Evolution and classification of cosmological perturbations

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The evolution of perturbations in a Friedmann universe is investigated by using a new method. In a first step fluctuations of the density, the pressure, and the four-velocity are removed by performing a gauge transformation. Subsequently the problem can be solved and that solution is finally transformed into some important gauges. With the help of this method the evolutions of all possible perturbations are classified; this leads to a physical interpretation of the obtained solutions. The complete solution of the density fluctuation in the de Donder gauge is given here for the first time; the result obtained by Rose, Rahmstorf, and Dehnen is not valid in the case of a really perturbed universe.

I. INTRODUCTION

The investigation of gravitational instability of perturbations in a homogeneous isotropic expanding universe is usually performed by employing linearized field equations; i.e., the relative smallness of the perturbations in pressure, density, and velocity as well as a coordinate system with a metric near the Robertson-Walker metric is presumed. We also proceed in such a manner: we are interested only in the beginning of the development of a small perturbation. One has to bear in mind that an instability analysis in such a way is not justified if effects of higher order, becoming more and more important when the perturbation grows, reverse this tendency.

Perturbation quantities are constructed by subtracting from the full quantity at a space-time point x^λ in the perturbed universe the background quantity at the corresponding space-time point x^λ in the fictitious Friedmann universe. The choice of such a correspondence defines a gauge. We define other gauges by performing infinitesimal transformations of the coordinates of the perturbed space-time, keeping the background coordinates fixed. By means of a so-called gauge condition—a condition concerning some of the perturbation quantities—we select a definite set of coordinate systems. Unfortunately, the perturbation quantities are gauge dependent; therefore, only some gauges yield a relative density fluctuation suitable as an indicator for stability of that perturbation.

In the literature one finds some gauges which were proposed in this context; Refs. 1–3 contain a selection of those. The results are systems of coupled differential equations for the perturbation quantities which are, mostly, soluble only in special cases. