Quantum probability distributions in the early Universe. IV. Stochastic dynamics in de Sitter space

F. R. Graziani
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
(Received 23 January 1989)

Using the Smoluchowski equation, we investigate the stochastic evolution of the horizon-averaged or coarse-grained scalar field (inflaton) in a pure de Sitter background. We clarify the effect quantum fluctuations have on the classical dynamics of relaxation. We consider two types of nonlinear potentials $[V(\phi)=\frac{1}{2}\gamma\phi^2+\frac{1}{4}g\phi^4$ and $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4]$ with inflaton probability distributions initially displaced from equilibrium. In the former case quantum fluctuations have only a minor effect on the classical inflaton dynamics. In the latter case the situation is different. Quantum fluctuations play a crucial part in the early-time behavior of the inflaton probability distribution. Using techniques borrowed from nonequilibrium statistical mechanics, we show how [for $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4$] macroscopic (classical) order originates from stochastic (quantum) initial conditions. We estimate the time scale at which this transition takes place. The work here extends and validates the conclusion of Guth and Pi that the long-time behavior of $f(\phi;t)$ can be described by a classical probability distribution.

I. INTRODUCTION

At the present time, the inflationary scenario[1,2] presents the most attractive solution to the standard collection of cosmological problems (e.g., flatness and horizon). Despite various criticisms,[3] the central feature of inflation, namely, that there is a period of exponential expansion associated with the early Universe, seems to remain intact. In fact, surveying the history of the inflationary scenario, one is immediately struck by the growth and development of Guth's original proposal.[1] For example, old inflation[1] became new inflation[2] which unfortunately was still plagued by density fluctuations that were several orders of magnitude too large.[4] Shortly thereafter, Linde[5] observed that once the Universe emerged from the Planck era, the field responsible for inflation [the so-called inflaton $\Phi(\mathbf{x},t)$] would exist as a nonequilibrium distribution of values satisfying $V(\Phi)\lesssim M_P^4$. Here $V(\Phi)$ is some potential for the field and M_P the Planck mass. Provided Φ were sufficiently large, inflation with sufficiently small density fluctuations could be realized with very natural $V(\Phi)$ (e.g., Φ^2 or Φ^4). This version of inflation is known as chaotic inflation.

In a somewhat parallel development various researchers[6–11] began to study the validity of using classical or semiclassical methods based on the effective potential for determining the dynamics of the inflaton field $\Phi(\mathbf{x},t)$. What spurred these developments were the observations[6–8] that the regulated quantum fluctuations associated with massless and massive free field theories in de Sitter space were stochastic in nature. Inspired by these results, Starobinsky[12] and Graziani and Olynyk[13] showed that the quantum dynamics of the inflaton field, when spatially averaged over a horizon, could be described by a Langevin or generalized Fokker-Planck equation (also known as a Wigner equation). The homogeneous (over a horizon volume) coarse-grained scalar field we denote by $\phi(t)$. Its evolution is governed by the classical (drift) force $-V'(\phi)$ and diffusive forces arising from the quantum fluctuations.[12–14] This latter force appears, to order $\hbar$, as a white-noise term in the homogeneous classical equations of motion or as a diffusive term in the drift equation. This approach we call stochastic inflation. Whereas $\phi(t)$ is homogeneous over a horizon, the stochastic nature of its evolution equation leads to the fact that globally (on scales greater than a horizon) the Universe is inhomogeneous. The Universe is made up of an ensemble of miniuniverses for which a $\phi(t)$ is defined in each.[14] Since this initial proposal, considerable attention has been devoted to the various implications of the stochastic approach to the dynamics of the coarse-grained inflaton.[15,16]

In this paper we wish to investigate the stochastic evolution of the inflaton (in a pure de Sitter environment) when

$$V(\phi)=\begin{cases}\dfrac{\gamma}{2}\phi^2+\dfrac{g}{4}\phi^4\,, & \text{(1a)}\\[2ex] -\dfrac{\lambda}{2}\phi^2+\dfrac{g}{4}\phi^4\,. & \text{(1b)}\end{cases}$$

We wish to see what effect the presence of quantum fluctuations has on the classical dynamics of the coarse-grained inflaton. In order to solve this problem we make the assumption of slow rollover [i.e., $\ddot{\phi}\ll 3H\dot{\phi}$, overdots refer to proper-time (as measured along the world line) derivatives]. Nambu and Sasaki[16] have shown that this is a good approximation provided $\langle\dot{\phi}^2\rangle\lesssim\langle V(\phi)\rangle$ and $g\langle\phi^4\rangle\ll\langle V(\phi)\rangle$ initially. Because of this approximation, the Fokker-Planck equation for the phase-space probability distribution becomes (to order $\hbar$) a Smoluchowski equation.[17] That is, if $f(\phi;t)d\phi$ is the probability that a measurement of ϕ will be in the range $(\phi,\phi+d\phi)$ at time t, then $f(\phi;t)$ obeys

$$\frac{\partial f(\phi;t)}{\partial t}=\frac{1}{3H}\frac{\partial}{\partial\phi}[V'(\phi)f(\phi;t)]+\frac{\hbar D}{9H^2}\frac{\partial^2 f(\phi;t)}{\partial\phi^2} . \quad (2)$$

D is the diffusion "constant" which we take to be arbitrary. We explicitly show $\hbar$ so as to emphasize the quantum origin of the diffusive term. In reality, D is determined by the microphysics[12] or equivalently, it is related to the steady-state fluctuations through the stationary distribution

$$f(\phi;\infty)=Ne^{(-3H/\hbar D)V(\phi)} . \quad (3)$$

For $D=9H^5/8\pi^2$, Eq. (2) becomes the Starobinsky equation.[12] In this paper we assume that $f(\phi;0)$ (i.e., the initial distribution) is Gaussian. This is only a simplifying assumption. It is possible to extend the calculations presented here to non-Gaussian initial distributions. In order to investigate the coarse-grained inflaton dynamics as described by Eq. (2), we borrow two techniques from nonequilibrium statistical mechanics. One is the van Kampen expansion.[18] This is a perturbative technique for small diffusion coefficients. It represents the stochastic variation of $\phi(t)$ as a small Gaussian fluctuation about a classical trajectory. As we shall see, this method tends to break down if the distribution $f(\phi;0)$ happens to be at a point of instability [e.g., Eq. (1b) with $\phi=0$]. In order to handle this situation we use the scaling theory of transient phenomena developed by Suzuki.[19] The purpose of these methods is to help us clarify the roles quantum fluctuations and classical behavior play in the stochastic evolution of the coarse-grained inflaton.

Guth and Pi[10] have looked at the quantum behavior of the slow-rollover phase transition. As they point out, quantum effects are important because it is they that give rise to energy-density fluctuations. In addition, they can determine the duration of the inflationary era. Two questions they raise are central to this paper. First, what is the correct order parameter for a system possessing reflection symmetry? Second, if ϕ is zero initially how long does it hover about this position? Classically, of course, this question is trivial; $\phi=0$ is a stationary state, it stays there forever.

In this paper we will investigate the quantum relaxation process when the inflaton distribution is initially displaced from equilibrium. We find that in potentials possessing no instability points [i.e., $V(\phi)=\frac{1}{2}\gamma\phi^2+\frac{1}{4}g\phi^4$] and assuming $f(\phi;0)$ is displaced far enough from its equilibrium configuration, quantum effects have little effect on the classical evolution of $\phi(t)$. However, when $f(\phi;0)$ finds itself in an unstable configuration, such as a Gaussian centered about $\phi=0$ in a potential $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4$, then quantum effects make a difference. By using Suzuki's scaling theory, we estimate the time at which ϕ hovers around zero. In addition, we find that this time determines the transition of $f(\phi;t)$ from quantum behavior to classical behavior. Finally, we compute an order parameter that clearly exhibits the following transient properties of the above system. $f(\phi;t)$ begins as an assumed Gaussian with a width $\langle\phi^2(0)\rangle$ which is assumed order $\hbar$ in smallness. The Smoluchowski equation is initially dominated by the quantum diffusive forces [i.e., the second term on the right-hand side of Eq. (2)]. At a time given by

$$\tilde{t}\simeq\frac{3H}{2\lambda}\ln\left[\frac{\lambda}{3g[\langle\phi^2(0)\rangle+\hbar D/3\lambda H]}\right] ,$$

the Gaussian breaks down, forming double peaks. It is at this time that Eq. (2) becomes dominated by classical drift forces. This is the transition from quantum to classical behavior. The distribution $f(\phi;t)$ for $t\gg\tilde{t}$ becomes essentially a classical probability distribution. This is an extension of the results of Guth and Pi.[10] They showed that the quantum mechanics of an upside-down harmonic oscillator could be described at late times by a classical probability distribution. The classical order parameter we define is zero for $t<\tilde{t}$ (i.e., the quantum regime) and then behaves for $t>\tilde{t}$ like a "classical" order parameter (assuming the "classical" ϕ were displaced from the origin by an amount set by the quantum fluctuations). This order parameter rises to $\pm\sqrt{\lambda/g}$ as it should.

