The use of such a factor e^{λ^2/M^2} instead of or in addition to $e^{-\lambda^2/M^2}$, however, hinders the comparison of Fujikawa's approach with the perturbation technique well established in high-temperature physics. When the Fujikawa factor is expanded in terms of $1/M^2$, the resultant infinite series is not assured to converge.

B. Renormalization group with cutoff parameter

Our renormalization equation (4.2) is based on the symmetry of $V_{\rm eff}$ for the transformation (5.3) of the parameters λ and M. In the early investigations on ϕ^4 theory with $\langle\phi\rangle=0$, the renormalization group is often formulated by taking the cutoff factor as a measure. For this case the basic symmetry is the invariance for the scaling $x\to ax$ and $\phi\to a^{-1}\phi$. Such a transformation is no longer possible because of the nonvanishing order parameter $\langle\phi\rangle=\phi_c$. Nevertheless we can argue for the renormalization equation for Λ, as we will explain below.

The effective potential (4.1) will be expressed in an integral form on the proper time s if we follow Schwinger's treatment.[3] This is also true in the ζ-function approach where $V_{\rm eff}=(d/ds)\zeta(s)|_{s=0}$. If one replaces s by $\hat{s}/\mu^2$ in such a $V_{\rm eff}$, the result is a function of μ^2 (see, for example, Ramond, Ref. 22, p. 118). This μ is just the arbitrary measure M appearing in Coleman and Weinberg's $V_{\rm eff}$ in (4.1). Then the transformation of (M,λ) into (M',λ') is to transform (s,λ) into (s',λ'), namely, a scaling of the proper time. As will be seen below, the transformation of (Λ,λ) into (Λ',λ') is also a scaling of the proper time.

To derive the renormalization equation in terms of Λ explicitly, we start from the Coleman-Weinberg form

$$V_{\rm eff}=\frac{\lambda}{4!}\phi_c^4+\tfrac{1}{2}B\phi_c^2+\frac{1}{4!}C\phi_c^4+\frac{\lambda\Lambda^2}{64}\phi_c^2+\frac{\lambda^2\phi_c^2}{256\pi^2}\left[\ln\frac{\lambda\phi_c^2}{2\Lambda^2}-\frac{1}{2}\right]\ , \tag{6.4}$$

which is equal to $V_{\rm eff}$ in (4.1). Here

$$C=-\frac{3\lambda^2}{32\pi^2}\left[\ln\frac{\lambda M^2}{2\Lambda^2}+\frac{11}{3}\right]$$

and

$$B=-\frac{\lambda\Lambda^2}{32}\ . \tag{6.5}$$

If one chooses a particular value for Λ such that C vanishes

$$-\frac{3\lambda^2}{32\pi^2}\left[\ln\frac{\lambda M^2}{2\Lambda^2}+\frac{11}{3}\right]=0\ , \tag{6.6}$$

then M disappears superficially from $V_{\rm eff}$ as

$$V_{\rm eff}=\frac{\lambda}{4!}\phi_c^4+\frac{\lambda^2\phi_c^2}{256\pi^2}\left[\ln\frac{\lambda\phi_c^2}{2\Lambda^2}-\frac{1}{2}\right]\ . \tag{6.7}$$

The symmetry for this $V_{\rm eff}$ can be deduced with the help of the relations (6.6) and (4.8). Namely, for another set of Λ' and λ' satisfying

$$C=-\frac{3\lambda'^2}{32\pi^2}\left[\ln\frac{\lambda' M'^2}{2\Lambda'^2}-\frac{1}{2}\right]=0\ , \tag{6.8}$$

we obtain the symmetry relation

$$\exp\left[2\left[\frac{16\pi^2}{3}\right]\left[\frac{1}{\lambda'}-\frac{1}{\lambda}\right]\right]=\frac{\lambda\Lambda'^2}{\lambda'\Lambda^2} . \tag{6.9}$$

This is an exact symmetry. One can deduce an infinitesimal transformation $\lambda'-\lambda$ from (6.10), but the result is not simple.

