Magnetic field induced by vacuum polarization in de Sitter spacetime

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We derive the effective Lagrangian of a magnetic field in the context of a massless scalar QED coupled to gravity in de Sitter spacetime. It is found that the magnetic field is produced at the beginning of the inflation in a minimally coupled scalar QED, and that the intensity B is proportional to eH^2 (H is the Hubble constant). When $H=10^{15}$ GeV, $B\sim 1.3\times 10^{47}$ G.

I. INTRODUCTION

There is now considerable interest in magnetic fields in the early Universe. At present, magnetic fields exist everywhere in our Universe. Many spiral galaxies have magnetic fields with a typical strength of $\sim 3\times 10^{-6}$ G (Ref. 1) and it is estimated that the present intergalactic magnetic fields have an intensity $\leq 10^{-9}$ G (Refs. 2 and 3). Many astrophysicists have investigated the origin of these cosmic magnetic fields.[4–8] These fields are believed to be formed as follows. A seed magnetic field, of which some production mechanisms have been proposed, is generated in a radiation-dominated universe. The generated magnetic field is relatively small, in the range $10^{-19}-10^{-21}$ G. By the dynamo mechanism this field is amplified to the value of the present magnetic field. But it seems that no satisfactory explanation about the origin of the magnetic field has yet been given.

Recently Turner and Widrow[9] proposed that the primeval magnetic field with sufficient strength can be produced in the inflationary universe model. Their proposal is that the magnetic field is excited by quantum fluctuations and the magnetic field flux is enhanced during inflation. They have shown that it leads to astrophysically significant results, though it is model dependent. Furthermore, it implies that the inflationary universe model, which has solved some important cosmological problems,[10–13] is also valid for solving the problem about the origin of cosmic magnetic fields. Therefore, their proposal is attractive. However, they did not sufficiently explain the creation of a magnetic field in inflation. Based on the analogy of a scalar field in de Sitter spacetime, they asserted that de Sitter-produced quantum fluctuations excite the primeval magnetic field with the following energy density ρ_B:

$$\frac{\rho_B}{\rho_{\text{total}}}\approx (H/m_{\text{Pl}})^2 ,$$

where ρ_{total}, H, and m_{Pl} are the total energy density of the Universe, Hubble constant, and Planck mass, respectively. This was only an assumption and not an investigated result.

The purpose of this paper is to calculate the magnitude of the primeval magnetic field which was assumed, for example, in the work of Turner and Widrow. We will derive and investigate the effective Lagrangian of magnetic fields induced by vacuum polarization of particle-antiparticle pairs in the context of a massless scalar QED coupled to gravity in de Sitter spacetime. We will show that the strong magnetic fields are produced by quantum fluctuation in a de Sitter vacuum.

The outline of this paper is as follows. In Sec. II we construct the effective Lagrangian of a magnetic field derived by quantum QED in de Sitter spacetime. In Sec. III we calculate and investigate the effective Lagrangian in two cases: (1) $\xi=\frac{1}{6}$ (gravitational conformal coupling) and (2) $\xi=0$ (gravitational minimal coupling). It is shown that the magnetic field can substantially not be produced at $\xi=\frac{1}{6}$, and that, on the other hand, at $\xi=0$ the strong magnetic field can be produced at the beginning of inflation, if the Hubble constant H has a large enough value. When $H=10^{15}$ GeV, $B\approx 1.3\times 10^{47}$ G. This produced magnetic field is proportional to eH^2 (e is the electromagnetic coupling) and the ratio $\rho_B/\rho_{\text{total}}$ coincides with one assumed by Turner and Widrow, up to e^2. Finally we summarize the work in Sec. IV.

Note that throughout this paper we use units in which $\hbar=\kappa=c=1$, except for the estimation of the intensity of the produced magnetic field in Sec. III.