II. STOCHASTIC DYNAMICS NEAR EQUILIBRIUM

A. The effective potential for coarse-grained systems undergoing slow-rolling behavior

Given a normalized stationary Wigner distribution $[W(x,p)]$ for a dissipationless quantum-mechanical system, we have shown[14] how to compute the effective potential. Construction begins with the generating function

$$T(J)=\int dx\,dp\,W(x,p)e^{Jx}=e^{W(J)} , \quad (4)$$

where x and p are the Weyl equivalent classical variables corresponding to the canonical position and momentum, respectively. The Legendre transform of $W(J)$ with respect to $\langle x\rangle$ yields the effective potential $U(\langle x\rangle)$. To order $\hbar$, the solution to $W(x,p)$ one uses in Eq. (4) is given by

$$W(x,p)\sim e^{-\alpha[p^2/2+V(x)]} . \quad (5)$$

The parameter α is determined by the Wigner constraint[17] and α^{-1} yields the leading contribution to the ground-state energy.

For the slow-rolling coarse-grained inflaton in de Sitter space [Eq. (2)] the stationary solution is given by

$$f(\phi;\infty)\sim e^{-(3H/\hbar D)V(\phi)} . \quad (6)$$

A natural question that arises is, what is the effective potential for this situation? Again, we can define a generating function

$$K(J)=\int d\phi\,f(\phi)e^{J\phi}=e^{F(J)} \quad (7)$$

for the moments and cumulants of ϕ. The reduced distribution $f(\phi)$ is actually derived from a full Wigner distribution $W(\phi,S)$, that is

$$f(\phi)=\int dS\,W(\phi,S) , \quad (8)$$

where S is the velocity (i.e., $S=\dot{\phi}$).

Substituting (8) into (7), we have

$$K(J)=\int d\phi\, dS\, W(\phi,S)e^{J\phi}=e^{F(J)} . \tag{9}$$

However, from (8) we know

$$W(\phi,S)\sim e^{-(3H/\hbar D)[S^2/2+V(\phi)]} . \tag{10}$$

Noting that (10) is functionally identical to (5), the computations of $K(J)$ and $F(J)$ are therefore identical in form with previous constructions for a dissipationless quantum-mechanical system. From paper I, we have, for a classical potential $\frac{1}{2}\omega^2x^2+\frac{1}{4}\lambda x^4$,

$$U(\langle x\rangle)=\frac{\omega^2}{2}\langle x\rangle^2+\frac{\lambda}{4}\langle x\rangle^4+\frac{3\lambda}{\omega^2\alpha}\langle x\rangle^2 . \tag{11}$$

Here Y and α^{-1} are energies and have dimensions of (length)$^{-1}$ (or L^{-1}). For the inflaton, by comparing (10) with (5), we see that by making the correspondences

$$x\rightarrow\phi,\quad \alpha\rightarrow 3H/\hbar D ,$$

$$p\rightarrow S,\quad \lambda\rightarrow g,\quad \omega\rightarrow\gamma .$$

The construction of $U(\langle\phi\rangle)$ is identical with the construction of $U(\langle x\rangle)$. We obtain the lowest order (in $\hbar g$) effective potential for the coarse-grained inflaton

$$U(\langle\phi\rangle)=\frac{\gamma}{2}\langle\phi\rangle^2+\frac{g}{4}\langle\phi\rangle^4+\frac{g\hbar D}{\gamma H}\langle\phi\rangle^2 . \tag{12}$$

Here, U is an energy density ($\sim L^{-4}$) and α has dimensions of L^4.

B. Equations of motion

From the Smoluchowski equation, we have

$$\frac{d\langle\phi(t)\rangle}{dt}=-\frac{1}{3H}\langle V'(\phi(t))\rangle . \tag{13}$$

If $V'(\phi)$ is nonlinear, then an infinite hierarchy of coupled moment equations is also generated. In addition there exist two types of evolution equations for $\langle\phi(t)\rangle$ that are sometimes stated in the inflation literature. One is the classical equation of motion which we denote by the subscript cl. It is given by

$$\frac{d\phi_{\rm cl}(t)}{dt}=-\frac{1}{3H}V'(\phi_{\rm cl}(t)) . \tag{14}$$

The second is based on a modification of (14) by replacing $V'(\phi_{\rm cl}(t))$ by the effective potential $U'(\langle\phi(t)\rangle)$. We denote this equation of motion by the subscript e:

$$\frac{d\langle\phi(t)\rangle_e}{dt}=-\frac{1}{3H}U'(\langle\phi(t)\rangle_e) . \tag{15}$$

The problem we wish to address in this paper is to what extent are Eqs. (13)–(15) similar? To answer this question we will use two techniques borrowed from nonequilibrium statistical mechanics. One is essentially an expansion about the deterministic trajectory developed by van Kampen.[18] The other is a fairly new method used by Suzuki[19] to look at the relaxation of stochastic systems from unstable initial configurations.

C. The van Kampen expansion

In 1961 van Kampen developed a systematic approximation method to solve the master equation. For our purposes we take this equation to be

$$\frac{\partial f(\phi;t)}{\partial t}=-\frac{\partial}{\partial\phi}[C_1(\phi;t)f(\phi;t)]+\frac{\partial^2}{\partial\phi^2}[C_2(\phi;t)f(\phi;t)] , \tag{16}$$

where C_1 and C_2 are given drift and diffusion coefficients, respectively, and $f(\phi;t)$ is the unknown probability distribution. In his paper, van Kampen showed how to solve (16) asymptotically for small diffusion coefficients via an expansion about the deterministic motion. If we write $C_2(\phi;t)=\hbar D_2(\phi;t)/2$, where $D_2(\phi;t)$ is a function of order 1, then

$$f(\phi;t)\sim\frac{1}{\sqrt{2\pi\hbar\sigma(t)}}\exp\left[-\frac{[\sigma-\langle\phi(t)\rangle]^2}{2\hbar\sigma(t)}\right] \tag{17}$$

obeys (16) to order $\hbar$. If we take $C_1(\phi;t)=-(1/3H)V'(\phi)=-(1/3H)(\gamma\phi+g\phi^3)$ [see Eq. (2)] then $\langle\phi(t)\rangle$ and $\sigma(t)$ obey, to order $\hbar$,

$$\frac{d\langle\phi(t)\rangle}{dt}=-\frac{1}{3H}\{[\gamma+3g\Sigma(t)]\langle\phi(t)\rangle+g\langle\phi(t)\rangle^3\} , \tag{18a}$$

$$\frac{d\Sigma(t)}{dt}=-\frac{2}{3H}[\gamma+3g\langle\phi(t)\rangle^2]\Sigma(t)+\frac{2\hbar D}{9H^2}$$

$$[\Sigma(t)=\langle\phi^2(t)\rangle-\langle\phi(t)\rangle^2] . \tag{18b}$$

The fluctuations are given by $\Sigma(t)$ and the mean is given by $\langle\phi(t)\rangle$. In Eqs. (18a) and (18b), $\Sigma(t)$ is identically equal to $\hbar\sigma(t)$. In order for the above equations to be valid, the assumption is that the stochastic behavior of $\phi(t)$ is approximately deterministic with small Gaussian fluctuations about the deterministic trajectory. This is reflected in the fact that $\Sigma(t)/\phi_{\rm cl}^2(t)$ must remain much less than one for all time. Finally, note that as $\hbar$ goes to zero $\Sigma(t)$ goes to zero and $\langle\phi(t)\rangle$ goes to $\phi_{\rm cl}(t)$. We are now in a position to discuss the solutions to the van Kampen equations (18a) and (18b), the classical equation (14), and the effective potential equation (15).

Setting $\Sigma(t)$ equal to zero in (18a) and integrating, we obtain

$$\phi_{\rm cl}^2(t)=\frac{f^2\delta^2}{(\delta^2+f^2)e^{2\gamma t/3H}-\delta^2}\simeq\delta^2e^{-2\gamma t/3H}\quad\text{if } \delta^2\ll f^2 , \tag{19}$$

where $\delta^2=\phi_{\rm cl}^2(0)$ and $f^2=\gamma/g$. Because $\Sigma(t)$ is already order $\hbar$ in smallness, $\phi_{\rm cl}^2(t)$ is to be used in place of $\langle\phi(t)\rangle^2$ in Eq. (18b). Again assuming that $\delta^2\ll f^2$ we obtain

$$\Sigma(t)\simeq\Sigma(0)e^{-2\gamma t/3H}+\frac{\hbar D}{3\gamma H}(1-e^{-2\gamma t/3H}) . \tag{20}$$

Note that $\lim_{t\rightarrow\infty}\Sigma(t)=\hbar D/3\gamma H$, the stationary value. In order for the van Kampen expansion to be valid we must have $\Sigma(t)/\phi_{\rm cl}^2(t) \ll 1$ or

$$\Sigma(0) \ll \delta^2 \quad \text{and} \quad \frac{\hbar D}{3\gamma H} \ll \delta^2 \ . \tag{21}$$

That is, the initial displacement of the inflaton from zero must be much greater than the scale set by the initial and final quantum fluctuations. Because of this, the evolution of $\langle \phi(t) \rangle$ proceeds semiclassically. That is, quantum-mechanical effects, if initially perturbative, remain perturbative to the classical motion throughout the evolution of $\langle \phi(t) \rangle$.

It should be emphasized that the analysis presented here is very much dependent on the fact that $V''(\delta)$ is greater than zero. That is, there are no instability points $[V''(\delta)<0]$ in the potential. We will consider in the next section situations in which the initial inflaton position is unstable. For now we will confine our studies to relaxation in so-called normal situations [i.e., $V''(\delta)>0$].

