Statistics from dynamics in curved spacetime

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We consider quantum fields of spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 with a nonzero mass in curved spacetime. We show that the dynamical Bogolubov transformations associated with gravitationally induced particle creation imply the connection between spin and statistics: By embedding two flat regions in a curved spacetime, we find that only when one imposes Bose-Einstein statistics for an integer-spin field and Fermi-Dirac statistics for a half-integer-spin field in the first flat region is the same type of statistics propagated from the first to the second flat region. This derivation of the flat-spacetime spin-statistics theorem makes use of curved-spacetime dynamics and does not reduce to any proof given in flat spacetime. We also show in the same manner that parastatistics, up to the fourth order, are consistent with the dynamical evolution of curved spacetime.

I. INTRODUCTION

The proof of the spin-statistics theorem was first given by Pauli in 1940 for free fields with arbitrary spin by arguments based on positivity of energy and microcausality.[1] This proof was generalized to the axiomatic quantum field formalism by Lüders and Zumino, and Burgoyne.[2] All these proofs are based on the kinematics of quantum fields. A review of the spin-statistics theorem was given by Streater and Wightman.[3] A proof based on the dynamics of quantum fields in curved spacetime was given by Parker[4] for spin-0 and spin-$\frac{1}{2}$ fields. Here we generalize that method of proof to massive fields of higher spin. The connection between spin and statistics and the inner product in curved spacetime has been noted by Wald.[5] Sorkin[6] has suggested that a spin-statistics correlation will exist whenever the underlying theory incorporates the possibility of pair creation. In our derivation below, the Bogolubov transformation that is responsible for determining the statistics is also the source of pair creation.

In a curved spacetime corresponding to a gravitational field, dynamical evolution is associated with a Bogolubov transformation of creation and annihilation operators of the quantum field.[4] As a consequence of this fact, one is able to infer the connection between spin and statistics in a novel way. If Fermi-Dirac statistics is imposed on a field of integer spin at a given time, then at a later time the field will no longer obey the same statistics. By contrast, Bose-Einstein statistics is consistent with the dynamical evolution of a field of integer spin. Similarly, Fermi-Dirac, but not Bose-Einstein statistics, is consistent with the dynamical evolution of a field of half-integer spin.

The essential elements of our derivation of statistics from dynamics are the existence of a conserved scalar product and of a nontrivial Bogolubov transformation resulting from the dynamical evolution of the system. We give the conserved scalar product in an arbitrary curved spacetime, but discuss the Bogolubov transformations in the context of an arbitrary spatially flat Robertson-Walker universe for simplicity.

The detailed derivation is given here for massive fields with spin values $(0,\frac{1}{2},1,\frac{3}{2},2)$. In principle, the spin-statistics theorem can be proved for any spin value with the method described in this paper. Problems which occur for higher-spin fields do not affect the particular aspects required in our derivation. The Lagrangian approach to quantum fields in flat spacetime is reviewed in Sec. II, Sec. III describes the vierbein formalism and the quantum theory for these fields in curved spacetime. In Sec. IV we study the temporal evolution of these fields in a model spacetime and prove the connection between spin and statistics. Finally, parastatistics is discussed in Sec. V and the results are summarized in Sec. VI.

II. FREE FIELDS ON MINKOWSKI SPACETIME

The physically relevant fields belong to finite-dimensional representations of the Lorentz group, with tensors corresponding to the single-valued representations and spinors to double-valued representations. In general, fields may carry both tensor and spinor indices. A complex field with a unique spin s has only $2(2s+1)$ independent solutions of a given momentum, so constraints must be imposed to obtain the correct number of independent solutions. (In counting solutions, note that for half-integer spin the complex conjugate of a solution is not another solution.) Lagrangians can be constructed which imply equations of motion with the appropriate constraints.

Listed in Table I are the Lagrangians, field equations, and plane-wave mode solutions for fields with physically interesting spins of $(0,1,2,\frac{1}{2},\frac{3}{2})$ (Ref. 7). In the table $\partial^2=\eta^{\mu\nu}\partial_\mu\partial_\nu$ with $\eta^{\mu\nu}=\text{diag}(-1,1,1,1)$, V is a cubical volume with periodic boundary conditions and quantities denoted by ψ and χ carry a suppressed spinor index as well as explicit tensor indices. For tensor fields the conserved scalar product is

TABLE I. Lagrangians, field equations, and mode solutions of free fields of spin 0, 1, 2, $\frac{1}{2}$, $\frac{3}{2}$ in Minkowski space. Positive and negative frequencies are denoted by $a=+1$ and -1. For a spin-s field there are $2s+1$ polarizations which correspond to $d=-s,-s+1,\ldots,s$. Other parameters are defined as $\omega_k=\sqrt{\mathbf{k}^2+m^2}$, $\{\gamma^\mu,\gamma^\nu\}=-2\eta^{\mu\nu}$, $b=1$ or $\frac{1}{3}$.

Spin	Lagrangian	Field equations	Mode solution $V^{-1/2}u^{ad}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}$ with $u^{ad}(\mathbf{k},t)$ as
0	$-(\partial^\mu\phi^*\partial_\mu\phi+m^2\phi^*\phi)$	$(-\partial^2+m^2)\phi=0$	$(2\omega_k)^{-1/2}e^{ia\omega_k t}$
1	$-[\frac{1}{2}(\partial_\mu\phi^*_\nu-\partial_\nu\phi^*_\mu)(\partial^\mu\phi^\nu-\partial^\nu\phi^\mu)+m^2\phi^{*\mu}\phi_\mu]$	$-\partial_\mu(\partial^\mu\phi^\nu-\partial^\nu\phi^\mu)+m^2\phi^\nu=0$	$(2\omega_k)^{-1/2}e^{ia\omega_k t}E^d_{\mathbf{k}\mu}$
		equivalently $(-\partial^2+m^2)\phi^\nu=0,\ \partial_\mu\phi^\mu=0$	
2	$-\Big\{\frac{1}{2}(\partial_\lambda\phi^*_{\mu\nu}\partial^\lambda\phi^{\mu\nu}+m^2\phi^*_{\mu\nu}\phi^{\mu\nu})$	$(-\partial^2+m^2)\phi^{\mu\nu}+\partial^\mu\partial_\lambda\phi^{\lambda\nu}+\partial^\nu\partial_\lambda\phi^{\lambda\mu}$	$(2\omega_k)^{-1/2}e^{ia\omega_k t}E^d_{\mathbf{k}\mu\nu}$
	$+\frac{b}{2}(-\partial_\lambda\phi^*\partial^\lambda\phi-m^2\phi^*\phi+\partial^\mu\phi^*_{\mu\nu}\partial^\nu\phi+\partial_\nu\phi^*\partial_\mu\phi^{\mu\nu})$	$-b\{g^{\mu\nu}[(-\partial^2+m^2)\phi+\partial_\kappa\partial_\lambda\phi^{\kappa\lambda}]+\partial^\mu\partial^\nu\phi\}=0$	
	$-\partial^\mu\phi^*_{\mu\nu}\partial_\lambda\phi^{\lambda\nu}\Big\}$ with $\phi=\phi^\mu{}_\mu,\ \phi^{\mu\nu}=\phi^{\nu\mu}$	equivalently $(-\partial^2+m^2)\phi^{\mu\nu}=0,\ \partial_\mu\phi^{\mu\nu}=0,\ \phi^\mu{}_\mu=0$	
$\frac{1}{2}$	$-[\bar\psi(-i\gamma\partial+m)\psi]$	$(-i\gamma\partial+m)\psi=0$	$e^{ia\omega_k t}\chi^{ad}_{\mathbf{k}}$
$\frac{3}{2}$	$-\{\bar\psi_\mu(-i\gamma\partial+m)\psi^\mu$	$(-i\gamma+m)\psi^\mu$	$e^{ia\omega_k t}\chi^{ad}_{\mathbf{k}\mu}$
	$+b[i\bar\psi_\mu(\gamma^\mu\partial_\nu+\gamma_\nu\partial^\mu)\psi^\nu+\bar\psi\gamma(i\gamma\partial+m)\gamma\psi]\}$	$+b[i(\gamma^\mu\partial_\nu+\gamma_\nu\partial^\mu)\psi^\nu+\gamma^\mu(m+i\gamma\partial)\gamma\psi]=0$	
		equivalently $(-i\gamma\partial+m)\psi^\mu=0,\ \gamma_\mu\psi^\mu=0,\ \partial_\mu\psi^\mu=0$	