II. EFFECTIVE LAGRANGIAN

In this section we are going to construct an effective Lagrangian of the magnetic field in de Sitter spacetime. Here we consider massless scalar QED coupled to gravity. However, it is noted that a magnetic field generally affects the structure of spacetime. Therefore, we assume that the magnetic energy ($B^2/2$) is sufficiently small compared to the vacuum energy ($\sim H^2m_{\text{Pl}}^2$), which leads to inflation of the Universe. The spacetime is then approximately described by de Sitter spacetime whose line element is given by

$$ds^2=dt^2-e^{2Ht}(dx^2+dy^2+dz^2) . \tag{2.1}$$

In conformal coordinates

$$ds^2=a^2(\eta)(d\eta^2-dx^2-dy^2-dz^2) , \tag{2.2}$$

where

$$a(\eta)=-(H\eta)^{-1} , \tag{2.3}$$

with

$$\eta=\int a^{-1}dt=-H^{-1}e^{-Ht} . \quad (2.4)$$

The scalar curvature is a constant:

$$R=12H^2 . \quad (2.5)$$

The model is defined by the Lagrangian density

$$L=L_0+L_1 , \quad (2.6)$$

where

$$L_0=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} , \quad (2.7)$$

$$L_1=D_\mu\phi^* D^\mu\phi-\xi R\phi^*\phi , \quad (2.8)$$

with a gauge-covariant derivative

$$D_\mu=\nabla_\mu+ieA_\mu , \quad (2.9)$$

and ξ is a coupling constant of the scalar field to the gravitational background. The equations of motion are

$$\nabla^\mu F_{\mu\nu}=iea^2\phi^*\overleftrightarrow{\partial}_\nu\phi-2e^2a^2\phi^*\phi A_\nu , \quad (2.10)$$

$$(D^\mu D_\mu+\xi R)\phi=0 . \quad (2.11)$$

For fixing a gauge freedom of A_μ, we choose the Coulomb gauge:

$$\nabla^i A_i=0, \quad A_0=0 . \quad (2.12)$$

For simplicity, we discuss the production of a homogeneous magnetic field. Furthermore, we assume that the magnetic field is along the z axis and the flux is conserved in de Sitter spacetime. So from Eqs. (2.10)–(2.12), we choose a classical solution

$$\phi=0, \quad A_\mu=\tfrac{1}{2}F_{\mu\nu}x^\nu , \quad (2.13)$$

where only F_{12} and F_{21} are nonvanishing:

$$F_{21}=-F_{12}=f=\text{const} . \quad (2.14)$$

On the other hand, the relation between the electromagnetic field-strength tensor and the electric and magnetic field in de Sitter spacetime is given by

$$F_{\mu\nu}=a^2\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} . \quad (2.15)$$

So the solution in Eq. (2.13) implies that the strength of the magnetic field is

$$B(\eta)=f/a(\eta)^2 . \quad (2.16)$$

We derive the effective Lagrangian due to quantum fluctuation $\hat{\phi}$ around the classical background given by Eq. (2.13). At first let us construct the effective Lagrangian in the functional-integral approach. The effective Lagrangian is defined by

$$\exp\left[i\int d^4x\sqrt{-g}\,L_{\text{eff}}\right] = \int[d\phi][d\phi^*]\exp\left[i\int d^4x\sqrt{-g}\,(L_0+L_1)\right] . \quad (2.17)$$

Thus the effective Lagrangian becomes

$$L_{\text{eff}}(A_\mu)=L_0+L_I , \quad (2.18)$$

where

$$L_I=-i(-g)^{-1/2}\left\langle x\left|\ln\left[\frac{1}{(-g)^{5/8}(D_\mu D^\mu+\xi R)(-g)^{-1/8}}\right]\right|x\right\rangle . \quad (2.19)$$