Knowing the approximate behavior of $\Sigma(t)$ we can see immediately how $\langle \phi(t) \rangle$ behaves. First, observe that if $\Sigma(0)<(>)\hbar D/3\gamma H$ then $\Sigma(t)$ increases (decreases) monotonically from $\Sigma(0)$ to $\hbar D/3\gamma H$. It is obvious from (18a) that $\langle \phi(t) \rangle$ obeys an equation similar to the classical equation satisfied by $\phi_{\rm cl}(t)$. The difference is while $\phi_{\rm cl}(t)$ decays from δ to zero on a relaxation time scale of $3H/\gamma$, $\langle \phi(t) \rangle$ decays with a time-changing scale denoted by

$$\tau(t) = \frac{3H}{\gamma + 3g\Sigma(t)} \ . \tag{22}$$

Because $\Sigma(t)$ is always greater than zero, $\tau(t)$ is less than $3H/\gamma$. Hence, $\langle \phi(t) \rangle$ approaches its equilibrium value of zero earlier than does $\phi_{\rm cl}(t)$. The effect is small because $\Sigma(t) \ll \delta^2 \ll f^2$. In Fig. 1 we show schematically how $\langle \phi(t) \rangle$ evolves and how this compares with $\phi_{\rm cl}(t)$. Note that $\langle \phi(t) \rangle$ behaves in two distinct ways, depending on which is larger, $\Sigma(0)$ or $\Sigma(\infty)$.

The use of the effective potential in determining the dynamics of relaxation of a quantum-mechanical system such as the coarse-grained inflaton requires setting

$$\frac{d\langle \phi(t) \rangle_e}{dt} = -\frac{1}{3H}\left[\gamma\langle \phi(t) \rangle_e + g\langle \phi(t) \rangle_e^3 + \frac{g\hbar D}{\gamma H}\langle \phi(t) \rangle_e\right] . \tag{23}$$

Notice that (23) is equivalent to (18a) if we set $\Sigma(t)$ (in the latter equation) equal to its stationary value of $\hbar D/3\gamma H$. It therefore seems that $U(\langle \phi(t) \rangle_e)$ governs the inflaton dynamics only in the long-time limit. How bad an error is made is a function of how far $\Sigma(0)$ differs from $\Sigma(\infty)$. Again, the error cannot be too large because the quantum fluctuations must always remain a perturbation to the classical trajectory. We show schematically in Fig. 1 how $\langle \phi(t) \rangle_e$ behaves as a function of time. Because $\tau(\infty)$ is the relevant time scale for $\langle \phi(t) \rangle_e$ and $\tau(\infty)<3H/\gamma$, $\langle \phi(t) \rangle_e$ tends to approach equilibrium earlier than $\phi_{\rm cl}(t)$.

It should be stressed that the results of this section depend on a Gaussian distribution of ϕ values. In a statistical equilibrium state the distribution of ϕ values takes on a nearly Gaussian distribution about the most probable value $\langle \phi(t) \rangle$ (Ref. 20). Hence, by its very nature, the van Kampen expansion tends to work only when $f(\phi,t)$ is near equilibrium.

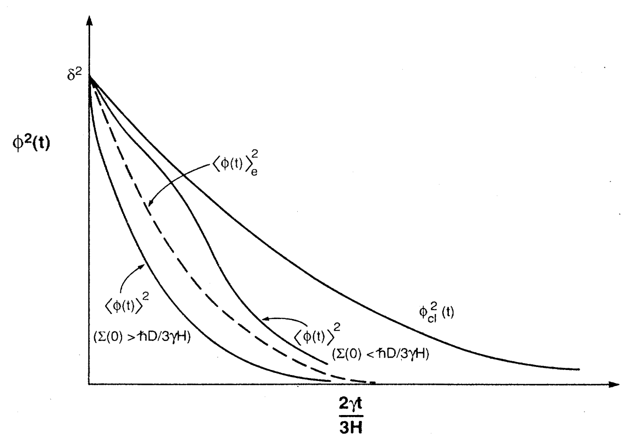

FIG. 1. The time evolution of $\langle \phi(t) \rangle$ [as solved from the van Kampen equations (18a) and (18b)] and its comparison with the classical and effective potential predictions. The potential is taken to be $V(\phi)=\frac{1}{2}\gamma\phi^2+\frac{1}{4}g\phi^4$.

III. RELAXATION NEAR THE INSTABILITY POINT

A. The van Kampen expansion

As in the preceding section, we deal with the stochastic dynamics of relaxation of the inflaton in the slow-rolling approximation. Here, however, we consider what happens when the initial distribution of ϕ values finds itself at or near an instability point [i.e., $V''(\delta)<0$]. As a concrete example we use the bistable potential $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4$.

Because the stationary distribution of ϕ values goes like $\exp[-V(\phi)]$, and hence is non-Gaussian, it is obvious that at some point the van Kampen expansion will break down. This assumes, of course, that the initial distribution is Gaussian, an assumption we make in this paper. This should not obscure the fact that the van Kampen expansion is of some use, especially in the early stages of the evolution of $f(\phi;t)$. In addition, it helps us to probe the transition from Gaussian to non-Gaussian behavior.

To obtain the equations of motion for $\langle \phi(t) \rangle$ and $\Sigma(t)$ all we have to do is use the analysis in Sec. II with the proviso $\gamma \to -\lambda$. We have

$$\frac{d\langle \phi(t) \rangle}{dt} = -\frac{1}{3H}\{[-\lambda+3g\Sigma(t)]\langle \phi(t) \rangle + g\langle \phi(t) \rangle^3\} \ , \tag{24a}$$

$$\frac{d\Sigma(t)}{dt} = -\frac{2}{3H}[-\lambda+3g\langle \phi(t) \rangle^2]\Sigma(t) + \frac{2\hbar D}{9H^2} \ , \tag{24b}$$

with the classical equation of motion

$$\frac{d\phi_{\rm cl}(t)}{dt} = \frac{\lambda}{3H}\phi_{\rm cl}(t) - \frac{g}{3H}\phi_{\rm cl}^3(t) \ . \tag{25}$$

Before we investigate the impact of Eqs. (24a), (24b), and (25) on the dynamics, let us look at the stationary states. We have

$$\phi_{\rm cl}^2=\lambda/g \tag{26}$$

for the classical system and, to order $\hbar$,

$$\langle\phi\rangle^2\simeq\lambda/g-\hbar D/2\lambda H\ , \tag{27}$$

$$\Sigma\simeq\hbar D/6\lambda H \tag{28}$$

for the quantum system. There do exist symmetric (i.e., $\langle\phi\rangle=\phi_{\rm cl}=0$) solutions to the stationary state. Classically, $\phi_{\rm cl}$ equal to zero is a viable yet highly unstable solution. Quantum mechanically, however, ϕ equal to zero is unallowed since quantum fluctuations will always drive the inflaton off of the instability point. This is suggestive of the importance quantum effects (or any fluctuations) have near points of instability. This, of course, does not mean that $\langle\phi\rangle$ cannot be zero. $\langle\phi\rangle$ equal to zero is a perfectly valid stationary solution and it is consistent with the stationary distribution

$$\exp\left[\frac{3H}{\hbar D}\left[\frac{\lambda}{2}\phi^2-\frac{g}{4}\phi^4\right]\right]\ .$$

If we try to locate this solution in the van Kampen expansion [Eqs. (24a) and (24b)] we immediately run into problems. For example, we obtain

$$\langle\phi\rangle=0,\quad \Sigma=-\frac{\hbar D}{3\lambda H}\ , \tag{29}$$

which is unphysical. The problem can immediately be traced to the Gaussian form for $f(\phi;t)$ used in the van Kampen expansion [Eq. (17)]. Because the symmetric stationary state is in reality non-Gaussian, the van Kampen expansion, by its very nature, precludes any accurate description of such a configuration. What then is the meaning of Eqs. (24a) and (24b)? This state corresponds to a single Gaussian centered approximately about $(\pm)\sqrt{\lambda/g}$ with a width $\hbar D/6\lambda H$. We can interpret this as a quasistationary state in the following sense. If the initial Gaussian were far enough away from the instability point and if the potential barrier separating the two vacuum states were wide enough, we would expect that the relaxation rate would be much larger than the tunneling rate. What this means is that our initial Gaussian would evolve intact as a Gaussian until it attained the stationary values given by (27) and (28). However, the Gaussian would not remain forever in this configuration. Ultimately, it would tunnel through the potential barrier until the true stationary state were reached. If, however, the tunneling probability were small, the probability distribution should remain in a Gaussian form with mean $\langle\phi\rangle$ and dispersion Σ for a considerable amount of time. This is why we refer to the state defined by (27) and (28) as quasistationary. Hence, the van Kampen expansion is only useful in describing the relaxation of a Gaussian distribution from a state which has slightly departed from "equilibrium." Equilibrium here refers, of course, to the quasistationary state. How far the initial distribution must be away from the instability point in order for the van Kampen expansion to be valid we will determine in a moment.