Corresponding to this symmetry, one can deduce a β-function $\Lambda(\partial\lambda/\partial\Lambda)$, denoted as β_Λ, from

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta_\Lambda\frac{\partial}{\partial\lambda}\right]V_{\text{eff}}[\text{Eq.}(6.7)]=0 . \tag{6.10}$$

The result is

$$\beta_\Lambda=\frac{-3\lambda^2}{16\pi^2}\left[\frac{1}{1-\dfrac{\lambda}{256\pi^2}}\right] , \tag{6.11}$$

which, for small λ, is approximately equal to β in (4.15) (Ref. 28).

There is another important choice of Λ. If one chooses Λ such that all counterterms disappear, the resultant V_{eff} is simply a bare potential. In this case Λ is related to M by

$$e^{11/3}M^2=\frac{2\Lambda^2}{\lambda}e^{-4\Lambda^2/\lambda\phi_c^2} , \tag{6.12}$$

and the symmetry of the Lagrangian is

$$\exp\left[2\left[\frac{16\pi^2}{3}\right]\left[\frac{1}{\lambda'}-\frac{1}{\lambda}\right]\right] =\frac{\lambda\Lambda'^2}{\lambda'\Lambda^2}\exp\left[-4\left[\frac{\Lambda'^2}{\lambda'\phi_c^2}-\frac{\Lambda^2}{\lambda\phi_c^2}\right]\right] . \tag{6.13}$$

It is probably worth making a further comment on this scheme. The relation (6.12) gives rise to a relationship between Λ and λ. For each chosen value of λ, Λ has a corresponding fixed value by this relationship, and cannot become arbitrarily large. In other words, instead of introducing counterterms in the Lagrangian, we just take a specific value of Λ, and we obtain a correct V_{eff}.

Correspondingly, for the anomaly one should be able to obtain the value (3.12) from $A(x)$ in (3.5) and $A_f(x)$ in (3.10) simply by choosing a suitable value for M (note that Fujikawa's M is the Λ). The second term in $A(x)$ in (3.5) should be canceled by the higher-order terms. Namely, M is a value such that

$$\frac{\lambda}{2}\frac{\phi_c^2}{M^2}-\frac{1}{3!}\left[\frac{\lambda}{2}\frac{\phi_c^2}{M^2}\right]^3+\frac{1}{4!}\left[\frac{\lambda}{2}\frac{\phi_c^2}{M^2}\right]^4\cdots =\exp\left[-\frac{\lambda}{2}\frac{\phi_c^2}{M^2}\right]-\frac{1}{2!}\left[\frac{\lambda}{2}\frac{\phi_c^2}{M^2}\right]^2=0 . \tag{6.14}$$

VII. CONCLUSION

It has been clarified that Fujikawa's prescription with the factor $e^{-(\lambda_\mu/M)^2}$ is equivalent to the authentic field-theoretical treatment, as it is equivalent to Schwinger's proper-time treatment Sec. III B. Such equivalence has been demonstrated explicitly by comparing Fujikawa's procedure with the one-loop calculation for the case of ϕ^4 theory Sec. III A.

The infinities appearing in Fujikawa's prescription can be eliminated by multiplying or by choosing a specific value for the measure M, instead of pushing it to infinity.

There is still one assumption left to justify this Fujikawa's prescription. One has to split the two-point function in terms of the proper time s. This is equivalent to inserting $e^{-(\lambda_\mu/M)^2}$ into the Fujikawa factor $A(x)$. This splitting process cannot be deduced as a natural consequence from the present field theory of path-integral form, even though it is a commonly used procedure.

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$$f(\lambda_n^2/M^2)=\frac{1-m^2/M^2}{1+(\lambda_n/M)^2}\ .$$

The factor $[1-(m^2/M^2)]$ reduces to 1 at $M\to\infty$, and can be neglected.

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