$$(\phi,\phi)=i\int d^3x\, \phi^* \overleftrightarrow{\partial}_t \phi \ .$$

Here ϕ denotes a scalar or a general tensor with indices suppressed. For spinor fields, correspondingly there is

$$(\psi,\psi)=\int d^3x\, \bar{\psi}\gamma^0\psi \ ,$$

where $\bar{\psi}=\psi^\dagger\gamma^0$, and ψ can be a Dirac spinor or a tensor-Dirac spinor which bears spacetime coordinate indices as well as spinor indices. The tensors $E_{\mathbf{k}}$ and spinors $\chi_{\mathbf{k}}$ appearing in Table I are defined in Appendix A.

It follows from the relations of Appendix A that

$$(\phi^{ad}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}},\phi^{a'd'}(\mathbf{k}',t)e^{i\mathbf{k}'\cdot\mathbf{x}})=-a\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'} \ ,$$

$$(\psi^{ad}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}},\psi^{a'd'}(\mathbf{k}',t)e^{i\mathbf{k}'\cdot\mathbf{x}})=\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'} \ .$$

For a given momentum, these mode functions with $a=\pm 1$ and $d=-s, -s+1, \ldots, s$, form a complete set of $2(2s+1)$ independent solutions.

In a suitably chosen curved spacetime, the field modes will undergo a dynamically induced Bogolubov transformation which mixes positive- and negative-frequency parts. The conserved scalar product then imposes a type of symmetry on the Bogolubov transformation, which in turn selects the proper commutation relation that is propagated consistently by the field equation.

A point of interest in Table I is that in the Lagrangians for spin 2 and spin $\frac{3}{2}$ there appears a factor b which may be chosen to be 1 or $\frac{1}{3}$. For either choice, one obtains the same equation of motion and constraints. In the literature, both values of b for spin $\frac{3}{2}$ have been used,[7,8] but conventionally b is chosen as 1, instead of $\frac{1}{3}$, for spin 2.

III. GENERALIZATION TO CURVED SPACETIME

The vierbein (or tetrad) formalism (see, for example, Ref. 9) is conveniently used to introduce the effects of gravitation on a physical system, once we know its Minkowski-space Lagrangian $\mathcal{L}_0(u,\partial u)$. The vierbein $V_\mu^A(x)$ is defined by

$$V^A{}_\mu(x)V^B{}_\nu(x)\eta_{AB}=g_{\mu\nu}(x) \ .$$

In this section we use $A,B,C,D,\ldots$ as Lorentz indices and $\mu,\nu,\kappa,\lambda,\ldots$ as spacetime coordinate indices. Indices $A,B,\ldots$ are lowered by η_{AB}, and $\mu,\nu,\ldots$ by $g_{\mu\nu}$. The covariant derivative D_A is given by

$$D_A=V_A{}^\mu(\partial_\mu+\tfrac{1}{2}\sigma^{BC}V_B{}^\nu V_{C\nu;\mu})\equiv V_A{}^\mu(\partial_\mu+\Gamma_\mu) \ ,$$

where the constant matrices σ^{AB} generate the matrix $D(\Lambda)$ representing the Lorentz transformation acting on a spin s field u. Thus if $x\rightarrow x'=\Lambda x$ and $u\rightarrow u'=D(\Lambda)u$ with $\Lambda^A{}_B=\delta^A{}_B+\omega^A{}_B$, then $D(\Lambda)=1+\frac{1}{2}\omega^{AB}\sigma_{AB}$, and the matrices σ_{AB} satisfy

$$[\sigma_{AB},\sigma_{CD}]=\sigma_{AD}\eta_{BC}-\sigma_{BD}\eta_{AC}+\sigma_{BC}\eta_{AD}-\sigma_{AC}\eta_{BD} \ .$$

Then the Lagrangian $\mathcal{L}$ in curved spacetime which replaces $\mathcal{L}_0(u,\partial u)$ is

$$\mathcal{L}=|g|^{1/2}\mathcal{L}_0(u,Du) \ .$$

The Lorentz tensors or tensor spinors appearing in $\mathcal{L}$ can be conveniently replaced by the corresponding world tensors or tensor spinors by using the relations (with $\nabla_\mu\phi^\nu$ and $\nabla_\mu\phi^{\nu\kappa}$ denoting the usual spacetime covariant derivatives)

$$\phi^A=V^A{}_\mu\phi^\mu,\quad D_A\phi^B=V_A{}^\mu V^B{}_\nu\nabla_\mu\phi^\nu,\quad \phi^{AB}=V^A{}_\mu V^B{}_\nu\phi^{\mu\nu},\quad D_A\phi^{BC}=V_A{}^\mu V^B{}_\nu V^C{}_\kappa\nabla_\mu\phi^{\nu\kappa} \ ,$$

$$D_A\psi=V_A{}^\mu(\partial_\mu+\Gamma_\mu{}^{1/2})\psi,\quad \Gamma_\mu{}^{1/2}=-\tfrac{1}{8}[\gamma^A,\gamma^B]V_A{}^\xi V_{B\xi;\mu},\quad \psi^A=V^A{}_\mu\psi^\mu,\quad \gamma^A=V^A{}_\mu\gamma^\mu \ ,$$

$$D_A\psi^B=V_A{}^\mu V^B{}_\nu d_\mu\psi^\nu,\quad \text{where } d_\mu\psi^\nu=\nabla_\mu\psi^\nu+\Gamma_\mu{}^{1/2}\psi^\nu,\quad \nabla_\mu\psi^\nu=\partial_\mu\psi^\nu+\Gamma^\nu{}_{\mu\xi}\psi^\xi \ .$$

Here $\Gamma_\mu{}^{1/2}$ is a matrix operating on spinors and $\Gamma^\nu{}_{\mu\xi}$ is the Christoffel symbol. These relations are derived in Appendix B.

It should be noticed that additional terms which describe possible local couplings between the field and gravity can be added to the above Lagrangian $\mathcal{L}$. Since only the conserved scalar product is important to our discussion, and it is not affected by these additions to $\mathcal{L}$, we will ignore them here.

Once the Lagrangian is known for a specific field u, the conserved current j^λ and conserved scalar product are given by the expressions

$$j^\lambda=-i\left[\frac{\partial\mathcal{L}}{\partial\partial_\lambda u}u-\frac{\partial\mathcal{L}}{\partial\partial_\lambda u^*}u^*\right] \ ,$$

$$(u,u)=\int d^3x\, j^0 \ ,$$

where j^λ satisfies $\partial_\lambda j^\lambda=0$.