With the above observations in mind, we are now in a position to discuss the solutions to (24a), (24b), and (25). The classical evolution equation gives

$$\phi_{\rm cl}^2(t)=\frac{\lambda/g}{(\lambda/g\delta^2-1)e^{-2\lambda t/3H}+1}\ . \tag{30}$$

Knowing $\phi_{\rm cl}^2(t)$, we can discuss the solution to (24b) by defining a time-dependent effective λ. That is

$$\lambda_{\rm eff}(t)=\lambda-3g\phi_{\rm cl}^2(t)\ . \tag{31}$$

Remember, Σ is only valid to order $\hbar$; that is why $\phi_{\rm cl}^2(t)$ is used in place of $\langle\phi(t)\rangle^2$. For $\phi_{\rm cl}^2(0)=\delta^2<\lambda/3g$, $\lambda_{\rm eff}$ is positive and $\Sigma(t)$ initially grows like $\exp(2\lambda_{\rm eff}t/3H)$. As $\phi_{\rm cl}^2(t)$ increases, $\lambda_{\rm eff}(t)$ decreases until at a time t_*, $\lambda_{\rm eff}(t_*)$ becomes zero. After t_*, $\lambda_{\rm eff}(t)$ becomes negative until finally approaching a value of -2λ. What this means is that $\Sigma(t)$ rises to a maximum value of $\Sigma(t_*)$ after which $\Sigma(t)$ falls off like $\exp(-4\lambda t/3H)$ finally attaining its stationary value of $\hbar D/6\lambda H$ as t approaches infinity. This rise in the fluctuations is called fluctuation enhancement and is a consequence of the nonlinearity of the system and the initial inflaton distribution being in the vicinity of the instability point (i.e., $\delta^2<\lambda/3g$).

It is easy to compute the time t_* at which $\Sigma(t)$ is maximized. Knowing that $\phi_{\rm cl}^2(t_*)=\lambda/3g$, we obtain

$$t_*=\frac{3H}{2\lambda}\ln\frac{1}{2}\left[\frac{\lambda}{g\delta^2}-1\right]$$

or

$$t_*\approx\frac{3H}{2\lambda}\ln\left[\frac{\lambda}{2g\delta^2}\right]\quad \text{if } \delta^2\ll\lambda/g\ . \tag{32}$$

To estimate the magnitude of the fluctuation enhancement, we can solve (24b) with $\langle\phi(t)\rangle^2\simeq\phi_{\rm cl}^2=0$. This will give us an upper bound on $\Sigma(t_*)$. We have

$$\Sigma(t_*)\sim\left[\Sigma(0)+\frac{\hbar D}{3\lambda H}\right]e^{2\lambda t_*/3H}-\hbar D/3\lambda H\ . \tag{33}$$

Assuming $\delta^2\ll\lambda/g$, we have

$$\Sigma(t_*)\sim\left[\Sigma(0)+\frac{\hbar D}{3\lambda H}\right]\frac{\lambda}{2g\delta^2}\ . \tag{34}$$

Equations (33) and (34) are explicit examples of the anomalous fluctuation theorem due to Kubo, Matsuo, and Kitahara.[20] It states that when any nonlinear stochastic system relaxes from the vicinity of an instability point, there is a K^2/δ^2 enhancement of the fluctuations at a time of $\ln(K/\delta)$ (K is some dimensionful constant). We show a schematic representation of $\Sigma(t)$ vs t in Fig. 2.

It should be pointed out that if $\lambda/3g<\delta^2<\lambda/g$, then $\lambda_{\rm eff}(t)$ remains negative for all time and $\Sigma(t)$ simply decays with no enhancement of the fluctuations.

It is obvious that as δ gets closer to zero (the instability point), $\Sigma(t_*)$ can get arbitrarily large, thus invalidating the spirit of the van Kampen expansion. To avoid this problem let us look at $\langle\phi^2(t)\rangle$ at $t=t_*$:

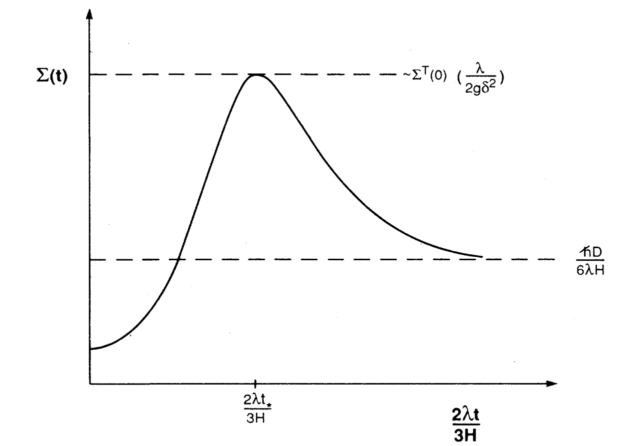

FIG. 2. The time evolution of $\Sigma(t)=\langle\phi^2(t)\rangle-\langle\phi(t)\rangle^2$ [as solved from the van Kampen equations (24a) and (24b)]. The $1/\delta^2$ rise in the fluctuations is an explicit example of Kubo's anomalous fluctuation theorem. The potential is taken to be $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4$.

$$\langle\phi^2(t_*)\rangle=\langle\phi(t_*)\rangle^2+\Sigma(t_*)$$
$$\simeq\frac{\lambda}{3g}+\Sigma^T(0)\left[\frac{\lambda}{2g\delta^2}\right], \tag{35}$$

where $\Sigma^T(0)=\Sigma(0)+\hbar D/3\lambda H$.

Because we must have $\Sigma(t_*)\ll\langle\phi(t_*)\rangle^2$ in order for the van Kampen expansion to work, δ^2 must obey

$$\delta^2\gg\Sigma^T(0) . \tag{36}$$

In addition we will be primarily interested in distributions in the vicinity of the instability point. Therefore, δ^2 actually obeys

$$\lambda/g\gg\delta^2\gg\Sigma^T(0) . \tag{37}$$

In conclusion, provided the initial Gaussian distribution is far enough away from the instability point that its width is much smaller than its mean position, the fluctuations remain a perturbation to the classical motion. That is, the evolution of the inflaton is primarily classical for all time provided $\delta^2\gg\Sigma^T(0)$.

Knowing $\phi_{\text{cl}}(t)$, $\Sigma(t)$, and t_*, we can now discuss the evolution of $\langle\phi(t)\rangle$. Define a time-dependent λ and denote it by $\Gamma(t)$. That is

$$\Gamma(t)=\lambda-3g\Sigma(t) . \tag{38}$$

Because $\Sigma(0)$ is order $\hbar$, $\Gamma(t)$ initially is slightly less than λ. For $t<t_*$, $\Gamma(t)$ decreases until a minimum value for $\Gamma(t)$ is reached at $t=t_*$. For $t>t_*$, $\Gamma(t)$ increases and as t approaches infinity, $\Gamma(t)$ approaches $\lambda-\hbar gD/3\lambda H$. The implication of all of this is that the characteristic time $3H\lambda$ associated with $\phi_{\text{cl}}(t)$ is always less than $\Gamma(t)$ and in fact at $t=t_*$ the difference between the two is at its most extreme. Hence, to the approximation presented here, it takes slightly longer for the quantum-mechanical inflaton to reach the ground state than it takes the classical inflaton. The effect is small (as it must be) since

$$\Gamma(t_*)=\lambda-3g\Sigma(t_*)\sim\lambda\left[1-\frac{3}{2}\frac{\Sigma^T(0)}{\delta^2}\right], \tag{39}$$

where $\delta^2\gg\Sigma^T(0)$.

To complete this section, we may ask the question, how do $\phi_{\text{cl}}(t)$, $\langle\phi(t)\rangle$, and $\Sigma(t)$ compare with the predictions based on the effective potential approach? The effective potential can be written[14]

$$U(\langle\phi\rangle_e)=-\frac{\lambda}{2}\langle\phi\rangle_e^2+\frac{g}{4}\langle\phi\rangle_e^4$$
$$+\frac{\hbar D}{3H}\left[-1+\frac{3g}{\lambda}\langle\phi\rangle_e^2\right]^{1/2}, \tag{40}$$

where

$$\frac{d\langle\phi(t)\rangle_e}{dt}=-\frac{1}{3H}U'(\langle\phi(t)\rangle_e) . \tag{41}$$

The first thing that comes to mind is that only for $\langle\phi(t)\rangle_e^2>\lambda/3g$ is $U(\langle\phi(t)\rangle_e)$ real. But it is precisely in the neighborhood of $\lambda/3g$ that $t=t_*$ and fluctuation enhancement occurs. Therefore, at the very least, Eq. (40) is able to describe the dynamics of $\langle\phi(t)\rangle_e$ only if $t>t_*$ or $\langle\phi(t=0)\rangle_e=\delta>\lambda/3g$. Hence, Eq. (40) can only describe the inflaton dynamics for slight departures from equilibrium.

In Fig. 3 we show qualitatively how $\phi_{\text{cl}}(t)$ and $\langle\phi(t)\rangle$ behave assuming initially that $\phi_{\text{cl}}(0)=\langle\phi(0)\rangle=\delta<\lambda/3g$.