Using the identity

$$\text{tr}[\ln(ab^{-1})]=\text{tr}\left[\int_0^\infty \frac{ds}{s}\left(e^{is(b+i\epsilon)}-e^{is(a+i\epsilon)}\right)\right] , \quad (2.20)$$

we rewrite L_I as

$$L_I=-i(-g)^{-1/2}\int_0^\infty \frac{ds}{s}\left(\langle x|e^{is(-\hat{H}+i\epsilon)}|x\rangle - \langle x|e^{is[(-g)^{-1/4}+i\epsilon]}|x\rangle\right) , \quad (2.21)$$

where

$$\hat{H}=-(-g)^{3/8}(D^\mu D_\mu+\xi R)(-g)^{-1/8} . \quad (2.22)$$

Substituting Eq. (2.13) into Eqs. (2.18) and (2.21), we obtain

$$L_0=-\tfrac{1}{2}B^2 , \quad (2.23)$$

$$L_I=-i(-g)^{-1/2}\int_0^{\infty e^{i\epsilon}}\frac{ds}{s}\langle x|e^{-is\hat{H}}|x\rangle , \quad (2.24)$$

where the integral in Eq. (2.24) is taken along the straight line from the origin to $\infty e^{i\epsilon}$, and

$$\hat{H}=\eta^{\mu\nu}(\hat{p}_\mu+eA_\mu)(\hat{p}_\nu+eA_\nu)-(\xi-\tfrac{1}{6})Ra^2(\hat{\eta}) \quad (2.25)$$

with $\eta^{\mu\nu}=\text{diag}(1,-1,-1,-1)$. Here, as the second term in Eq. (2.21) is a magnetic-independent term, we drop it. Because we dropped the second term, we have divergences in the above effective Lagrangian. But the divergences are independent of the magnetic field and are subtracted later. So we regularize and calculate the effective Lagrangian by the point-splitting method, i.e., $\langle x|e^{-is\hat{H}}|x\rangle\to\langle x|e^{-is\hat{H}}|x'\rangle$. According to Ref. 14, the element $\langle x|e^{-is\hat{H}}|x'\rangle$ can be calculated. L_I is given by

$$L_I = e^{i(\nu+1/2)\pi/2} U \frac{H^4(\eta\eta')^{5/2}}{32\pi^{3/2}} \times \int_0^{\infty e^{i\epsilon}} ds\, s^{-5/2} H_\nu^{(2)}\left[\frac{\eta\eta'}{2s}\right] \times \exp\left[\frac{i}{4s}[\eta^2+\eta'^2-(z-z')^2]\right] \times \frac{|\omega|}{\sin(|\omega|s)} F_B\ , \tag{2.26}$$

where

$$\nu = (\tfrac{9}{4} - 12\xi)^{1/2}\ , \tag{2.27}$$

$$\omega = ef\ , \tag{2.28}$$

$$U = \exp\left[-ie\int_{x'}^{x} d\mathbf{x}\cdot\mathbf{A}\right]\ , \tag{2.29}$$

and

$$F_B = \exp\left[-\left[1+e^{i(|\omega|s+\pi/2)}\frac{1}{\sin(|\omega|s)}\right]\delta^2\right] \tag{2.30}$$

with

$$\delta^2 = |\omega|[(x-x')^2+(y-y')^2]/4\ . \tag{2.31}$$

Note that we take the limit $x'\to x$ after the integration of Eq. (2.26). Here we rotate the contour to the imaginary axis, and make the replacement $s\to(1/2|\omega|)e^{i\pi/2}s$. Thus using the relation between the Hankel function $H_\nu^{(2)}$ and the modified Bessel function K_ν,

$$H_\nu^{(2)}(-i\zeta) = \frac{2}{\pi} e^{\pi(1+\nu)i/2} K_\nu(\zeta)\ , \tag{2.32}$$

we finally obtain

$$L_{\text{eff}} = L_0 + L_I\ ,$$

where

$$L_I = -UH^4\left[\frac{C}{2\pi}\right]^{5/2}\int_0^\infty ds\, s^{-5/2} K_\nu\left[\frac{C}{s}\right] \times e^{C(1-\Sigma)/s}\frac{1}{\sinh(s/2)} F_B\ , \tag{2.33}$$

with

$$\Sigma = [-(\eta-\eta')^2+(z-z')^2]/2\eta\eta'\ ,$$
$$C = |ef|\eta\eta'\ . \tag{2.34}$$

