# Bosonic zero-frequency modes and initial conditions

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Zero-frequency modes for massless scalar and vector fields are considered. Although these modes lack a particle interpretation, they may nonetheless be quantized. It is shown how the quantum field theory of such massless fields contains an arbitrary parameter which determines the energy associated with the zero mode. We then show how this parameter may be related to initial conditions in a model theory with a time-dependent mass which vanishes in the future. The energy carried by the zero mode is determined by the original particle content of the quantum state and the details of how the mass varies in time. These considerations are applied to scalar and vector fields in flat spacetime with periodic boundary conditions and in Robertson-Walker universes. The connection between scalar zero modes and global symmetry breaking is discussed, especially the conditions under which broken-symmetry states decay in time due to zero-mode effects.

## I. INTRODUCTION

It is well known that massless wave equations may have solutions which are independent of the spatial coordinates. These are the zero-frequency modes, or zero modes. Here we are concerned with bosonic zero modes, as opposed to fermionic zero modes. Bosonic zero modes and their quantization have been discussed, for example, by Fulling,[1] DeWitt,[2] and by Kuchar.[3] A simple example of such a mode occurs for the massless scalar field in flat spacetime, $\Box\phi=\partial_\mu\partial^\mu\phi=0$, which has a solution which grows linearly in time:

$$f_0=At+B\ , \tag{1}$$

where $A$ and $B$ are constants. In the context of quantum field theory, such modes are usually excluded on the grounds that they are not normalizable. If, however, the space has a finite volume, this objection no longer applies. Consider a flat spacetime with torus topology $S^1\times S^1\times S^1$ and spatial volume $V$. Now the mode of Eq. (1) will have unit Klein-Gordon norm if

$$A^*B-AB^*=\frac{i}{V}\ . \tag{2}$$

If we wish to quantize the massless scalar field on this space, we must include the zero mode in order to have a complete set of $c$-number solutions of the wave equation. This may be done by writing the field operator as

$$\phi=\frac{q+pt}{\sqrt{V}}+\phi_N,\quad \phi_N=\sum_{\mathbf{k}\neq 0}(a_{\mathbf{k}}f_{\mathbf{k}}+a_{\mathbf{k}}^\dagger f_{\mathbf{k}}^*)\ , \tag{3}$$

where $q$ and $p$ are operators and $\phi_N$ is the expansion of the field operator in terms of nonzero-momentum modes, $f_{\mathbf{k}}=e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}/\sqrt{2V\omega}$. The equal-time commutation relation for $\phi$ and its conjugate momentum, $\pi=\dot\phi$, is $[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]=i\delta(\mathbf{x}-\mathbf{y})$; it will be satisfied if $q$ and $p$ satisfy the commutation relation

$$[q,p]=i\ . \tag{4}$$

This follows from the completeness relation for plane waves on a torus:

$$V\delta(\mathbf{x}-\mathbf{y})=1+\sum_{\mathbf{k}\neq 0}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\ . \tag{5}$$

The zero mode does not have a particle interpretation in the sense that other modes do; it is not meaningful to talk about there being a certain number of zero-frequency particles in a given quantum state. Nonetheless, the zero mode does have physical effects and carries nonzero energy. Classically, the energy density of the zero mode of Eq. (1) is $U=\dot\phi^2/2=|A|^2/2$. In the quantized theory, the expectation value of the energy density in an arbitrary quantum state contains a term $\langle p^2\rangle/2$ due to the zero mode. In a theory in which the mass is identically zero for all time, this energy is arbitrary; there is an undetermined parameter in the theory introduced by the zero mode. At the classical level, this arbitrariness arises because the constant $A$ is undetermined. At the quantum level, it arises because of freedom to choose the quantum state even after the particle content of all modes other than zero modes has been specified. This freedom causes $\langle p^2\rangle/2$ to be undetermined. Most discussions of the Casimir energy for massless fields in compact spaces of which we are aware ignore this contribution to the total energy.