B. Suzuki scaling theory

So far, we have discussed the effect quantum fluctuations have on the evolution of the coarse-grained inflaton field. We have found with certain restrictions [namely,

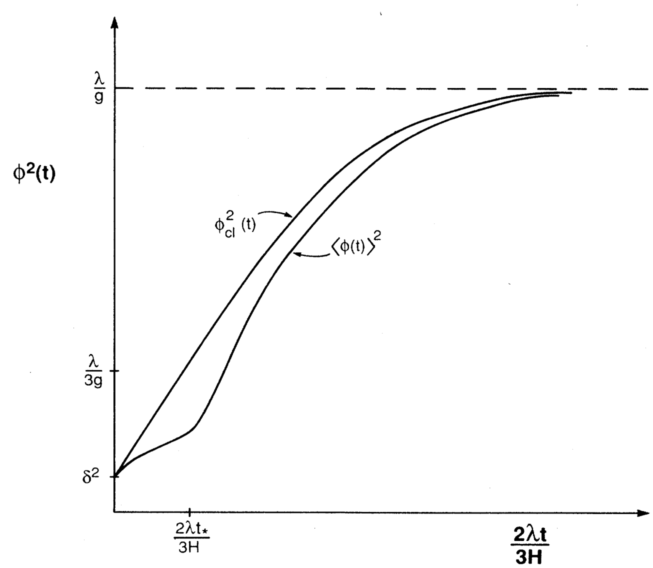

FIG. 3. The time evolution of $\langle\phi(t)\rangle$ [as solved from the van Kampen equations (24a) and (24b)] and its comparison with the classical prediction. The potential is taken to be $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4$.

$f(\phi;0)$ not be too near any instability point], that quantum-mechanical effects remain perturbative to the evolution of $\phi_{\rm cl}(t)$. That is, the evolution of the inflaton [as represented by $\langle\phi(t)\rangle$] is primarily classical. We now wish to investigate the case where δ^2 is no longer much greater than $\Sigma^T(0)$ but is, in fact, zero.

As previously discussed, for $\delta^2 \ll \Sigma^T(0)$, the van Kampen expansion breaks down. Fortunately, Suzuki[19] has devoted considerable effort to the study of how nonlinear stochastic systems evolve from unstable configurations. The theory which attempts to deal with this problem is known as the scaling theory of transient phenomena. It is based on the following observations [assuming $f(\phi;0)$ is Gaussian]: initially, fluctuations are the important feature since it is they that drive $f(\phi;t)$ away from the instability point. In this first phase $f(\phi;t)$ remains nearly Gaussian. Eventually, the Gaussian form for $f(\phi;t)$ breaks down and, as we explicitly demonstrate later, drift forces become dominant during this second phase or so-called intermediate time regime. Suzuki's theory is an asymptotic evaluation technique that seeks to determine $f(\phi;t)$ during the intermediate phase. Finally, $f(\phi;t)$ settles to its stationary non-Gaussian form. In terms of the inflaton dynamics the above observations are important. The initial regime is where the inflaton behaves quantum mechanically. The intermediate regime is where the inflaton (or rather, its distribution) behaves classically. This is because (as we shall see) the Smoluchowski equation initially is dominated by the quantum-mechanical diffusive forces. This phase gradually dies out until the Smoluchowski equation is dominated by the classical drift forces. The Suzuki theory determines the transition between the two types of behavior.

In this section, by using a self-consistent Hartree-Fock approach in the van Kampen expansion, we probe the inflaton dynamics during the initial Gaussian (quantum) phase. We then discuss and use the scaling theory of Suzuki to probe the intermediate (classical) non-Gaussian phase. The calculations in this section are based on the work of Suzuki. Because his scaling theory is not well known in the astrophysics and particle-physics community, we show considerable detail. See Refs. 13 and 16 for additional detail on Suzuki's theory.

1. Gaussian approach to scaling

As a simple introduction to the scaling ideas of Suzuki, we perform a Hartree-Fock approximation[17] to the Smoluchowski equation for the coarse-grained inflaton. In this approximation (which is Gaussian) we take $\langle\phi(t)\rangle=0$. Therefore, the initial configuration for $f(\phi;t)$ assumes δ equals zero. In this approximation,[17]

$$\phi^3=3\langle\phi^2(t)\rangle\phi \ .$$

Therefore, if $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4$ we have

$$V'(\phi)|_{\rm HF}=[-\lambda+3g\langle\phi^2(t)\rangle\phi] \ . \tag{42}$$

Substituting (42) into Eq. (2) we obtain

$$\frac{d\langle\phi^2(t)\rangle_{\rm HF}}{dt}=\frac{2\Gamma(t)}{3H}\langle\phi^2(t)\rangle_{\rm HF}+\frac{2\hbar D}{9H^2} \ , \tag{43}$$

where $\Gamma(t)=-\lambda+3g\langle\phi^2(t)\rangle_{\rm HF}$. Equation (43) is a *nonlinear* differential equation for the fluctuations. Note the difference between it and linear equation (24b) derived via the van Kampen approach. If $\langle\phi(t)\rangle$ were chosen to be zero in (24b) we would have $\langle\phi^2(t)\rangle\sim\exp(2\lambda t/3H)$, a completely erroneous result when one considers the fact that $\langle\phi^2(t)\rangle$ must ultimately attain a stationary value. Solving (43) we obtain

$$\langle\phi^2(t)\rangle_{\rm HF}\simeq\left[\frac{\lambda}{3g}\right]\frac{\tau_{\rm HF}(t)}{1+\tau_{\rm HF}(t)} \ , \tag{44}$$

where

$$\tau_{\rm HF}(t)=\frac{3g}{\lambda}\left[\langle\phi^2(0)\rangle+\frac{\hbar D}{3\lambda H}\right]e^{2\lambda t/3H} \ . \tag{45}$$

Equation (44) is an asymptotic solution in the sense that the diffusive term is small $[(\lambda/g)^2 \gg 4\hbar D/gH]$ and $2\lambda t/3H$ is much greater than one. Our results deviate slightly from Suzuki's. Our Hartree-Fock equivalent of ϕ^3 contains a factor of 3 which arises from the vanishing of the third-order cumulant. Hence, for us

$$\lim_{t\to\infty}\langle\phi^2(t)\rangle_{\rm HF}=\lambda/3g \ ,$$

while for Suzuki

$$\lim_{t\to\infty}\langle\phi^2(t)\rangle_{\rm HF}=\lambda/g \ .$$

Neglecting for the moment this discrepancy, it is important to realize that although initially $\langle\phi^2(0)\rangle$ is assumed to be order $\hbar$, by the time $\tau_{\rm HF}(t)$ becomes order 1 $\langle\phi^2(t)\rangle$ also becomes order 1. That is, the fluctuations go from being perturbative and quantum mechanical to being classical on a time scale determined by where $\tau_{\rm HF}(t)$ is order 1. This is the transition from quantum to classical behavior alluded to earlier. In addition, the asymptotic solution can be written entirely as a function of a stationary value and Suzuki's scaling variable $\tau_{\rm HF}(t)$. $\tau_{\rm HF}(t)$ expresses a cooperative effect between nonlinearity (i.e., g), initial fluctuation $[\langle\phi^2(0)\rangle]$, and diffusion (i.e., D). These observations regarding $\tau(t)$ and $\langle\phi^2(t)\rangle_{\rm HF}$ are common to all stochastic systems relaxing from instability points and are not specific to the case studied here.

2. Stationary-state fluctuations

Is there any significance to the value $\lambda/3g$ for the stationary fluctuations? First, let us look at the stationary distribution associated with substituting $\phi^3=3\langle\phi^2(t)\rangle\phi$ into $\exp[-V(\phi)]$. Via the Hartree-Fock approximation, the stationary distribution $f(\phi;\infty)\sim\exp[(-3H/\hbar D)(-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4)]$ gets transformed to $f_{\rm HF}(\phi;\infty)\sim\exp[(-3H/2\hbar D)\Delta\phi^2]$, where Δ is defined by

$$\Delta=\lim_{t\to\infty}\Gamma(t) \ .$$

If we compute $\langle\phi^2(\infty)\rangle$ from $f_{\rm HF}(\phi;\infty)$ we obtain the quadratic equation

$$\langle\phi^2(\infty)\rangle_{\rm HF}^2-\frac{\lambda}{3g}\langle\phi^2(\infty)\rangle_{\rm HF}-\frac{\hbar D}{9gH}=0 \ , \tag{46}$$

or if $(\lambda/g)^2 \gg 4\hbar D/gH$ we have

$$\langle \phi^2(\infty)\rangle_{\rm HF} \simeq \frac{\lambda}{3g}\left[1+\frac{4\hbar g D}{\lambda^2 H}\right] . \tag{47}$$

What value for $\langle \phi^2(\infty)\rangle$ do we obtain if we use the correct non-Gaussian $f(\phi;\infty)$? Defining $v^2=\lambda/g$, the stationary distribution becomes

$$f(\phi;\infty)=N\exp\left[-s\left[\frac{-\phi^2}{v^2}+\frac{1}{2}\frac{\phi^4}{v^4}\right]\right] , \tag{48}$$

where $s=3H\lambda^2/2g\hbar D \gg 1$. It proves convenient if we introduce a parameter ϵ into $f(\phi;\infty)$,

$$f_\epsilon(\phi;\infty)=N\exp\left[-s\left[\frac{-\epsilon\phi^2}{v^2}+\frac{1}{2}\frac{\phi^4}{f^4}\right]\right] , \tag{49}$$

and define a moment integral by

$$I_\epsilon(s)=N\int_{-\infty}^{\infty} d\phi\,\exp\left[-s\left[\frac{-\epsilon\phi^2}{v^2}+\frac{1}{2}\frac{\phi^4}{v^4}\right]\right] . \tag{50}$$

Evaluating (50) by standard saddle-point techniques, we obtain

$$I_\epsilon(s)\simeq 2\frac{\sqrt{2\pi}Ne^{s\epsilon^2/2}}{4\epsilon s^{1/2}/v^2} . \tag{51}$$