Table II gives in a general curved spacetime the Lagrangians, field equations, and scalar products for spins up to 2. Notice from the table that, for pure tensors, if ϕ is a solution to the field equation so is its complex conjugate ϕ^*, and that $(\phi^*,\phi^*)=-(\phi,\phi)$.

IV. SOLUTION IN A MODEL UNIVERSE: STATISTICS DERIVED FROM DYNAMICS

In order to show the connection between statistics and dynamical evolution, consider fields in a spatially flat Robertson-Walker universe. The metric $g_{\mu\nu}$ and vierbein $V^A{}_\mu$ for this model spacetime are

$$g_{\mu\nu}=\mathrm{diag}(-1,a^2(t),a^2(t),a^2(t)) \ ,$$

$$V^A{}_\mu=\mathrm{diag}(1,a(t),a(t),a(t)) \ .$$

TABLE II. Lagrangians, field equations, and scalar products of fields of spin 0, 1, 2, $\frac{1}{2}$, $\frac{3}{2}$ in curved spacetime. The coordinate-dependent matrices γ^μ satisfy $\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}$. $\bar{\psi}=\psi^\dagger V^0_\mu\gamma^\mu$, $\bar{\psi}_\mu=\psi^\dagger_\mu V^0_\nu\gamma^\nu$. $b=1$ or $\frac{1}{3}$ as in Table I.

Spin	Lagrangian	Field equation	Scalar product (u,u)
0	$-\lvert g\rvert^{1/2}(\partial^\mu\phi^*\partial_\mu\phi+m^2\phi^*\phi)$	$(-\Box+m^2)\phi=0$	$i\int d^3x\lvert g\rvert^{1/2}(\phi^*\overleftrightarrow{\partial}_t\phi)$
1	$-\lvert g\rvert^{1/2}[\frac{1}{2}(\nabla_\mu\phi^*_\nu-\nabla_\nu\phi^*_\mu)(\nabla^\mu\phi^\nu-\nabla^\nu\phi^\mu)+m^2\phi^{*\mu}\phi_\mu]$	$-\nabla_\mu(\nabla^\mu\phi^\nu-\nabla^\nu\phi^\mu)+m^2\phi^\nu=0$	$-i\int d^3x\lvert g\rvert^{1/2}(\phi^*_\nu\nabla^0\phi^\nu-\nabla^0\phi^*_\nu\phi^\nu)$
		equivalently $(-\Box+m^2)\phi^\nu+g^{\nu\sigma}R_{\rho\sigma}\phi^\rho=0$, $\nabla_\nu\phi^\nu=0$	
2	$-\lvert g\rvert^{1/2}\Big\{\frac{1}{2}(\nabla_\lambda\phi^*_{\mu\nu}\nabla^\lambda\phi^{\mu\nu}+m^2\phi^*_{\mu\nu}\phi^{\mu\nu})$	$(-\Box+m^2)\phi^{\mu\nu}+\nabla^\mu\nabla_\lambda\phi^{\lambda\nu}+\nabla^\nu\nabla_\lambda\phi^{\lambda\mu}$	$-i\int d^3x\lvert g\rvert^{1/2}\{(\phi^*_{\mu\nu}\nabla^0\phi^{\mu\nu}-\nabla^0\phi^{*\mu\nu}\phi_{\mu\nu})$
	$+\frac{b}{2}(-\nabla_\lambda\phi^*\nabla^\lambda\phi-m^2\phi^*\phi+\nabla^\mu\phi^*_{\mu\nu}\nabla^\nu\phi+\nabla^\nu\phi^*\nabla^\mu\phi_{\mu\nu})$	$-b\{g^{\mu\nu}[(-\Box+m^2)\phi+\nabla_\kappa\nabla_\lambda\phi^{\kappa\lambda}]+\nabla^\mu\nabla^\nu\phi\}=0$	$+b[-(\phi^*\partial^0\phi-\partial^0\phi^*\phi)+(\phi^*\nabla_\kappa\phi^{\kappa0}-\nabla_\kappa\phi^{*\kappa0}\phi)$
	$-\nabla^\mu\phi^*_{\mu\nu}\nabla_\lambda\phi^{\lambda\nu}\Big\}$ with $\phi=\phi^\mu{}_\mu$, $\phi^{\mu\nu}=\phi^{\mu\nu}$		$+(\phi^{*0}{}_\nu\nabla^\nu\phi-\nabla^\nu\phi^*\phi^0{}_\nu)]$
			$-2(\phi^{*0}{}_\nu\nabla_\kappa\phi^{\kappa\nu}-\nabla_\kappa\phi^{*\kappa\nu}\phi^0{}_\nu)\}$
$\frac{1}{2}$	$-\lvert g\rvert^{1/2}[-i\bar{\psi}\gamma^\mu(\partial_\mu+\Gamma^{1/2}_\mu)\psi+m\bar{\psi}\psi]$	$[-i\gamma^\mu(\partial_\mu+\Gamma^{1/2}_\mu)+m]\psi=0$	$\int d^3x\lvert g\rvert^{1/2}\bar{\psi}\gamma^a V_a{}^0\psi$
$\frac{3}{2}$	$-\lvert g\rvert^{1/2}\{\bar{\psi}_\nu(-i\gamma^\mu d_\mu+m)\psi^\nu$	$(-i\gamma^\mu d_\mu+m)\psi^\nu$	$\int d^3x\lvert g\rvert^{1/2}[\psi_\mu\gamma^a V_a{}^0\psi^\mu-b(\psi_\sigma\gamma^\sigma\psi^0+\bar{\psi}^0\gamma_\sigma\psi^\sigma$
	$+b[i\bar{\psi}_\mu(\gamma^\mu d_\nu+\gamma_\nu d^\mu)\psi^\nu$	$+b[i(\gamma^\nu d_\mu+\gamma_\mu d^\nu)\psi^\mu+\gamma^\nu(i\gamma^\kappa d_\kappa+m)\gamma_\mu\psi^\mu]=0$	$+\bar{\psi}_\rho\gamma^\rho\gamma^a V_a{}^0\gamma_\sigma\psi^\sigma)]$
	$+\bar{\psi}_\nu\gamma^\nu(i\gamma^\kappa d_\kappa+m)\gamma^\mu\psi_\mu]\}$		

We consider a statistically bounded situation in a period of time $[T_1,T_2]$, which is divided into three intervals $[T_1,t_1]$, $[t_1,t_2]$, and $[t_2,T_2]$. The scale factor $a(t)$ is a constant a_1 in the first interval $[T_1,t_1]$, another constant a_2 in the last interval $[t_2,T_2]$, and changes in an arbitrary manner in the middle interval $[t_1,t_2]$. As the Universe starts expanding or contracting, field quanta are excited out of the original vacuum in $[T_1,t_1]$. If we compare the fields at the beginning and the end, we find that the fields undergo a Bogolubov transformation which is described by the D matrix (defined below). Because there is mixing between positive- and negative-frequency modes, the D matrix is nondiagonal. This nondiagonality of the D matrix determines that only one of the two physically observed statistics is valid for a field with a specific spin.