## II. ZERO MODES AND INITIAL CONDITIONS

### A. Scalar modes in flat space

In this paper we wish to show how this ambiguity is related to initial conditions in a theory where the mass is a function of time. We consider a scalar field with a time-dependent mass which was nonzero in the past and asymptotically vanishes in the future. This type of time dependence is very natural in a cosmological context: interactions can generate an effective mass at high temperatures which disappears when the Universe cools. Let us first consider a scalar field in a flat spacetime (but with torus topology and volume $V$) which satisfies the wave equation

$$\Box\phi+m^2(t)\phi=0 \ , \tag{6}$$

where $m(t)\rightarrow m_0$ as $t\rightarrow-\infty$ and $m(t)\rightarrow 0$ as $t\rightarrow+\infty$. To proceed further we need to assume an explicit form for $m(t)$, which we take to be

$$m^2(t)=\tfrac{1}{2}m_0^2[1-\tanh(\rho t)] \ , \tag{7}$$

where $m_0$ and $\rho$ are constants.

The solutions of the wave equation with this form for $m^2(t)$ may be given in terms of hypergeometric functions. These solutions were discussed by Bernard and Duncan[4] and by Birrell and Davies[5] in the context of particle creation in an expanding universe. A solution of Eq. (6) which depends upon time only is

$$f_0=\frac{1}{\sqrt{2Vm_0}}e^{-im_0\rho^{-1}\{\rho t-\ln[2\cosh(\rho t)]\}/2}F\left[1-\frac{im_0}{2\rho},-\frac{im_0}{2\rho},1-\frac{im_0}{\rho};\frac{1+\tanh(\rho t)}{2}\right] , \tag{8}$$

where $F$ is a Gaussian hypergeometric function. The asymptotic forms of $f_0$ are given by

$$f_0\sim\frac{1}{\sqrt{2Vm_0}}e^{-im_0t}, \quad t\rightarrow-\infty \tag{9}$$

and

$$f_0\sim A_0t+B_0, \quad t\rightarrow+\infty \ . \tag{10}$$

Here

$$A_0=\frac{1}{\sqrt{2Vm_0}}\frac{2\rho\Gamma\left[1-\frac{im_0}{\rho}\right]}{\Gamma\left[1-\frac{im_0}{2\rho}\right]\Gamma\left[-\frac{im_0}{\rho}\right]} \tag{11}$$

and

$$B_0=\frac{A_0}{2\rho}\left[2\psi(1)-\psi\left[1-\frac{im_0}{2\rho}\right]-\psi\left[-\frac{im_0}{2\rho}\right]\right] , \tag{12}$$

where $\psi(x)$ denotes $\Gamma'(x)/\Gamma(x)$. We have chosen the solution which is a positive-frequency exponential in the past. It naturally goes over into a linearly growing function in the future.

Because the space has a finite spatial volume $V$, this solution has a finite Klein-Gordon norm, which has been normalized to unity. In the past, this is the mode function for a massive particle in a state of zero spatial momentum. In this region, there is a well-defined particle interpretation for this mode. However, in the future this is not the case. We may write the quantized field operator $\phi$ as

$$\phi=a_0f_0+a_0^\dagger f_0^*+\phi_N \ . \tag{13}$$

Again $\phi_N$ is the expansion of $\phi$ in terms of nonzero momentum modes, and $a_0$ and $a_0^\dagger$ are operators satisfying

$$[a_0,a_0^\dagger]=1, \quad [a_0,\phi_N]=0, \quad [a_0^\dagger,\phi_N]=0 \ . \tag{14}$$

In the past, $a_0$ and $a_0^\dagger$ are annihilation and creation operators, respectively, for particles in the zero-momentum mode. In the future, they may be related to the operators $p$ and $q$ which appear in Eq. (3) by

$$p=\sqrt{V}(A_0a_0+A_0^*a_0^\dagger) \tag{15}$$

and

$$q=\sqrt{V}(B_0a_0+B_0^*a_0^\dagger) \ . \tag{16}$$

The commutation relations for $p$ and $q$, Eq. (4), are satisfied because the constants $A_0$ and $B_0$ satisfy Eq. (2) with $A=A_0$ and $B=B_0$. Of course Eqs. (15) and (16) hold at all times because the operators in them are time independent; but at late times, $\phi$ has the form of Eq. (3).

We are now in a position to interpret the zero-frequency mode in terms of the initial conditions: the quantum state of the system at early times and the form of $m(t)$. At early times, this quantum state may be interpreted in terms of its particle content, as all modes have a particle interpretation. If we use the Heisenberg picture, the state does not change in time. At late times, the zero-momentum mode has become a zero-frequency mode and we can no longer meaningfully assign a particle number to it. However, we can calculate expectation values of observable quantities in this quantum state and identify the contribution of the zero-frequency mode. For example, suppose that the state is a coherent state of zero-momentum particles and that no other modes are excited. Then the state is $|z\rangle$, where $a_0|z\rangle=z|z\rangle$, and $z$ is some complex number. The expectation value of the field operator $\langle\phi\rangle=zf_0+z^*f_0^*$ has the asymptotic form

$$\langle\phi\rangle\sim(zA_0+z^*A_0^*)t, \quad t\rightarrow\infty \ . \tag{17}$$

The coefficient of the linearly growing term is not an arbitrary constant, as it was when we considered the strictly massless theory. It is now determined by the parameters of quantum state (e.g., $z$) and those related to the past history (e.g., $m_0$ and $\rho$). Other observables which are influenced by the zero mode include the energy density; in our example of a coherent state for the zero mode, it is

$$U=\langle T_{00}\rangle=\tfrac{1}{2}(zA_0+z^*A_0^*)^2 \ . \tag{18}$$

Note that here we are giving only the energy density of the zero mode. In addition, there is always the usual Casimir energy density in a compact space and any energy from the excitation of nonzero-momentum modes.

The parameter $\rho$ in Eq. (7) is the inverse time scale over which the mass goes to zero. From Eq. (11) we can obtain the limiting forms for $A_0$:

$$A_0 \sim \frac{-im_0}{\sqrt{2Vm_0}}, \quad \rho \gg m_0 \tag{19}$$

and

$$A_0 \sim \left[\frac{-i\rho}{\pi V}\right]^{1/2} e^{-im_0\rho^{-1}\ln 2}, \quad \rho \ll m_0 \ . \tag{20}$$

These limits correspond to the transition time being very short or very long, respectively, compared to the Compton time, $m_0^{-1}$. As we might expect, a longer transition time (smaller $\rho$) leads to a slower growth rate for the zero mode.

The time dependence of the mass also influences the modes with nonzero momentum. Particles are created into these modes in numbers determined by the initial particle number and the rate at which the mass changes. This is very similar to the particle creation by expansion of the Universe discussed in Refs. 4 and 5.

Although the zero mode does not have a particle interpretation, there is a relationship between the energy carried by the zero mode at late times and the energy that was present as particles at early times. In the limit of rapid change in the mass, $\rho \gg m_0$, the energy density in the zero mode at late times is approximately $U \approx m_0|z|^2/V$. Recall that $|z|^2$ is the mean number of particles present in the zero-momentum mode when the field is massive. We can interpret the energy density of the zero mode at late times as typically being of the same order as the rest mass energy density that was present before the mass goes to zero. This correspondence is not exact and depends upon the phase of the complex number $z$. In the limit of slow transition, $\rho \ll m_0$, the energy density is approximately $U \approx \rho|z|^2/V$. This corresponds to an energy of $\rho$ for each of the preexisting particles.

### B. Scalar zero modes in expanding universes

We now wish to turn to the question of zero modes in an expanding universe. Consider again a massless scalar field in a Robertson-Walker universe, for which the metric may be written as

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \tag{21}$$

where $k = 0, +1, -1$, which denote the flat, closed, and open cases, respectively. The generally covariant Klein-Gordon equation for the massless scalar field in these metrics, $\Box\phi = \nabla_\mu \nabla^\mu \phi = 0$, has a spatially homogeneous solution of the form

$$f_0 = c_1 \int a^{-3} dt + c_2 \ , \tag{22}$$

where $c_1$ and $c_2$ are constants. If, for example, the scale factor is of a power-law form, $a(t) \propto t^\alpha$, then this mode is of the form

$$f_0 = Ct^{1-3\alpha} + D \ . \tag{23}$$

Thus $f_0$ grows in time only if the expansion rate is not too great: $\alpha < \frac{1}{3}$. As in the case of flat spacetime, this mode is normalizable only if the spatial sections have finite volume. In the case of a closed ($k = +1$) universe, this arises naturally. In the open or flat universes, this is the case only if we put a torus topology on the space. We are always free to consider spaces with a metric of the Robertson-Walker form, but with periodic identifications imposed upon it. In either case, the condition that $f_0$ have unit norm becomes

$$c_1^* c_2 - c_1 c_2^* = \frac{i}{V} \ . \tag{24}$$

Here, $V$ is the coordinate volume of the compact spatial section. In the absence of any knowledge of the initial conditions, the coefficient $c_1$ is not uniquely determined. However, it can be calculated in models where the effective mass of the field was nonzero in the past. Finally, let us note that the energy density of the zero mode scales as $a^{-6}$:

$$U = \frac{\dot\phi^2}{2} = \frac{|c_1|^2}{2a^6(t)} \ . \tag{25}$$

### C. Global-symmetry breaking

We now wish to examine the relationship of zero modes to the breaking of global symmetries. In spaces where the zero mode for the massless scalar field is growing in time, the broken-symmetry configuration is unstable and decays in time. This phenomenon was discussed in a different context by Ford and Vilenkin.[6] Global-symmetry breaking arises when a scalar field acquires a nonzero vacuum expectation value. We restrict our attention to the case of U(1)-symmetry breaking, where this expectation value is spatially homogeneous. The complex scalar field $\Phi$ may be represented as

$$\Phi = \sigma e^{i\phi/\sigma} \ , \tag{26}$$

where $\sigma$ is a constant and $\phi$ is the Goldstone field. If $\Phi$ and $\phi$ are quantized fields, then the vacuum expectation value of $\Phi$ is

$$\langle \Phi \rangle = \sigma e^{-\langle \phi^2 \rangle/(2\sigma^2)} \ . \tag{27}$$

In Ref. 6, symmetry breaking in two-dimensional flat spacetime and four-dimensional de Sitter spacetime were discussed. In both of these cases, $\langle \phi^2 \rangle$ is necessarily a growing function of time.[7] This means that $\langle \Phi \rangle$ must decay in time. The same conclusion applies to spaces with growing scalar zero modes because $\langle \phi^2 \rangle$ must grow as $f_0^2$. For example, in a Robertson-Walker universe with a power-law scale factor, $a(t) \propto t^\alpha$, with $\alpha < \frac{1}{3}$, the expectation value of $\Phi$ must decay as

$$\langle \Phi \rangle \propto e^{-ct^{2(1-3\alpha)}} \ , \tag{28}$$

where $c$ is a constant that depends upon the rate of growth of the zero mode and hence upon the initial conditions on the theory. From Eq. (22) we can see that it is possible for a zero mode which initially does not grow to begin growth at a later time. This will happen if the expansion rate of the universe decelerates from faster than to slower than $t^{1/3}$. Thus we would have a situation where the spontaneously broken-symmetry state is stable in the rapidly expanding universe, but decays in the slowly expanding universe.

### III. VECTOR ZERO MODES

Scalars are not the only boson field which may have zero modes in compact spaces. Other bosonic fields, such as vectors, may also have such solutions.[8] Let us consider a massless vector field $A_\mu$ in flat spacetime. If the Lorentz gauge condition, $\partial_\mu A^\mu=0$, is imposed, then the wave equation is

$$\Box A^\mu=0 . \qquad (29)$$

In rectangular coordinates, the vector and scalar wave operators are identical in flat space. Thus, there are three linearly growing vector modes of the form

$$F^\mu{}_{(j)}=e^\mu{}_{(j)}(At+B) , \qquad (30)$$

where $A$ and $B$ are constants and the $e_{(j)}{}^\mu$ are three linearly independent unit vectors labeled by $j=1$, 2, and 3. The gauge condition requires that these vectors have no time components, so we may take them to be the basis vectors in rectangular coordinates. Note that these zero modes break Lorentz invariance. This is no cause for concern because the modes are only normalizable in a space with torus topology; such a space does not have Lorentz invariance. We obtain the correct normalization for our modes if they have unit Klein-Gordon norm, so $A^*B-AB^*=i/V$.

As in the scalar case, we may relate the rate of growth of the zero modes to initial conditions. If $A^\mu$ satisfies the equation

$$\Box A^\mu+m^2(t)A^\mu=0 , \qquad (31)$$

where $m^2(t)$ is as given, for example, by Eq. (7), then the modes which become zero modes in the future are

$$F^\mu{}_{(j)}=e^\mu{}_{(j)}f_0 . \qquad (32)$$

Here $f_0$ is the function given in Eq. (8). Of course, the time-dependent mass term breaks gauge invariance, but we can think of it as a simplified model for interactions which alter the propagation characteristics of the zero-momentum modes. The expansion for the quantized field operator may be expressed as

$$A^\mu=A_N{}^\mu+\sum_j [a_jF^\mu{}_{(j)}+a_j^\dagger(F^\mu{}_{(j)})^*] . \qquad (33)$$

Here $A_N{}^\mu$ is the contribution to the field operator from modes other than zero modes, and $a_j$ and $a_j^\dagger$ are the annihilation and creation operators for the zero modes. At late times, when $f_0$ is a linear function of time, these modes do not have a particle interpretation. Nonetheless, we may calculate expectation values to which these modes contribute. For example, there is an electric field and energy density associated with the zero modes. If the only mode excited is $F_{(j)}{}^\mu$, and it is in a coherent state, $|z\rangle$, then the expectation value of the electric field is

$$\langle \mathbf{E}\rangle=-\mathbf{e}_{(j)}(z\dot f_0+z^*\dot f_0^*)\sim-\mathbf{e}_{(j)}(zA_0+z^*A_0^*) , \quad t\to\infty . \qquad (34)$$

Here $\mathbf{e}_{(j)}$ is the polarization vector of mode $j$, and $A_0$ is given in Eq. (11).

We can also have zero modes for the electromagnetic field in an expanding universe if it is spatially flat but has torus topology. The field equations are conformally invariant, so the solutions (for the covariant vector, $A_\mu$) in a $k=0$ Robertson-Walker metric are just the flat-space solutions. Thus the zero modes are of the form

$$F_{\mu(j)}=e_{\mu(j)}(A\eta+B) . \qquad (35)$$

Here $\eta$ is the conformal time defined by $d\eta=a^{-1}dt$. The gauge conditions $\nabla^\mu F_{\mu(j)}=0$ and $F_{0(j)}=0$ have been imposed. The equation for a field with a time-dependent mass, Eq. (31), is not conformally invariant, so the determination of the constants $A$ and $B$ in terms of initial conditions does depend in a nontrivial way on the scale factor $a(t)$.

In a closed Robertson-Walker universe, the situation is quite different. Here there are no growing zero modes for the electromagnetic field. The wave equation in a curved spacetime is

$$\nabla_\mu\nabla^\mu A_\nu+R^\mu{}_\nu A_\mu=0 , \qquad (36)$$

where the gauge condition $\nabla^\mu A_\mu=0$ has been imposed, and $R^\mu{}_\nu$ is the Ricci tensor. First consider an Einstein universe, the static closed universe, of unit radius. A zero mode must be a spatially homogeneous solution; this homogeneity may be expressed as the conditions

$$\nabla_r A_\nu=\nabla_\theta A_\nu=\nabla_\varphi A_\nu=0 . \qquad (37)$$

Here the metric is given by Eq. (21) with $k=1$ and $a=1$. The wave equation with these conditions becomes

$$\ddot A_t=0 \text{ and } \ddot A_i+2A_i=0 , \qquad (38)$$

where the overdots denote differentiation with respect to $t$. The solution for $A_t$ is a linear function of time; however, this is a pure gauge mode. The solutions for the spatial components, $A_i$, are oscillatory in time. Thus, there are no nonoscillatory zero modes for a massless vector field in an Einstein universe. If we replace $t$ by $\eta$ in the solutions of Eq. (38) that have only spatial components, then we obtain solutions for $A_\mu$ in an arbitrary expanding universe. These are also oscillatory modes. The situation here is similar to that for a massive field in flat spacetime. Indeed, the Ricci tensor term in the wave equation behaves as an effective mass term.

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