Realizing that

$$\lim_{\epsilon\to 1} I_\epsilon(s)\simeq 1 ,$$

we obtain

$$N\simeq\frac{1}{v}\left[\frac{s}{2\pi}\right]^{1/2}e^{-s/2} .$$

Therefore,

$$\langle \phi^2(\infty)\rangle \simeq \frac{1}{v}\left[\frac{s}{2\pi}\right]^{1/2}e^{-s/2}\frac{v^2}{s}\frac{\partial}{\partial\epsilon}\left.\frac{\sqrt{2}e^{s\epsilon^2/2}v}{\sqrt{\epsilon s}}\right|_{\epsilon=1}$$

or $\qquad$ (52)

$$\langle \phi^2(\infty)\rangle \simeq v^2\left[1-\frac{1}{2s}\right]\simeq\frac{\lambda}{g}\left[1-\frac{\hbar g D}{3\lambda^2 H}\right] .$$

It is also possible to compute $\langle \phi^4(\infty)\rangle$. From (50),

$$\langle \phi^4(\infty)\rangle \simeq v^4\left[1+\frac{3}{4s^2}\right] ,$$

$$\langle \phi^4(\infty)\rangle \simeq \frac{\lambda^2}{g^2}\left[1+\frac{\hbar^2 g^2 D^2}{3\lambda^4 H^2}\right] . \tag{53}$$

It is important to note that $\langle \phi^4\rangle_{\rm HF}=3\langle\phi^2\rangle^2_{\rm HF}$ while $\langle\phi^4(\infty)\rangle=\langle\phi^2(\infty)\rangle^2$. The former condition is satisfied by Gaussian distributions. Therefore, it makes sense that λ/g is the stationary fluctuation value for the non-Gaussian distribution. $\lambda/3g$ is the largest value $\langle\phi^2(t)\rangle$ can attain and still have the distribution identified with a Gaussian. Therefore, the time at which $\langle\phi^2(t)\rangle$ equals $\lambda/3g$ corresponds to the transition from the Gaussian to the non-Gaussian regime. As we shall see, it corresponds approximately to the time at which the Gaussian breaks down and forms double peaks.

3. *Scaling approach to stochastic dynamics*

We wish to generalize the results of Sec. I so as not to rely on an assumed Gaussian form for $f(\phi;t)$. To do this, consider again the Smoluchowski equation for the coarse-grained inflaton field:

$$\frac{\partial f(\phi;t)}{\partial t}=\frac{1}{3H}\frac{\partial}{\partial\phi}[V'(\phi)f(\phi;t)]+\frac{\hbar D}{9H^2}\frac{\partial^2 f(\phi,t)}{\partial\phi^2} , \tag{54}$$

where now $V(\phi)$ is any function of ϕ. Define the nonlinear change of variables

$$\xi=F(\phi)-\exp\left[\int_{\phi_0}^{\phi}\frac{-\lambda\,ds}{V'(s)}\right] . \tag{55}$$

Then Eq. (54) can be transformed into

$$\frac{\partial\tilde f(\xi;t)}{\partial t}=\frac{-\lambda}{3H}\frac{\partial}{\partial\xi}[\xi\tilde f(\xi;t)] +\frac{\hbar D}{9H^2}\frac{\partial}{\partial\xi}\left[G(\xi)\frac{\partial}{\partial\xi}[G(\xi)\tilde f(\xi;t)]\right] , \tag{56}$$

where

$$G(\xi)=\frac{d\xi}{d\phi}=\left[\frac{-\lambda\xi}{\frac{d}{d\xi}V[F^{-1}(\xi)]}\right]^{1/2} \tag{57}$$

and $\phi=F^{-1}(\xi)$.

Here $\tilde f(\xi;t)d\xi=f(\phi;t)d\phi$ so that $G(\xi)$ represents the Jacobian of the transformation. As Eq. (56) stands, it represents a stochastic process with unstable linear drift and complicated diffusion; unfortunately it is an equation that is no more easier to solve than (2). Suzuki's idea is that it is possible to extract an asymptotically correct solution to (56) if $G(\xi)$ is linearized. Take, for example, our case of $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4$. Therefore,

$$\xi=F(\phi)=\frac{\phi}{\left[\frac{\lambda}{g}-\phi^2\right]^{1/2}} \tag{58}$$

and

$$G^2(\xi)=(g/\lambda)(1+\xi^2)^3 . \tag{59}$$

If we ignore order-ξ^2 contributions to $G(\xi)$, Eq. (56) ends up describing an unstable Ornstein-Uhlenbeck[17] process, namely,

$$\frac{\partial\tilde f_{\rm sc}(\xi;t)}{\partial t}=-\frac{\lambda}{3H}\frac{\partial}{\partial\xi}[\xi\tilde f_{\rm sc}(\xi;t)] +\frac{\hbar g D}{9\lambda H^2}\frac{\partial^2}{\partial\xi^2}\tilde f_{\rm sc}(\xi;t) , \tag{60}$$

where we have denoted the solution to such an equation by the subscript sc for scaling. We will see in a moment the significance of such a label, but first, what is the meaning of dropping order-ξ^2 terms and higher from $G(\xi)$?

Consider Eq. (56) with $G(\xi)$ given by

$$G(\xi)=\sqrt{g/\lambda}(1+\epsilon\xi^2)^{3/2} . \quad (61)$$

The smallness parameter ϵ is only an artifact used to keep track of the expansion of $G(\xi)$ about $\xi=0$. It will be set equal to one at the end of all calculations. Rewriting (56) with (61) to order ϵ, we have

$$\frac{\partial \tilde{f}(\xi;t)}{\partial t} \simeq \frac{-\lambda}{3H}\frac{\partial}{\partial \xi}[\xi\tilde{f}(\xi;t)]+\frac{\hbar g D}{9\lambda H^2}\frac{\partial^2 \tilde{f}(\xi;t)}{\partial \xi^2} +\epsilon\frac{\partial^2}{\partial\xi^2}[\tfrac{3}{2}\xi^2\tilde{f}(\xi;t)]+\epsilon\frac{\partial}{\partial\xi}\left[\tfrac{3}{2}\xi^2\frac{\partial\tilde{f}(\xi;t)}{\partial\xi}\right] . \quad (62)$$

Let us now compute the fluctuations from both Eqs. (62) and (60). We denote the latter by the subscript sc. We obtain

$$\frac{d\langle \xi^2(t)\rangle}{dt}=\frac{2\lambda}{3H}\left[1+\frac{2\hbar g D}{\lambda^2 H}\right]\langle\xi^2(t)\rangle+\frac{2\hbar g D}{9\lambda H^2} , \quad (63a)$$

$$\frac{d\langle \xi^2(t)\rangle_{sc}}{dt}=\frac{2\lambda}{3H}\langle\xi^2(t)\rangle_{sc}+\frac{2\hbar g D}{9\lambda H^2} . \quad (63b)$$

Therefore, we see that $\langle \xi^2(t)\rangle_{sc}$ represents an asymptotic solution in the same sense that $\langle\phi^2(t)\rangle_{HF}$ [Eq. (44)] represents an asymptotic solution of (43). Namely, if $\hbar g D/\lambda^2 H$ is much less than one then $\langle\xi^2(t)\rangle\simeq\langle\xi^2(t)\rangle_{sc}$. This example is by no means rigorous proof that $\tilde{f}(\xi;t)$ is approximately equal to $\tilde{f}_{sc}(\xi;t)$ under the same conditions. However, it does illustrate the plausibility of such an assignment. It turns out, under closer scrutiny,[19] that $\tilde{f}_{sc}(\xi;t)$ is indeed the solution to (56) under the condition that $\hbar g D/\lambda^2 H$ is much less than one. We refer the reader to the work of Suzuki[19] for details.

The observation that $\tilde{f}_{sc}(\xi;t)$ is the approximate solution to (56) is a very useful one. The reason is, because $\tilde{f}_{sc}(\xi;t)$ describes an Ornstein-Uhlenbeck process, its solution is well known.[17] To obtain the solution $f_{sc}(\phi;t)$ from $\tilde{f}_{sc}(\xi;t)$ we merely use the inverse of the nonlinear transformation (58) and the fact that $f(\phi;t)=(d\xi/d\phi)\tilde{f}(\xi;t)$. It is also important to realize that initially $\xi\simeq\sqrt{g/\lambda}\phi$. What this means is that if the ϕ distribution is initially Gaussian, then the ξ distribution is also approximately Gaussian. With this in mind, Eq. (60) can be solved using the method of characteristics.[17] We obtain the normalized distribution (centered about $\phi=0$)

$$\tilde{f}_{sc}(\xi;t)=\frac{1}{\sqrt{2\pi j(t)}}e^{-\xi^2/2j(t)} , \quad (64)$$

where

$$j(t)=\frac{g}{\lambda}\left[\langle\phi^2(0)\rangle+\frac{\hbar D}{3\lambda H}\right]e^{2\lambda t/3H}-\frac{\hbar g D}{3\lambda^2 H} . \quad (65)$$

Notice that the nonlinearity, initial fluctuation, diffusion constant, etc., are now contained in a single function $j(t)$. It is the so-called Suzuki scaling variable of time. It plays a role similar to $\tau_{HF}(t)$. The major difference, of course, between $j(t)$ and $\tau_{HF}(t)$ is that the former is associated with a non-Gaussian distribution. $\tilde{f}_{sc}(\xi;t)$ depends only on $j(t)$ and ξ. If we assume $2\lambda t \gg 3H$ then we obtain

$$j(t)\simeq\tau(t)=\frac{g}{\lambda}\left[\langle\phi^2(0)\rangle+\frac{\hbar D}{3\lambda H}\right]e^{2\lambda t/3H} =\tfrac{1}{3}\tau_{HF}(t) . \quad (66)$$

Therefore, although technically not correct, the general principles behind the Gaussian approach to scaling are correct.