A. Case I. Bose-Einstein statistics for integer spin fields

It is clear from the previous section that solutions to tensor field equations appear in complex-conjugate pairs. The number of these pairs is equal to $2s+1$, where s is the spin of the field. Hence, we can write the mode solutions

$$\phi^{ad}(\mathbf{x},t)=[Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi^{ad}(\mathbf{k},t) , \tag{3.1}$$

where $a=\pm1$ and $d=-s, -s+1, \ldots, s$, and ϕ is a scalar for spin 0, a vector for spin 1, and a tensor for spin 2. Now $\phi^{ad}(\mathbf{k},t)$ satisfies a second-order ordinary differential equation obtained from the field equation. From Sec. III, if $\phi(\mathbf{x},t)$ is a solution, so is $\phi^*(\mathbf{x},t)$; hence if $\phi^{+1d}(\mathbf{k},t)$ is a solution to the ordinary second-order differential equation having positive-frequency asymptotic behavior, then its complex conjugate with inverse momentum $\phi^{+1d}(-\mathbf{k},t)^*$ is a solution with negative-frequency asymptotic behavior. Therefore, we can choose $\phi^{+1d}(\mathbf{k},t)$ and $\phi^{-1d}(\mathbf{k},t)=\phi^{+1d}(-\mathbf{k},t)^*$ as the complete set of independent solutions to the second-order differential equation which determines the mode solutions with a given momentum. To normalize the mode solution in (3.1), we can choose the normalization constraint to be independent of d and $\mathbf{k}$:

$$N_a=([Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi^{ad}(\mathbf{k},t) , [Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi^{ad}(\mathbf{k},t)) ,$$

but N_a necessarily depends on a, as can be seen from

$$(\phi^*(\mathbf{x},t),\phi^*(\mathbf{x},t))=-(\phi(\mathbf{x},t),\phi(\mathbf{x},t)) .$$

We can choose $N_a=-a$, which implies that

$$([Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi^{ad}(\mathbf{k},t),[Va^3(t)]^{-1/2}e^{i\mathbf{k}'\cdot\mathbf{x}}\phi^{a'd'}(\mathbf{k}',t))=-a\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'} . \tag{3.2}$$

The definition of the scalar product for each spin is given in Table II.

Let $I=1,2$ and define $\phi_I^{ad}(\mathbf{k},t)$ in the following way:

$$\phi_I^{ad}(\mathbf{k},t)=(2\omega_{k_I})^{-1/2}e^{ia\omega_{k_I}t} \quad \text{for spin } s=0 ,$$

$$[\phi_I^{ad}(\mathbf{k},t)]_\mu=\zeta_{(I)\mu}{}^{\nu}(2\omega_{k_I})^{-1/2}e^{ia\omega_{k_I}t}E^d_{\mathbf{k}_I\nu} \quad \text{for spin } s=1 ,$$

$$[\phi_I^{ad}(\mathbf{k},t)]_{\mu\nu}=\zeta_{(I)\mu}{}^{\rho}\zeta_{(I)\nu}{}^{\sigma}(2\omega_{k_I})^{-1/2}e^{ia\omega_{k_I}t}E^d_{\mathbf{k}_I\rho\sigma} \quad \text{for spin } s=2 ,$$

with $\mathbf{k}_I=\mathbf{k}/a_I$, $\zeta_{(I)\mu}{}^{\nu}=\mathrm{diag}(1,a_I,a_I,a_I)$ and the tensors E defined in Table I. In this way, the functions $(Va_I^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_I^{ad}(\mathbf{k},t)$ have the normalization of Eq. (3.2) with t in the intervals $[t_I,T_I]$ and $a(t)=a_I$.

We take $\phi^{ad}(\mathbf{x},t)$ to have the following form in the initial interval:

$$\phi^{ad}(\mathbf{x},t)=(Va_1^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_1^{ad}(\mathbf{k},t) \quad \text{when } t \text{ is in } [T_1,t_1] .$$

Then the solution $\phi^{ad}(\mathbf{x},t)$ having the above specified form in the initial interval, must be a linear combination of ϕ_2^{ad} of the following form in the final interval:

$$\phi^{ad}(\mathbf{x},t)=\sum_{a'd'}D_{a'd',ad}(Va_2^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_2^{a'd'}(\mathbf{k},t) \quad \text{when } t \text{ is in } [t_2,T_2] .$$

The constant D matrix introduced here depends on $\mathbf{k}$, but does not mix different $\mathbf{k}$ values because of the spatial homogeneity. In general $D_{a'd',ad}$ will not reduce to $\delta_{aa'}\delta_{dd'}$ because of mixing positive- and negative-frequency solutions. Conservation of the scalar product (3.2) implies that

$$\left[\sum_{a'd'}D_{a'd',ad}(Va_2^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_2^{a'd'}(\mathbf{k}_2,t), \sum_{b'e'}D_{b'e',be}(Va_2^3)^{-1/2}e^{i\mathbf{k}'\cdot\mathbf{x}}\phi_2^{b'e'}(\mathbf{k}_2't,t)\right]$$

$$=\sum_{a'd'}\sum_{b'e'}D^*_{a'd',ad}D_{b'e',be}(-a')\delta_{a'b'}\delta_{d'e'}\delta_{\mathbf{k}\mathbf{k}'}=-a\delta_{ab}\delta_{de}\delta_{\mathbf{k}\cdot\mathbf{k}'} ,$$

where the last line is the scalar product between $[Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi^{ad}(\mathbf{k},t)$ and $[Va^3(t)]^{-1/2}e^{i\mathbf{k}'\cdot\mathbf{x}}\phi^{be}(\mathbf{k}',t)$. Therefore,

$$\sum_{a'd'}(-a')D^*_{a'd',ad}D_{a'd',be}=-a\delta_{ab}\delta_{de}\ . \tag{3.3}$$

Through a little bit of algebra, one can show from (3.3) that

$$\sum_{a'd'}(-a')D_{ad,a'd'}D^*_{be,a'd'}=-a\delta_{ab}\delta_{de}\ . \tag{3.4}$$

Furthermore, if $D_{ad,a'd'}$ does not commute with the matrix $(-a)\delta_{aa'}\delta_{dd'}$, which is the case when $D_{ad,a'd'}$ is not diagonal in aa' due to mixing between positive- and negative-frequency solutions, one has

$$\sum_{a'd'}D_{ad,a'd'}D^*_{be,a'd'}\neq\delta_{ab}\delta_{de}\ . \tag{3.5}$$

The field operator ϕ can be expressed in terms of creation and annihilation operators of particles in the modes specified earlier:

$$\phi=\sum_{ad\mathbf{k}}[Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi^{ad}(\mathbf{k},t)A^{ad}_{\mathbf{k}}\ . \tag{3.6}$$

In the early and late universe the field can be written as

$$\phi=\sum_{ad\mathbf{k}}(Va_1^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_1^{ad}(\mathbf{k},t)A^{ad}_{\mathbf{k}}\quad\text{when } t \text{ is in } [T_1,t_1]\ , \tag{3.7}$$

$$\phi=\sum_{ad\mathbf{k}}(Va_2^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_2^{ad}(\mathbf{k},t)a^{ad}_{\mathbf{k}}\quad\text{when } t \text{ is in } [t_2,T_2]\ , \tag{3.8}$$

where the $a^{ad}_{\mathbf{k}}$ annihilate particles at late times. Equation (3.6) reduces to Eq. (3.7) when t is in $[T_1,t_1]$, but after the system evolves, so that t is in $[t_2,T_2]$, Eq. (3.6) assumes the form

$$\phi=\sum_{ad\mathbf{k}}\sum_{a'd'}(Va_2^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}D_{a'd',ad}\phi_2^{a'd'}(\mathbf{k},t)A^{ad}_{\mathbf{k}}\quad\text{when } t \text{ is in } [t_2,T_2]\ .$$

This must be the same as (3.8), so

$$a^{ad}_{\mathbf{k}}=\sum_{a'd'}D_{ad,a'd'}A^{a'd'}_{\mathbf{k}}\ . \tag{3.9}$$

Our fields are complex, so that $A^{+1d}_{\mathbf{k}}$ is the creation operator for an antiparticle in mode $\phi_1^{+1d}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}$ and $A^{-1d}_{\mathbf{k}}$ is the annihilation operator for a particle in mode $\phi_1^{-1,d}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}$. The commutation relations can be written in a compact form.