III. PRODUCTION OF MAGNETIC FIELD

In Sec. II we constructed the integral form of the effective Lagrangian. We shall now calculate the effective Lagrangian and investigate if a magnetic field is produced by the quantum effect. Before integrating L_I, we rewrite L_I in order to be able to integrate it easily. First we require that the x component and y component of two points x'^μ and x^μ coincide:

$$x'\to x\ ,\quad y'\to y\ ,$$

where $x'^\mu = (\eta', x', y', x')$ and $x^\mu = (\eta, x, y, z)$ are the arguments of the effective Lagrangian in Eq. (2.33). In this coincidence, $x = x'$ and $y = y'$, L_I takes a simpler form:

$$L_I = -H^4\left[\frac{C}{2\pi}\right]^{5/2}\int_0^\infty ds\, s^{-5/2}K_\nu\left[\frac{C}{s}\right] \times e^{C(1-\Sigma)/s}\frac{1}{\sinh(s/2)}\ . \tag{3.1}$$

Here we assume that Σ is a positive value so that the point x'^μ approaches in a spacelike manner another point x^μ in the (η, z) plane. In the above integration we use the relation

$$\frac{1}{\sinh(s)} = \frac{1}{s} + \sum_{n=1}^{\infty}(-1)^n\frac{2s}{s^2+n^2\pi^2}\ . \tag{3.2}$$

Then we rewrite L_I as

$$L_I = L_I^{(1)} + L_I^{(2)}\ , \tag{3.3}$$

where

$$L_I^{(1)} = -2H^4\left[\frac{C}{2\pi}\right]^{5/2}\int_0^\infty ds\, s^{-7/2}K_\nu\left[\frac{C}{s}\right]e^{C(1-\Sigma)/s} \tag{3.4}$$

and

$$L_I^{(2)} = -4H^4\left[\frac{C}{2\pi}\right]^{5/2}\sum_{n=1}^{\infty}(-1)^n \times\int_0^\infty ds\, s^{-3/2}\frac{1}{s^2+4n^2\pi^2} \times K_\nu\left[\frac{C}{s}\right]e^{C(1-\Sigma)/s}\ . \tag{3.5}$$

In the following discussion we calculate the effective Lagrangian in two typical cases: (1) $\xi=\frac{1}{6}$ (gravitational conformal coupling) and (2) $\xi=0$ (gravitational minimal coupling).

A. $\xi=\frac{1}{6}$

This case corresponds to the condition $\nu=\frac{1}{2}$. First we calculate $L_I^{(1)}$ in Eq. (3.4). In Appendix A we calculate it for an arbitrary value of ν. Substituting $\nu=\frac{1}{2}$ into the formula in Appendix A, we obtain

$$L_I^{(1)} = -\frac{H^4}{4\pi^2\Sigma^2}\ . \tag{3.6}$$

Next we calculate $L_I^{(2)}$. For $\nu=\frac{1}{2}$, the modified Bessel function becomes

$$K_\nu(u) = \left[\frac{\pi}{2u}\right]^{1/2} e^{-u}\ . \tag{3.7}$$

Substituting the above equation into Eq. (3.5), we get

$$L_I^{(2)}=-\frac{C^2H^4}{2\pi^2}\sum_{n=1}^{\infty}(-1)^n\int_0^{\infty}du\frac{u}{4n^2\pi^2u^2+C^2}e^{-\Sigma u} . \tag{3.8}$$

$L_I^{(2)}$ is calculated in Appendix B. The result is

$$L_I^{(2)}=-\frac{C^2H^4}{96\pi^2}\left[\ln\Sigma+\gamma+\ln(C/\pi)+\frac{6}{\pi^2}\zeta'(2)\right]+O(\Sigma) , \tag{3.9}$$

where γ is Euler's constant. From Eqs. (3.3), (3.6), and (3.9), the effective Lagrangian is given by

$$L_{\text{eff}}=-\tfrac{1}{2}B^2-\frac{H^4}{4\pi^2\Sigma^2}-\frac{C^2H^4}{96\pi^2}\left[\ln\Sigma+\gamma+\ln(C/\pi)+\frac{6}{\pi^2}\zeta'(2)\right]+O(\Sigma) . \tag{3.10}$$

It is noted that the parameter C can be expressed in terms of the magnetic field

$$C=|eB|/H^2+O(\Sigma) . \tag{3.11}$$

Being a B-independent term, the Σ^{-2} term can be dropped in the effective Lagrangian.

Next we renormalize the effective Lagrangian according to Ref. 14. Here we rewrite the effective Lagrangian in terms of σ by the relation

$$(2\sigma)^{1/2}=\frac{1}{H}\arccos(1-\Sigma) , \tag{3.12}$$

where the square of the geodesic distance between the points x and x' is denoted by 2σ. From Eqs. (3.10)–(3.12), we get

$$L_{\text{eff}}=-\frac{1}{2}\left[1+\frac{e^2}{48\pi^2}\ln(\sigma\kappa^2)\right]B^2-\frac{e^2B^2}{192\pi^2}[\ln(e^2B^2/\kappa^4)+\alpha]+O(\Sigma) , \tag{3.13}$$

where the redundant parameter κ^2 is introduced to adjust the dimensionality of the theory, and

$$\alpha=2\gamma+\frac{\pi^2}{3}\zeta'(2)-2\ln(\pi) .$$

In our model the physical parameter is only the Hubble constant H. So we can assume that $\kappa=O(H)$.

The logarithmically divergent factor of the first term in Eq. (3.13) can be absorbed by a change of scale of B and a charge renormalization according to

$$B_R^2=\left[1+\frac{e^2}{48\pi^2}\ln(\sigma\kappa^2)\right]B^2 ,$$

$$e_R^2=\left[1+\frac{e^2}{48\pi^2}\ln(\sigma\kappa^2)\right]^{-1}e^2 ,$$

where B_R and e_R represent a renormalized magnetic field and a renormalized charge, respectively. Thus, after performing the limit $(x'\to x)$, we obtain the final result for L_{eff}:

$$L_{\text{eff}}=-\tfrac{1}{2}B^2-\frac{e^2B^2}{192\pi^2}[\ln(e^2B^2/\kappa^4)+\alpha] , \tag{3.14}$$

where we have dropped the subscript R in B and e for simplicity.

Now we are going to discuss the production of the magnetic field. Here we investigate if the effective potential of B has a minimum away from the point $B=0$. It is noted that if the energy minimum emerges in the state $B\neq 0$, the value of B is the intensity of the magnetic field produced in the vacuum state. Then we look for an effective potential of B. As $B=fe^{-2Ht}$, our effective Lagrangian L_{eff} is time dependent. In general, we may not treat $-L_{\text{eff}}$ as the effective potential. However, B varies slowly at $t\approx 0$, i.e., at the beginning of the inflation and then we can treat $-L_{\text{eff}}$ as the effective potential of B as in a static case. So the effective potential at $t\approx 0$ is given by

$$V_{\text{eff}}=\tfrac{1}{2}B^2+\frac{e^2}{192\pi^2}B^2\left[\ln\frac{e^2B^2}{\kappa^4}+\alpha\right] . \tag{3.15}$$

This effective potential shows an existence of the minimum at $B\neq 0$. The minimum is given by

$$B=\frac{\kappa^2}{e}\exp\left[-\frac{48\pi^2}{e^2}-\frac{\alpha}{2}-\frac{1}{2}\right] . \tag{3.16}$$

In Gauss units,

$$B=\hbar c\frac{\kappa^2}{e}\exp\left[-\frac{12\pi}{e^2}\hbar c-\frac{\alpha}{2}-\frac{1}{2}\right]\approx 10^{-2237}\kappa^2\text{ G} . \tag{3.17}$$

As the factor 10^{-2237} is too small to be physically acceptable, the above value B may be taken to be zero in a physical sense.

Thus we see that at $\xi=\frac{1}{6}$ (the gravitational conformal coupling), the magnetic field cannot be produced due tc vacuum polarization at early time of the inflationary universe, $t\approx 0$.

B. $\xi=0$

This case corresponds to $\nu=\frac{3}{2}$. Using the formula in Appendix A, $L_I^{(1)}$ in Eq. (3.4) is given by

$$L_I^{(1)}=-\frac{H^4}{4\pi^2}\left[\frac{1}{\Sigma^2}+\frac{1}{\Sigma}\right] . \tag{3.18}$$

Next we calculate $L_I^{(2)}$. In this case, the modified Bessel function becomes

$$K_\nu(u)=\left[\frac{\pi}{2u}\right]^{1/2}e^{-u}\left[1+\frac{1}{u}\right] . \tag{3.19}$$

Substituting the above equation into Eq. (3.5), we get

$$L_I^{(2)} = -\frac{C^2H^4}{2\pi^2}\sum_{n=1}^{\infty}(-1)^n\int_0^\infty du\frac{u}{4n^2\pi^2u^2+C^2}e^{-\Sigma u} - \frac{C^2H^4}{2\pi^2}\sum_{n=1}^{\infty}(-1)^n\int_0^\infty du\frac{1}{4n^2\pi^2u^2+C^2}e^{-\Sigma u} . \tag{3.20}$$

The first term in Eq. (3.20) is the same one as in the $\xi=\frac{1}{6}$ case and the second term can be easily calculated. Thus we obtain

$$L_I^{(2)} = -\frac{C^2H^4}{96\pi^2}\left[\ln(\Sigma)+\gamma+\ln(C/\pi)+\frac{6}{\pi^2}\zeta'(2)\right] + \frac{CH^4}{8\pi^2}\ln 2+O(\Sigma) . \tag{3.21}$$

From Eqs. (3.3), (3.18), and (3.21), the effective Lagrangian is given by

$$L_{\text{eff}} = -\tfrac{1}{2}B^2-\frac{H^4}{4\pi^2}\left[\frac{1}{\Sigma^2}+\frac{1}{\Sigma}\right] - \frac{C^2H^4}{96\pi^2}\left[\ln(\Sigma)+\gamma+\ln(C/\pi)+\frac{6}{\pi^2}\zeta'(2)\right] + \frac{CH^4}{8\pi^2}\ln 2+O(\Sigma) . \tag{3.22}$$

After the renormalization, we obtain

$$L_{\text{eff}} = -\tfrac{1}{2}B^2-\frac{e^2B^2}{192\pi^2}\left[\ln\left[\frac{e^2B^2}{\kappa^2}\right]+\alpha\right]+\frac{H^2\ln 2}{8\pi^2}|eB| . \tag{3.23}$$

Hence, the effective potential at $t\approx 0$ is given by

$$V_{\text{eff}} = \tfrac{1}{2}B^2+\frac{e^2B^2}{192\pi^2}\left[\ln\left[\frac{e^2B^2}{\kappa^2}\right]+\alpha\right]-\frac{H^2\ln 2}{8\pi^2}|eB| . \tag{3.24}$$

Here we note that the linear term of B appears in V_{eff}. Owing to the linear term, the magnetic field which gives the energy minimum has a strong intensity. As $e^2/192\pi^2 \ll 1$, we approximately obtain the minimum

$$B\approx\frac{eH^2}{8\pi^2}\ln 2 \tag{3.25}$$

and in Gauss units

$$B\approx\frac{eH^2}{2\pi}\ln 2 . \tag{3.26}$$

Therefore, the magnetic field can be produced in proportion to eH^2 at $t\approx 0$. The ratio of the free energy density ρ_B in the produced magnetic field to the total energy ρ_{total} in the inflationary universe is

$$\frac{\rho_B}{\rho_{\text{total}}}\approx\frac{e^2}{48\pi^3}(\ln 2)^2\left[\frac{H}{m_{\text{Pl}}}\right]^2 , \tag{3.27}$$

where $\rho_B=\frac{1}{2}B^2$ and we use natural units. This ratio coincides with Turner and Widrow's assumption, described in the Introduction, up to e^2.

By the way, it is known that in the inflationary universe model the Hubble constant H has the following constraint:[15]

$$\frac{H}{m_{\text{Pl}}}<10^{-4} , \tag{3.28}$$

where the Planck mass $m_{\text{Pl}}=1.22\times 10^{19}$ GeV.

When we choose typical values $H=10^{14}$ and 10^{15} GeV we can estimate the values of the produced magnetic field, respectively:

$$B\approx 1.3\times 10^{45}\ \text{G} \quad (\text{for } H=10^{14}\ \text{GeV}\approx 5\times 10^{27}\ \text{cm}^{-1}) \tag{3.29}$$

and

$$B\approx 1.3\times 10^{47}\ \text{G} \quad (\text{for } H=10^{15}\ \text{GeV}\approx 5\times 10^{28}\ \text{cm}^{-1}) . \tag{3.30}$$

We see that the strong magnetic field can be produced at $t\approx 0$, if the Hubble constant has a large enough value. Here we point out that our produced magnetic field $B\approx(eH^2/8\pi^2)\ln 2$ is not inconsistent with our assumption $\frac{1}{2}B^2 \ll H^2m_{\text{Pl}}^2$ in Sec. II. Since $\frac{1}{2}B^2/H^4\approx e^2/128\pi^4 \ll 1$ and from Eq. (3.28) $m_{\text{Pl}}^2/H^2>10^8$, our assumption is satisfied.

Thus we see that the strong magnetic field is generated at the beginning of the inflation in a massless and minimally coupled scalar field and its ratio $\rho_B/\rho_{\text{total}}$ coincides with one assumed by Turner and Widrow up to e^2.

IV. SUMMARY

We have calculated an effective Lagrangian of a magnetic field B in the context of a massless scalar QED coupled to gravity in a de Sitter spacetime, and have investigated the production of a magnetic field due to quantum effect in two cases: (1) $\xi=\frac{1}{6}$ (gravitational conformal coupling), (2) $\xi=0$ (gravitational minimal coupling). This effective Lagrangian leads to the effective potential of a magnetic field at the beginning of inflation ($t\approx 0$). It is shown that the effective potential has a minimum away from the point $B=0$ in both cases. Though the minimum at $\xi=\frac{1}{6}$ is nearly zero, the minimum at $\xi=0$ is very large, $B\approx 1.3\times 10^{47}$ G, when the Hubble constant $H=10^{15}$ GeV. Thus we have seen that the magnetic field is produced due to vacuum-polarization effects in a massless and minimally coupled scalar field at the beginning of the de Sitter phase, and that the magnitude is proportional to eH^2 and the ratio $\rho_B/\rho_{\text{total}}$ coincides with one assumed by Turner and Widrow up to e^2.

We have determined the initial magnitude of the magnetic field during the de Sitter phase. In this paper we do not discuss the magnetic field after production. But our result will be used to determine the power spectrum at the present time in a realistic model of inflation.

APPENDIX A

We calculated $L_I^{(1)}$ in Eq. (3.4) for $|\nu|\le\frac{5}{2}$. First we evaluate the integral

$$I^{(\nu)}=\int_0^\infty ds\, s^{-7/2}K_\nu\left[\frac{C}{s}\right]e^{C(1-\Sigma)/s}\ . \tag{A1}$$

Using the formulas

$$\int_0^\infty dx\, x^{\mu-1}K_\nu(x)e^{-ax}=\left[\frac{\pi}{2}\right]^{1/2}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{(a^2-1)^{(\mu/2)-(1/4)}}\times P_{\nu-1/2}^{-\mu+1/2}(a)$$

and (A2)

$$P_\nu^\mu(a)=\frac{1}{\Gamma(1-\mu)}\left[\frac{a+1}{a-1}\right]^{\mu/2}F\left[-\nu,\nu,1,1-\mu;\frac{1-a}{2}\right]\ ,$$

we obtain

$$I^{(\nu)}=C^{-5/2}\left[\frac{\pi}{2}\right]^{1/2}\frac{\Gamma(\frac{5}{2}+\nu)\Gamma(\frac{5}{2}-\nu)}{2\Sigma^2}\times F(\tfrac{1}{2}-\nu,\nu+\tfrac{1}{2},1,3;1-\tfrac{1}{2}\Sigma)\ . \tag{A3}$$

The hypergeometric function can be expanded in terms of Σ:

$$F(\tfrac{1}{2}-\nu,\nu+\tfrac{1}{2},1,3;1-\tfrac{1}{2}\Sigma)=\frac{2}{\Gamma(\frac{5}{2}-\nu)\Gamma(\frac{5}{2}+\nu)}\times\left[1-(\tfrac{1}{4}-\nu^2)\frac{\Sigma}{2}+(\tfrac{1}{4}-\nu^2)(\tfrac{9}{4}-\nu^2)\frac{\Sigma^2}{4}\times[k_0-\ln(\Sigma/2)]\right]+O(\Sigma^3)\ , \tag{A4}$$

where

$$k_0=\tfrac{3}{2}-2\gamma-\psi(\tfrac{5}{2}-\nu)-\psi(\tfrac{5}{2}+\nu)\ .$$

Thus we obtain

$$I^{(\nu)}=C^{-5/2}\left[\frac{\pi}{2}\right]^{1/2}\left[\frac{1}{\Sigma^2}-(\tfrac{1}{4}-\nu^2)\frac{1}{2\Sigma}+\tfrac{1}{4}(\tfrac{1}{4}-\nu^2)(\tfrac{9}{4}-\nu^2)\times[k_0-\ln(\Sigma/2)]\right]+O(\Sigma)\ , \tag{A5}$$

so that $L_I^{(1)}$ is given by

$$L_I^{(1)}=-\frac{H^4}{4\pi^2}\left[\frac{1}{\Sigma^2}-(\tfrac{1}{4}-\nu^2)\frac{1}{2\Sigma}+\tfrac{1}{4}(\tfrac{1}{4}-\nu^2)(\tfrac{9}{4}-\nu^2)\times[k_0-\ln(\Sigma/2)]\right]+O(\Sigma)\ . \tag{A6}$$

APPENDIX B

In this appendix we calculate $L_I^{(2)}$ in Eq. (3.8):

$$L_I^{(2)}=-\frac{H^4C^2}{2\pi^2}\sum_{n=1}^\infty(-1)^nI_n\ , \tag{B1}$$

where

$$I_n=\int_0^\infty\frac{u}{4n^2\pi^2u^2+C^2}e^{-\Sigma u}\ . \tag{B2}$$

First let us calculate the integral I_n. We alter the variables, $v=2n\pi u/C$, and obtain

$$I_n=\frac{1}{4\pi^2n^2}\int_0^\infty dv\frac{v}{v^2+1}e^{-\alpha_n v} \tag{B3}$$

$$=\frac{1}{4\pi^2n^2}\left[\frac{\pi}{2C}\right]^{1/2}(-1)[\cos(\alpha_n)\mathrm{ci}(\alpha_n)+\sin(\alpha_n)\mathrm{si}(\alpha_n)]\ , \tag{B4}$$

where si(x) is sine integral and ci(x) cosine integral, and $\alpha_n=C\Sigma/2n\pi$. By virtue of the expansions

$$\mathrm{ci}(x)=\gamma+\ln(x)+\sum_{n=1}^\infty\frac{(-1)^nx^{2n}}{2n(2n)!}$$

and (B5)

$$\mathrm{si}(x)=-\frac{\pi}{2}+\sum_{n=0}^\infty\frac{(-1)^nx^{2n+1}}{(2n+1)(2n+1)!}\ ,$$

we get

$$I_n=-\frac{1}{4\pi^2n^2}[\ln(\Sigma)+\ln(C/2\pi)+\gamma-\ln(n)]+O(\Sigma)\ . \tag{B6}$$

The summation in Eq. (B1) can be carried out using the equalities

$$\sum_{n=1}^\infty(-1)^{n-1}\frac{1}{n^2}=\frac{\pi^2}{12}$$

and (B7)

$$\sum_{n=1}^\infty(-1)^n\frac{\ln(n)}{n^2}=\frac{\pi^2}{12}\ln(2)+\tfrac{1}{2}\zeta'(2)\ ,$$

and we obtain

$$L_I^{(2)}=-\frac{C^2H^4}{96\pi^2}\left[\ln(\Sigma)+\ln(C/\pi)+\gamma+\frac{6}{\pi^2}\zeta'(2)\right]+O(\Sigma)\ . \tag{B8}$$

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