We are now in a position to show that if $2\lambda t$ is much greater than $3H$ then $\tilde{f}_{sc}(\xi;t)$ satisfies the drift equation [i.e., Eq. (60) with $D=0$]. Consider the following:

$$\frac{\partial F(\xi;t)}{\partial t}=\frac{-\lambda}{3H}\frac{\partial}{\partial\xi}[\xi F(\xi;t)] . \quad (67)$$

If initially

$$F(\xi;0)=\frac{1}{\sqrt{2\pi\Delta(0)}}e^{-\xi^2/2\Delta(0)} \quad (68)$$

[where $\Delta(0)$ is an arbitrary width parameter], then the solution to (67) is

$$F(\xi;t)=\frac{1}{\sqrt{2\pi\Delta(t)}}e^{-\xi^2/2\Delta(t)} , \quad (69)$$

where

$$\Delta(t)=\Delta(0)e^{2\lambda t/3H} .$$

By choosing $\Delta(0)=(g/\lambda)[\langle\phi^2(0)\rangle+\hbar D/3\lambda H]$ it is seen that $F(\xi;t)=\tilde{f}_{sc}(\xi;t)$ if $2\lambda t\gg 3H$.

What does all this have to do with the distribution we are actually interested in: $f(\phi;t)$? Although $\tilde{f}_{sc}(\xi;t)$ describes an Ornstein-Uhlenbeck process, because of the relationship between ξ and ϕ, $f_{sc}(\phi;t)$ describes a stochastic process that is nonlinear and very different from Ornstein-Uhlenbeck. In fact, using (58), we obtain

$$f_{sc}(\phi;t)=\frac{1}{\sqrt{2\pi j(t)}}\frac{\lambda/g}{\left[\frac{\lambda}{g}-\phi^2\right]^{3/2}} \times\exp\left[\frac{-\phi^2}{2j(t)\left[\frac{\lambda}{g}-\phi^2\right]}\right] . \quad (70)$$

In the limit that t approaches zero, this distribution is approximately Gaussian. Notice that if we set $D=9H^5/8\pi^2$ and $\langle\phi^2(0)\rangle=0$, then $f_{sc}(\phi;t)$ is identical to the distributions computed by Rey[15] and Sasaki, Nambu, and Nakao.[15] Equation (70) does not adequately describe the stationary state distribution $f(\phi;\infty)$ but it does describe the initial Gaussian phase, the intermediate time regime, and the transition between the two. For $2\lambda t/3H\gg 1$, $j(t)$ goes to $\tau(t)$ and (70) describes a drift-dominated intermediate phase.

To get an idea of the usefulness of (70) [or (64)] the moments of ϕ can be computed. We obtain a simple extension of Suzuki's result:[19]

$$\langle \phi^{2n}(t) \rangle \simeq (\lambda/g)^n Z_n(j(t)) , \tag{71}$$

where

$$Z_n(j(t)) = \frac{1}{\sqrt{2\pi}} \int^{\infty} d\sigma \, e^{-\sigma^2/2} \frac{j^n(t)\sigma^{2n}}{[1+j(t)\sigma^2]^n} . \tag{72}$$

The first noticeable feature is that the moments are non-Gaussian because $\langle \phi^{2n}(t) \rangle$ does not equal some function of $\langle \phi^2(t) \rangle^n$ (as determined by the vanishing of all cumulants greater than second order). The second is that $j(t)$ equal to $\frac{1}{3}$ yields $\langle \phi^2(t) \rangle \simeq \frac{1}{3}(\lambda/g)$, that is, the fluctuations have reached 33% of their non-Gaussian stationary value but have reached their final Gaussian value. What this means is that at a time $\tilde{t}$ where $j(\tilde{t})$ is of order 1, one determines the transition from Gaussian to non-Gaussian behavior. In other words, it describes the inflatons' transition from quantum behavior to classical behavior. We will make this statement more quantitative in a moment. Last, Eq. (70) expresses the central feature of Suzuki's theory. In the intermediate time regime where the distribution is dominated by classical drift forces, the moments can be written as a function of a stationary value and a universal function $Z(j(t))$. The universal function itself depends only on the scaling variable $j(t)$. All nonlinearities, initial fluctuations, time, etc., are scaled into a single variable $\tau(t)$. These features of $\langle \phi^{2n}(t) \rangle$ are general and do not depend on the specific form of $V(\phi)$ (Ref. 19). The only thing that changes is the universal function $Z(j(t))$ and $j(t)$ itself. Note that the Hartree-Fock calculation clearly exhibits the traits mentioned above. The universal function for the fluctuations is in this case $Z(\tau_{\rm HF}(t)) = \tau_{\rm HF}(t)/[1+\tau_{\rm HF}(t)]$ and the scaling variable $\tau_{\rm HF}(t)$ is given by (45).

The transition from Gaussian to non-Gaussian behavior can be analyzed by looking at the extrema of $\ln[\sqrt{2\pi j(t)} f_{\rm sc}(\phi;t)]$. Denoting such a quantity by $\phi_{\rm ex}(t)$, we obtain

$$\phi_{\rm ex}^2(t) \left[\left[\frac{3\lambda}{g} - \frac{\lambda}{g j(t)} \right] - 3\phi_{\rm ex}^2(t) \right]^2 = 0 , \tag{73}$$

which has solutions

$$\phi_{\rm ex}^2(t) = 0 \tag{74a}$$

or

$$\phi_{\rm ex}^2(t) = \frac{\lambda}{g} \left[1 - \frac{1}{3j(t)} \right] . \tag{74b}$$

In other words, initially there exists a single maxima at $\phi_{\rm ex}^2(t)=0$ up until a time determined by where $j(t)=\frac{1}{3}$. We denote this time by $\tilde{t}$. For $t > \tilde{t}$, the distribution forms double peaks [given by (74b)] and hence is no longer Gaussian. It is important to note that the transition takes place for $j(t)=\frac{1}{3}$ *or equivalently,* $\tau_{\rm HF}(t)=1$. Therefore, although the Gaussian approach to scaling does not give quite the correct value for the moments, it does yield correct qualitative information on the nature of the fluctuations and the transition from Gaussian to non-Gaussian behavior.

The dimensionless time $H\tilde{t}$ is easily calculated to be

$$H\tilde{t} = \frac{3H^2}{2\lambda} \ln \frac{\lambda}{3g \left[\langle \phi^2(0) \rangle + \dfrac{\hbar D}{3\lambda H} \right]} \tag{75}$$

if $\hbar g D/\lambda^2 H \ll 1$.

Is there a significance to $\tilde{t}$ other than its role as differentiating the initial and intermediate time regimes? Essentially, $j(t)$ [or $\tau_{\rm HF}(t)$] signal the onset of macroscopic order.[19] This macroscopic order originates from the quantum fluctuations [$\langle \phi^2(0) \rangle$ and $\hbar D/3\lambda H$]. Imagine a particular value of the coarse-grained inflaton field $\phi(t)$ associated with a particular shade of grey. We will call ϕ equal to zero grey, ϕ equal to $-\sqrt{\lambda/g}$ black, and ϕ equal to $\sqrt{\lambda/g}$ white. The calculations presented here apply to a situation in which the Universe (on scales greater than a horizon) finds itself in a nonequilibrium single phase state with all fluctuations quantum mechanical in origin. Therefore, initially the Universe (on scales greater than the horizon) looks like a vast sea of grey with some regimes lighter than others. This reflects the fact than quantum mechanics dictates that there be an initial distribution of ϕ values denoted by $\langle \phi^2(0) \rangle$. Relaxation then proceeds from this symmetrical quantum configuration to a stationary state consisting of two coexisting phases ($\phi = \pm\sqrt{\lambda/g}$, white and black, respectively). The pattern formation that arises is essentially a phase separation process[21] whose dynamics are governed by a Wiener (quantum) process for early times ($t \ll \tilde{t}$) and eventually by a classical drift equation for $t \gg \tilde{t}$. Once the separate phases start appearing, that time is called the onset of macroscopic order.[19] Such a time is provided by scaling theory and is approximately the time at which $f(\phi;t)$ exhibits twin peaks, that is, $H\tilde{t}$ [Eq. (75)]. $H\tilde{t}$ differentiates the quantum behavior of the inflaton from its classical behavior. Hence, $\tilde{t}$ gives quantitative information on the nonlinear (non-Gaussian) nonequilibrium process by which classical order arises out of essentially stochastic (quantum) initial conditions.