Bose-Einstein:

$$[A^{ad}_{\mathbf{k}},A^{a'd'\dagger}_{\mathbf{k}}]=-a\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'}\ ,\quad [A^{ad}_{\mathbf{k}},A^{a'd'}_{\mathbf{k}'}]=0\ . \tag{3.10}$$

Fermi-Dirac:

$$\{A^{ad}_{\mathbf{k}},A^{a'd'\dagger}_{\mathbf{k}'}\}=\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'}\ ,\quad \{A^{ad}_{\mathbf{k}},A^{a'd'}_{\mathbf{k}'}\}=0\ . \tag{3.11}$$

Using (3.4), (3.5), and (3.9),

$$[a^{ad}_{\mathbf{k}},a^{a'd'\dagger}_{\mathbf{k}'}]=\delta_{\mathbf{k}\mathbf{k}'}\sum_{be}(-b)D_{ad,be}D^*_{a'd',be}=-a\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'}\ ,$$

$$[a^{ad}_{\mathbf{k}},a^{a',d'}_{\mathbf{k}'}]=0\ ,$$

$$\{a^{ad}_{\mathbf{k}},a^{a'd'\dagger}_{\mathbf{k}'}\}=\delta_{\mathbf{k}\mathbf{k}'}\sum_{be}D_{ad,be}D^*_{a'd',be}\neq\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'}\ ,$$

if there is mixing between negative and positive modes.

Therefore we claim that the Bose-Einstein commutation relation is the invariant relation that should be imposed to quantize the field. By invariant we mean the commutation relation is carried along with the physical system's evolution without changing form.

This argument applies to neutral fields as well, where $A^{+1d}_{-\mathbf{k}}$ and $A^{-1d}_{\mathbf{k}}$ are the creation and annihilation operator for particle in mode $e^{i\mathbf{k}\cdot\mathbf{x}}\phi^{-1d}(\mathbf{k},t)$.

B. Case II. Fermi-Dirac statistics for half-integer-spin fields

Because the coefficients of the spinor field equations are complex matrix functions, it is no longer true that solutions exist as complex-conjugate pairs. The spinor equation is of first order. It is known from the theory of first-order differential equation system that the eigenvectors $[Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi^{ad}(\mathbf{k},t)$, called mode functions here, can be normalized such that

$$([Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi^{ad}(\mathbf{k},t),[Va^3(t)]^{-1/2}e^{i\mathbf{k}'\cdot\mathbf{x}}\psi^{a'd'}(\mathbf{k}',t))=\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'}\ . \tag{3.12}$$

Similar to case I, let $I=1,2$, define $\psi_I^{ad}(\mathbf{k},t)$ as

$$\psi_I^{ad}(\mathbf{k},t)=e^{ia\omega_{k_I}t}\chi^{ad}_{\mathbf{k}_I}\quad\text{for spin } s=\tfrac{1}{2}\ ,$$

$$[\psi_I^{ad}(\mathbf{k},t)]_\mu=\zeta_{(I)\mu}{}^{\nu}e^{ia\omega_{k_I}t}\chi^{ad}_{\mathbf{k}_I\nu}\quad\text{for spins } s=\tfrac{3}{2}\ ,$$

where χ are defined in Table I. Then functions $(Va_I^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi_I^{ad}(\mathbf{k},t)$ have the normalization of Eq. (3.12) where t is in intervals $[t_I,T_I]$.

We take the mode functions to have initial form

$$\psi^{ad}(\mathbf{x},t)=(Va_I^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi_1^{ad}(\mathbf{k},t)\quad\text{when } t \text{ is in } [T_1,t_1]\ .$$

Then

$$\psi^{ad}(\mathbf{x},t)=\sum_{a'd'}(Va_2^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi_2^{a'd'}(\mathbf{k},t)D_{a'd',ad} \quad \text{when } t \text{ is in } [t_2,T_2] \ .$$

From (3.12) one has

$$\left[\sum_{a'd'}D_{a'd',ad}(Va_2^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi_2^{a'd'}(\mathbf{k},t),\ \sum_{b'e'}D_{b'e',be}(Va_2^3)^{-1/2}e^{i\mathbf{k}'\cdot\mathbf{x}}\psi_2^{b'e'}(\mathbf{k}',t)\right]=\delta_{\mathbf{k}\mathbf{k}'}\delta_{ab}\delta_{de} \ .$$

Therefore,

$$\sum_{a'd'}D^*_{a'd',ad}D_{a'd',be}=\delta_{ab}\delta_{de}$$

or

$$\sum_{a'd'}D_{ad,a'd'}D^*_{be,a'd'}=\delta_{ab}\delta_{de} \tag{3.13}$$

and

$$\sum_{a'd'}(-a')D_{ad,a'd'}D^*_{be,a'd'}\neq -a\delta_{ab}\delta_{de} \ , \tag{3.14}$$

if $D_{ad,a'd'}$ does not commute with $-a\delta_{aa'}\delta_{dd'}$.

Like tensor field operators, the spinor field operators are also undergoing Bogolubov transformations:

$$\begin{aligned}
\psi&=\sum_{ad\mathbf{k}}[Va^3(t)]^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi^{ad}(\mathbf{k},t)A_{\mathbf{k}}^{ad} \ ,\\
\psi&=\sum_{ad\mathbf{k}}(Va_1^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi_1^{ad}(\mathbf{k},t)A_{\mathbf{k}}^{ad} \quad \text{when } t \text{ is in } [T_1,t_1] \ ,\\
\psi&=\sum_{ad\mathbf{k}}(Va_2^3)^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\psi_2^{ad}(\mathbf{k},t)a_{\mathbf{k}}^{ad} \quad \text{when } t \text{ is in } [t_2,T_2] \ ,\\
a_{\mathbf{k}}^{ad}&=\sum_{a'd'}D_{ad,a'd'}A_{\mathbf{k}}^{a'd'} \ .
\end{aligned} \tag{3.15}$$

One now finds by using (3.13)–(3.15), that of the two quantization schemes (3.10) and (3.11), only the Fermi-Dirac quantization is invariant under transformation induced by the field's temporal development:

$$\begin{aligned}
\{a_{\mathbf{k}}^{ad},a_{\mathbf{k}'}^{a'd'\dagger}\}&=\delta_{\mathbf{k}\mathbf{k}'}\sum_{be}D_{ad,be}D^*_{a'd',be}\\
&=\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'} \ ,\\
\{a_{\mathbf{k}}^{ad},a_{\mathbf{k}'}^{a'd'\dagger}\}&=0 \ ,\\
[a_{\mathbf{k}}^{ad},a_{\mathbf{k}'}^{a'd'\dagger}]&=\delta_{\mathbf{k}\mathbf{k}'}\sum_{be}(-b)D_{ad,be}D^*_{a'd',be}\\
&\neq -a\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'}
\end{aligned}$$

because of induced mixing between negative and positive modes.

V. THE POSSIBILITY OF PARASTATISTICS

A more general method of quantization called paraquantization has been studied in the literature.[10] The basic commutation relation for paraquantization of a given order p are

$$\begin{aligned}
&[a_k,[a_l^\dagger,a_m]_\mp]=2\delta_{kl}a_m \ ,\\
&[a_k,[a_l,a_m]_\mp]=0 \ ,
\end{aligned} \tag{4.1}$$

$$a_ka_l^\dagger|0\rangle=p\delta_{kl}|0\rangle \ . \tag{4.2}$$

Here $|0\rangle$ is defined by $a_k|0\rangle=0$ for all field modes k; $[A,B]_\mp=AB\mp BA$, with upper sign referring to para-Fermi statistics and the lower sign referring to the para-Bose statistics; and p is a positive integer. For $p=0,1,2$ (4.1) together with (4.2) are equivalent to the following self-contained sets of commutation relations:[11]

$$p=0: \quad a_k=a_k^\dagger=0 \ ,$$

$$p=1: \quad [a_k,a_l^\dagger]_\pm=\delta_{kl'}, \quad [a_k,a_l]_\pm=0 \ , \tag{4.3}$$

$$\begin{aligned}
p=2: \quad &\langle a_k,a_l^\dagger,a_m\rangle_\pm=2\delta_{kl}a_m\pm 2\delta_{lm}a_k \ ,\\
&\langle a_k,a_l,a_m^\dagger\rangle_\pm=2\delta_{lm}a_k \ ,\\
&\langle a_k,a_l,a_m\rangle_\pm=0 \ ,
\end{aligned} \tag{4.4}$$

where $\langle A,B,C\rangle_\pm=ABC\pm CBA$.

In the last section, we have seen that the ordinary commutation relations, which correspond to $p=1$, are dynamically invariant. Indeed, one can also see this by examining (4.1) and (4.2) if they are dynamically invariant when $p=1$. For example, consider the Bose case. In the notation of the previous section, Eqs. (4.1) and (4.2) with $p=1$ take the form:

$$\begin{aligned}
&[a_{\mathbf{k}}^{ad},\{a_{\mathbf{k}'}^{a'd'\dagger},a_{\mathbf{k}''}^{a''d''}\}]=2(-a)\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}''}^{a''d''} \ ,\\
&[a_{\mathbf{k}}^{ad},\{a_{\mathbf{k}'}^{a'd'},a_{\mathbf{k}''}^{a''d''}\}]=0 \ ,
\end{aligned} \tag{4.5}$$

$$a_{\mathbf{k}'}^{a'd'}a_{\mathbf{k}}^{ad\dagger}|0\rangle=\delta_{aa'}\delta_{dd'}\delta_{\mathbf{k}\mathbf{k}'}|0\rangle \ . \tag{4.6}$$

When (4.5) is subjected to (3.4), a Bogolubov transformation of Bose type, its form does not change; but it does change when subjected to (3.13), a Bogolubov transformation of Fermi type. At the same time, (4.6) is invariant. Consider the scalar field. The Bogolubov transformation is $a_{\mathbf{k}}=\alpha_k A_{\mathbf{k}}+\beta_k^* A_{-\mathbf{k}}^\dagger$ with $|\alpha_k|^2-|\beta_k|^2=1$, and the late-time vacuum is[4]

$$|0)\equiv\prod_{\mathbf{k}}\alpha_k^{-1}\sum_{n_{\mathbf{k}}=0}^{\infty}(-\beta_k^*/\alpha_k)^{n_{\mathbf{k}}}|n_{\mathbf{k}},n_{-\mathbf{k}}\rangle \ .$$

From $A_{\mathbf{k}}A_{\mathbf{k}}^\dagger|0\rangle=|0\rangle$ it is easy to see that $a_ka_k^\dagger|0)=|0)$.

The above method cannot be easily generalized to prove the dynamical invariance of the commutation relation of an arbitrary order p. It is obvious that (4.1) is invariant, but invariance of (4.2) cannot be straightforwardly justified due to the fact that, to our knowledge, the matrix elements of the creation and annihilation operators for $p>1$ have not been calculated.

However, we can verify that the self-contained sets of commutation relations of order $p=2$ are dynamically invariant. That can be seen by writing (4.4) in the notation of Sec. IV. Then one has

$$\langle a_{\mathbf{k}'}^{a'd'}, a_{\mathbf{k}}^{ad\dagger}, a_{\mathbf{k}''}^{a''d''} \rangle_+ = 2(\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}\delta_{dd'}a_{\mathbf{k}''}^{a''d''} + \delta_{\mathbf{k}\mathbf{k}''}\delta_{aa''}\delta_{dd''}a_{\mathbf{k}'}^{a'd'}) ,$$

$$\langle a_{\mathbf{k}'}^{a'd'}, a_{\mathbf{k}}^{ad\dagger}, a_{\mathbf{k}''}^{a''d''} \rangle_- = 2(-a)(\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'}\delta_{dd'}a_{\mathbf{k}''}^{a''d''} - \delta_{\mathbf{k}\mathbf{k}''}\delta_{aa''}\delta_{dd''}a_{\mathbf{k}'}^{a'd'}) ,$$

$$\langle a_{\mathbf{k}'}^{ad}, a_{\mathbf{k}'}^{a'd'}, a_{\mathbf{k}''}^{a''d''} \rangle_+ = 2\delta_{\mathbf{k}'\mathbf{k}''}\delta_{a'a''}\delta_{d'd''}a_{\mathbf{k}}^{ad} ,$$

$$\langle a_{\mathbf{k}}^{ad}, a_{\mathbf{k}'}^{a'd'}, a_{\mathbf{k}''}^{a''d''\dagger} \rangle_- = 2(-a')\delta_{\mathbf{k}'\mathbf{k}''}\delta_{a'a''}\delta_{d'd''}a_{\mathbf{k}}^{ad} ,$$

$$\langle a_{\mathbf{k}}^{ad}, a_{\mathbf{k}}^{a'd'}, a_{\mathbf{k}''}^{a''d''} \rangle_\pm = 0 .$$

The above sets of relations are invariant when subjected to the Bogolubov transformation corresponding to the appropriate spin fields as can be found by using para-Bose commutation relations for integer spin fields and para-Fermi commutation relations for half-integer spin fields.

The self-contained sets of commutation relations of order $p=3,4$ have also been checked to be invariant. See Appendix C.

VI. SUMMARY AND DISCUSSION

In conclusion, we have shown that the symmetries and nondiagonality of the dynamically induced Bogolubov transformation determine the statistics for a field propagating in a curved spacetime with two embedded flat regions. Various commutation relations were considered in the flat regions, and the spin-statistics theorem in flat spacetime was inferred by showing that only when the appropriate commutation relations were imposed were they consistent with the dynamics. One can also infer the flat spin-statistics theorem from physical continuity by taking the limit in which the deviation from flat spacetime becomes arbitrarily small, and requiring that the dynamically consistent commutation relation does not discontinuously change in the limit of flat spacetime. We note, however, that the demonstration in this paper is nonperturbative. Our proof makes essential use of curved spacetime dynamics because in flat spacetime the Bogolubov transformation reduces to the identity. The quantum fields we consider are interacting with a gravitational field of arbitrary strength, but are free in other respects.

Finally, we note that the commutation or anticommutation relations of the creation and annihilation operator are equivalent to the corresponding commutation or anticommutation relations of the fields and their conjugate moments.

In this paper we have considered massive fields with spin value up to 2. Our method of proof should work for a field of any spin if its Lagrangian is known.

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APPENDIX A

We define the tensors $E_{\mathbf{k}}$ and spinors $\chi_{\mathbf{k}}$ appearing in Table I in the following. $E_{\mathbf{k}\mu}^{d}$ for spin-1 field:

$$E_{\mathbf{k}}^{0} = (|\mathbf{k}|/m, k^0\mathbf{k}/m|\mathbf{k}|) ,$$

$$E_{\mathbf{k}}^{\pm 1} = (0, \mp 2^{-1/2}(\mathbf{e}_{\mathbf{k}1} \pm i\mathbf{e}_{\mathbf{k}2})) ,$$

$$|\mathbf{e}_{\mathbf{k}1}|^2 = |\mathbf{e}_{\mathbf{k}2}|^2 = 1, \quad \mathbf{e}_{\mathbf{k}1}\cdot\mathbf{e}_{\mathbf{k}2} = \mathbf{e}_{\mathbf{k}1}\cdot\mathbf{k} = \mathbf{e}_{\mathbf{k}2}\cdot\mathbf{k} = 0 ,$$

$$E_{\mathbf{k}\mu}^{*d}E_{\mathbf{k}\nu}^{d'}\eta^{\mu\nu} = \delta_{dd'}, \quad E_{\mathbf{k}\mu}^{*d} = (-1)^d E_{\mathbf{k}\mu}^{-d} ,$$

$$k^\mu E_{\mathbf{k}\mu}^{d} = 0 .$$

$E_{\mathbf{k}\mu\nu}^{d}$ for spin-2 field:

$$E_{\mathbf{k}\mu\nu}^{\pm 2} = E_{\mathbf{k}\mu}^{\pm 1}E_{\mathbf{k}\nu}^{\pm 1} ,$$

$$E_{\mathbf{k}\mu\nu}^{\pm 1} = 2^{-1/2}(E_{\mathbf{k}\mu}^{\pm 1}E_{\mathbf{k}\nu}^{0} + E_{\mathbf{k}\mu}^{0}E_{\mathbf{k}\nu}^{\pm 1}) ,$$

$$E_{\mathbf{k}\mu\nu}^{0} = 6^{-1/2}(E_{\mathbf{k}\mu}^{+1}E_{\mathbf{k}\nu}^{-1} + E_{\mathbf{k}\mu}^{-1}E_{\mathbf{k}\nu}^{+1} + 2E_{\mathbf{k}\mu}^{0}E_{\mathbf{k}\nu}^{0}) ,$$

$$E_{\mathbf{k}\mu\nu}^{*d}\, E_{\mathbf{k}\mu'\nu'}^{d'}\eta^{\mu\mu'}\eta^{\nu\nu'} = \delta_{dd'} ,$$

$$k^\mu E_{\mathbf{k}\mu\nu}^{d} = 0, \quad \eta^{\mu\nu}E_{\mathbf{k}\mu\nu}^{d} = 0 .$$

$\chi_{\mathbf{k}}^{ad}$ for spin-$\frac{1}{2}$ field:

$$\chi_{\mathbf{k}}^{ad} = \begin{bmatrix} \eta_{\mathbf{k}}^{d} \\ \dfrac{\sigma\cdot\mathbf{k}}{m - a\omega_k}\eta_{\mathbf{k}}^{d} \end{bmatrix}, \quad \eta_{\mathbf{k}}^{d} = \begin{bmatrix} 1 \\ \dfrac{k_1 + ik_2}{\mathrm{sgn}(d)|\mathbf{k}| + k_3} \end{bmatrix} .$$

Note that the a,d in the denominator above have the same value as the a,d in the superscript on $\chi_{\mathbf{k}}^{ad}$:

$$\chi_{\mathbf{k}}^{\dagger ad}\chi_{\mathbf{k}}^{a'd'} = \delta_{aa'}\delta_{dd'} .$$

$\chi_{\mathbf{k}\mu}^{ad}$ for spin-$\frac{3}{2}$ field:

$$\chi_{\mathbf{k}\mu}^{a\pm 3/2} = E_{\mathbf{k}\mu}^{\pm 1}\chi_{\mathbf{k}}^{a\pm 1/2} ,$$

$$\chi_{\mathbf{k}\mu}^{a\pm 1/2} = 3^{-1/2}(2^{1/2}E_{\mathbf{k}\mu}^{0}\chi_{\mathbf{k}}^{a\pm 1/2} + E_{\mathbf{k}\mu}^{\pm 1}\chi_{\mathbf{k}}^{a\mp 1/2}) ,$$

$$\chi_{\mathbf{k}\mu}^{\dagger ad}\chi_{\mathbf{k}\nu}^{a'd'}\eta^{\mu\nu} = \delta_{aa'}\delta_{dd'} ,$$

$$\gamma^\mu\chi_{\mathbf{k}\mu}^{ad} = 0, \quad k^\mu\chi_{\mathbf{k}\mu}^{ad} = 0 .$$

APPENDIX B

In this appendix we derive the covariant derivative relations used in the vierbein formalism. For a Lorentz vector ϕ^A, $D_A\phi^B = V_A{}^{\mu}(\partial_\mu + \Gamma_\mu^{(1)})^B{}_C\phi^C$,

$$(\Gamma_\mu^{(1)})^B{}_C = \tfrac{1}{2}(\sigma_{DE}^{(1)})^B{}_C V^{D\xi}V^{E}{}_{\xi;\mu} ,$$

$$(\sigma_{DE}^{(1)})^B{}_C = \delta_D{}^B\eta_{EC} - \delta_E{}^B\eta_{DC} = 2\delta_{[D}{}^B\eta_{E]C} ,$$

$$(\Gamma_\mu^{(1)})^B{}_C\phi^C = \tfrac{1}{2}V^D{}_\xi V^{E\xi}{}_{;\mu}2\delta_{[D}{}^B\eta_{E]C}\phi^C = V^B{}_\xi V_C{}^{\xi}{}_{;\mu}\phi^C = -V^B{}_{\xi;\mu}\phi^\xi ,$$

$$(\partial_\mu+\Gamma^{(1)}_\mu)^B{}_C\phi^C=\partial_\mu(V^B{}_\nu\phi^\nu)-V^B{}_{\xi;\mu}\phi^\xi$$
$$=V^B{}_\nu(\phi^\nu{}_{,\mu}+\Gamma^\nu{}_{\xi\mu}\phi^\xi)\ .$$

Then

$$D_A\phi^B=V_A{}^\mu V^B{}_\nu\nabla_\mu\phi^\nu\ .$$

For a Lorentz tensor ϕ^{AB}, one has $D_A\phi^{BC}=V_A{}^\mu V^B{}_\nu V^C{}_\kappa\nabla_\mu\phi^{\nu\kappa}$, which can be obtained similarly by noting that

$$(\sigma^{(2)}_{AB})^{CD}{}_{EF}=(\sigma^{(1)}_{AB})^C{}_E\delta^D{}_F+\delta^C{}_E(\sigma^{(1)}_{AB})^D{}_F\ .$$

For a Dirac spinor,

$$(\sigma^{(1/2)}_{AB})^\alpha{}_\beta=-\tfrac{1}{4}([\gamma_A,\gamma_B])^\alpha{}_\beta$$

with $\{\gamma_A,\gamma_B\}=-2\eta_{AB}$. For a spinor-tensor ψ^A, we have

$$(\sigma^{(3/2)}_{AB})^{\alpha D}{}_{\beta F}=(\sigma^{(1/2)}_{AB})^\alpha{}_\beta\delta^D{}_F+\delta^\alpha{}_\beta(\sigma^{(1)}_{AB})^D{}_F\ ,$$

where α,β are the spinor indices. Then the relations listed in Sec. II follow.