Is it possible to define an order parameter for the above system even though $\langle \phi(t) \rangle$ equals zero? A similar problem has arisen before in the inflationary context.[4] For example, if $\delta^2 \gg \Sigma^T(0)$ there exists a nonzero $\langle \phi(t) \rangle^2$, which deviates perturbatively from $\phi_{\rm cl}^2(t)$, which evolves from δ^2 to λ/g. Here fluctuations play a secondary role. $\langle \phi(t) \rangle$ gives information on the transition from the semidisordered state [i.e., $\Sigma^T(0) \ll \delta^2 \ll \lambda/g$] to the ordered state. For $\delta=0$, where the system admits a $\phi \to -\phi$ symmetry, the even moments of the coarse-grained inflaton field seem to offer the only candidates for an order parameter. However, the extrema [$\phi_{\rm ex}^2(t)$] of $f_{\rm sc}(\phi;t)$ offers another possibility. $\phi_{\rm ex}^2(t)$ remains zero during the initial Gaussian (quantum) phase and then at $\tilde{t}$ increases monatonically to λ/g. $\phi_{\rm ex}(t)$ is useful because it clearly exhibits the three stages of evolution of $f(\phi;t)$. In addition, it plays a role analogous to $\langle \phi(t) \rangle$ for systems where $\delta \neq 0$. Using (30) for $\delta^2 \ll \lambda/g$ but nonzero, we make a plot of $\phi_{\rm cl}^2(t)$ and $\phi_{\rm ex}^2(t)$ vs Ht in Fig. 4. It clearly demonstrates a

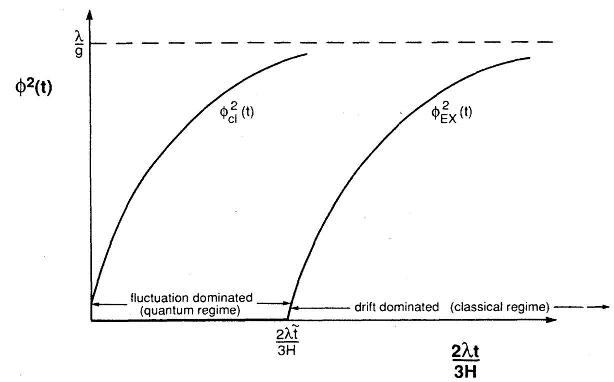

FIG. 4. The time evolution of the position of the maxima $[\phi^2_{\rm ex}(t)]$ of the inflaton probability distribution. The potential is given by $V(\phi)=-\frac{1}{2}\lambda\phi^2+\frac{1}{4}g\phi^4$. For comparison we show the classical behavior of $\phi^2(t)$, assuming $\phi^2(0)$ is slightly displaced from zero.

time lag $(H\tilde{t})$ between the relaxation of the stochastic and classical inflaton fields. It is interesting to note that the classical description provides a lower bound for the relaxation time. This is in agreement with Evans and McCarthy.[3]

Is it possible to make $H\tilde{t}$ small enough so that there is no appreciable difference between $\phi^2_{\rm cl}(t)$ and $\phi^2_{\rm ex}(t)$? First, if $H\tilde{t}\ll 1$ or

$$(g/\lambda)\Sigma^T(0)\gg\tfrac{1}{3}e^{-2\lambda/3H^2}\,, \tag{76}$$

then the time lag is negligible. Next, we realize that if $\lambda t/3H\gg 1$ then

$$\phi^2_{\rm ex}(t)\sim\frac{\lambda}{g}\left[1-\frac{\lambda}{3g\Sigma(0)}e^{-2\lambda t/3H}\right] \tag{77a}$$

and

$$\phi^2_{\rm cl}(t)\sim\frac{\lambda}{g}\left[1-\frac{\lambda}{g\delta^2}e^{-2\lambda t/3H}\right]. \tag{77b}$$

Therefore, if $\delta^2\sim 3\Sigma(0)$ the large Ht behavior of $\phi_{\rm ex}(t)$ is indistinguishable from a classical order parameter. The only difference is that $\phi_{\rm ex}(t)$ is a time translated version of $\phi_{\rm cl}(t)$. If this time translation is small then there is essentially no difference [as far as $\phi_{\rm ex}(t)$ and $\phi_{\rm cl}(t)$ are concerned] between the relaxations of the quantum-mechanical and classical descriptions of the inflaton dynamics. Hence, long-time behavior of $\phi_{\rm ex}(t)$ is indistinguishable from a classical order parameter.

IV. CONCLUSION

By using the van Kampen expansion and the Suzuki scaling theory of transient phenomena we have investigated how quantum fluctuations modify the classical dynamics of relaxation of the coarse-grained inflaton. We have shown that provided the initial inflaton distributions' displacement were much greater than the scale set by the initial and final quantum fluctuations, the evolution of the inflaton is primarily classical. However, if the inflaton distribution $f(\phi;t)$ evolves from an unstable configuration, then quantum fluctuations play an important role in the early-time evolution of $f(\phi;t)$. In fact, for $t\ll\tilde{t}$ the fluctuations of $f(\phi;t)$ are quantum mechanical. At a time $t\simeq\tilde{t}$, the evolution of $f(\phi;t)$ goes from being dominated by quantum diffusive forces to classical drift forces. Around this time, moments of ϕ go from being quantum mechanical to classical. It is at this point that the initial Gaussian distribution [with initial width $\langle\phi^2(0)\rangle$] breaks down and forms double peaks [with $\langle\phi^2(t)\rangle$ now classical with a value of $\lambda/3g$]. Even though $\langle\phi(t)\rangle^2$ is zero we show it is possible to define an "order parameter" for the early Universe. $\phi^2_{\rm ex}(t)$ is the position of the peak of the inflaton distribution. It is zero for $t<\tilde{t}$, where quantum effects dominate. At the point that classical drift forces dominate, $\phi^2_{\rm ex}(t)$ rises monatonically, behaving like a classical order parameter. The evolution of $f(\phi;t)$ for $t\gg\tilde{t}$ is dominated by classical drift forces and supports Guth and Pi[10] that $f(\phi;t)$ can be described by a classical probability distribution.

ACKNOWLEDGMENTS

I wish to thank Keith Olive and Kirk Olynyk for many useful discussions. In addition, I wish to thank A. D. Linde for encouragement. This research was supported in part by DOE Grant No. DE-AC02-83ER-40105.

[1]A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[2]A. D. Linde, Phys. Lett. **108B**, 389 (1982); **114B**, 431 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
[3]G. F. Mazenko, W. G. Unruh, and R. M. Wald, Phys. Rev. D **31**, 273 (1985); M. Evans and J. G. McCarthy, *ibid.* **31**, 1799 (1985).
[4]A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. W. Hawking, Phys. Lett. **115B**, 295 (1982); A. A. Starobinsky, *ibid.* **117B**, 175 (1982); J. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).
[5]A. D. Linde, Phys. Lett. **123B**, 177 (1983); Zh. Eksp. Teor. Fiz. **87**, 369 (1984) [Sov. Phys. JETP **60**, 211 (1984)]; Lett. Nuovo Cimento **39**, 401 (1984); Phys. Lett. **162B**, 281 (1985).
[6]A. Vilenkin and L. H. Ford, Phys. Rev. D **26**, 1231 (1982); A. Vilenkin, Nucl. Phys. **B226**, 527 (1983).
[7]A. D. Linde, Phys. Lett. **116B**, 335 (1982).
[8]A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).
[9]S. W. Hawking and I. G. Moss, Nucl. Phys. **B224**, 180 (1983).
[10]A. H. Guth and S.-Y. Pi, Phys. Rev. D **32**, 1899 (1985).
[11]G. Semenoff and N. Weiss, Phys. Rev. D **31**, 689 (1985); **31**, 699 (1985).
[12]A. A. Starobinsky, in *Field Theory, Quantum Gravity and Strings,* proceedings of a Seminar Series, Meudon and Paris, France, 1984–1985, edited by H. J. de Vega and N. Sanchez (Lecture Notes in Physics, Vol. 246) (Springer, Berlin, 1986),

p. 107.

[13]F. R. Graziani and K. O. Olynyk, Fermilab Report No. 85/175-T, 1985 (unpublished).

[14]F. R. Graziani, Phys. Rev. D **38**, 1122 (1988); **38**, 1131 (1988); **38**, 1802 (1988).

[15]J. M. Bardeen and G. J. Bublik, Class. Quantum Gravit. **4**, 573 (1987); S.-Y. Rey, Nucl. Phys. **B284**, 706 (1987); D. H. Coule and K. O. Olynyk, Mod. Phys. Lett. A **2**, 811 (1987); M. D. Pollack, Phys. Lett. B **192**, 59 (1987); M. D. Pollack, Nucl. Phys. B (to be published); M. Sasaki, Y. Nambu, and K.-I. Nakao, *ibid.* **B308**, 868 (1988); Phys. Lett. B **209**, 197 (1988); R. H. Brandenberger, H. A. Feldman, and J. H. MacGibbon, Brown University Report No. HET-648, 1987 (unpublished); J. M. Cornwall and R. Bruinsma, Phys. Rev. D **38**, 3146 (1988).

[16]Y. Nambu and M. Sasaki, Phys. Lett. B **205**, 441 (1988).

[17]H. Risken, *The Fokker-Planck Equation* (Springer, New York, 1984).

[18]N. G. van Kampen, Can. J. Phys. **39**, 551 (1961).

[19]M. Suzuki, Adv. Chem. Phys. **46**, 195 (1981).

[20]R. Kubo, K. Matsuo, and K. Kitahara, J. Stat. Phys. **14**, 129 (1976).

[21]P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977); J. D. Gunton, M. San Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1983), Vol. 8, p. 269.