APPENDIX C

To show the higher-order paracommutation relations invariant under a Bogolubov transformation induced by the field's time evolution, we introduce the following notations:[11]

$$a_{\hat k}=(a_k,a_k^\dagger)$$

with a_k the annihilation operator and $a_k^\dagger$ the creation operator:

$$\delta^{(\pm)}(\hat k,\hat l)=\begin{cases}0 & \text{if } a_{\hat k}=a_k, a_{\hat l}=a_l\ ,\\ 0 & \text{if } a_{\hat k}=a_k^\dagger, a_{\hat l}=a_l^\dagger\ ,\\ \delta_{kl} & \text{if } a_{\hat k}=a_k, a_{\hat l}=a_l^\dagger\ ,\\ \pm\delta_{kl} & \text{if } a_{\hat k}=a_k^\dagger m, a_{\hat l}=a_l\ ,\end{cases}$$

with the upper sign for para-Fermi case and the lower sign for para-Bose case:

$$[a_{\hat 1},a_{\hat 2},\dots,a_{\hat n}]_\pm=\sum_{\pi\in S_n}\begin{bmatrix}1\\ \mathrm{sgn}(\pi)\end{bmatrix}a_{\pi\hat 1}a_{\pi\hat 2}\cdots a_{\pi\hat n}\ ,$$

with S_n the permutation group, and 1 referring to para-Fermi case and $\mathrm{sgn}(\pi)$ referring to para-Bose case.

Recalling (3.13) and (3.3), one has

$$\sum_{\substack{a'd'\\ b'e'}}D_{ad,a'd'}\begin{bmatrix}1\\ -a'\end{bmatrix}\delta_{a'b'}\delta_{d'e'}\delta_{\mathbf{k}\cdot\mathbf{k}'}D^*_{be,b'e'}=\begin{bmatrix}1\\ -a\end{bmatrix}\delta_{ab}\delta_{de}\delta_{\mathbf{k}\cdot\mathbf{k}'}\ .$$

Here upper 1 refers to para-Fermi case and lower $-a$ refers to para-Bose case. The above

$$\begin{bmatrix}1\\ -a\end{bmatrix}\delta_{ab}\delta_{de}\delta_{\mathbf{k}\cdot\mathbf{k}'}$$

is just the $\delta^{(\pm)}(\hat k,\hat l)$. Therefore $\delta^{(\pm)}(\hat k,\hat l)$ is invariant under Bogolubov transformation. Using the analogy with Lorentz invariance, the Bogolubov invariance of the following self-contained commutation relations follow:[11]

$$p=0:\quad a_{\hat k}=0\ ,$$

$$p=1:\quad [a_{\hat k},a_{\hat l}]_\pm=\delta^{(\pm)}(\hat k,\hat l)\ ,$$

$$p=2:\quad [a_{\hat k},a_{\hat l},a_{\hat m}]_\pm=4[\delta^{(\pm)}(\hat k,\hat l)a_{\hat m}+\delta^{(\pm)}(\hat l,\hat m)a_{\hat k}+\delta^{(\pm)}(\hat m,\hat k)a_{\hat l}]\ ,$$

$$\begin{aligned}p=3:\quad [a_{\hat k},a_{\hat l},a_{\hat m},a_{\hat n}]_\pm=10\{&\delta^{(\pm)}(\hat k,\hat l)[a_{\hat m},a_{\hat n}]_\pm+\delta^{(\pm)}(\hat k,\hat m)[a_{\hat n},a_{\hat l}]_\pm+\delta^{(\pm)}(\hat k,\hat n)[a_{\hat l},a_{\hat m}]_\pm\\ &+\delta^{(\pm)}(\hat l,\hat m)[a_{\hat k},a_{\hat n}]_\pm+\delta^{(\pm)}(\hat l,\hat n)[a_{\hat m},a_{\hat k}]_\pm+\delta^{(\pm)}(\hat m,\hat n)[a_{\hat k},a_{\hat l}]_\pm\}\\ &-18[\delta^{(\pm)}(\hat k,\hat l)\delta^{(\pm)}(\hat m,\hat n)+\delta^{(\pm)}(\hat k,\hat m)\delta^{(\pm)}(\hat n,\hat l)+\delta^{(\pm)}(\hat k,\hat n)\delta^{(\pm)}(\hat l,\hat m)]\ ,\end{aligned}$$

$$\begin{aligned}p=4:\quad [a_{\hat k},a_{\hat l},a_{\hat m},a_{\hat n},a_{\hat r}]_\pm=20\{&\delta^{(\pm)}(\hat k,\hat l)[a_{\hat m},a_{\hat n},a_{\hat r}]_\pm+\delta^{(\pm)}(\hat k,\hat m)[a_{\hat n},a_{\hat l},a_{\hat r}]_\pm\\ &+\delta^{(\pm)}(\hat k,\hat n)[a_{\hat l},a_{\hat m},a_{\hat r}]_\pm+\delta^{(\pm)}(\hat k,\hat r)[a_{\hat n},a_{\hat m},a_{\hat l}]_\pm\\ &+\delta^{(\pm)}(\hat l,\hat m)[a_{\hat k},a_{\hat n},a_{\hat r}]_\pm+\delta^{(\pm)}(\hat l,\hat n)[a_{\hat r},a_{\hat m},a_{\hat k}]_\pm\\ &+\delta^{(\pm)}(\hat l,\hat r)[a_{\hat k},a_{\hat m},a_{\hat n}]_\pm+\delta^{(\pm)}(\hat m,\hat n)[a_{\hat k},a_{\hat l},a_{\hat r}]_\pm\\ &+\delta^{(\pm)}(\hat m,\hat r)[a_{\hat n},a_{\hat l},a_{\hat k}]_\pm+\delta^{(\pm)}(\hat n,\hat r)[a_{\hat k},a_{\hat l},a_{\hat m}]_\pm\}\\ -128[&\delta^{(\pm)}(\hat k,\hat l)\delta^{(\pm)}(\hat m,\hat n)a_{\hat r}+\delta^{(\pm)}(\hat k,\hat l)\delta^{(\pm)}(\hat r,\hat m)a_{\hat n}+\delta^{(\pm)}(\hat k,\hat l)\delta^{(\pm)}(\hat n,\hat r)a_{\hat m}\\ &+\delta^{(\pm)}(\hat k,\hat m)\delta^{(\pm)}(\hat n,\hat l)a_{\hat r}+\delta^{(\pm)}(\hat k,\hat m)\delta^{(\pm)}(\hat l,\hat r)a_{\hat n}+\delta^{(\pm)}(\hat k,\hat m)\delta^{(\pm)}(\hat r,\hat n)a_{\hat l}\\ &+\delta^{(\pm)}(\hat k,\hat n)\delta^{(\pm)}(\hat l,\hat m)a_{\hat r}+\delta^{(\pm)}(\hat k,\hat n)\delta^{(\pm)}(\hat r,\hat l)a_{\hat m}+\delta^{(\pm)}(\hat k,\hat n)\delta^{(\pm)}(\hat m,\hat r)a_{\hat l}\\ &+\delta^{(\pm)}(\hat k,\hat r)\delta^{(\pm)}(\hat m,\hat l)a_{\hat n}+\delta^{(\pm)}(\hat k,\hat r)\delta^{(\pm)}(\hat l,\hat n)a_{\hat m}+\delta^{(\pm)}(\hat k,\hat r)\delta^{(\pm)}(\hat n,\hat m)a_{\hat l}\\ &+\delta^{(\pm)}(\hat l,\hat m)\delta^{(\pm)}(\hat n,\hat r)a_{\hat k}+\delta^{(\pm)}(\hat l,\hat n)\delta^{(\pm)}(\hat r,\hat m)a_{\hat k}+\delta^{(\pm)}(\hat l,\hat r)\delta^{(\pm)}(\hat m,\hat n)a_{\hat k}]\ .\end{aligned}$$

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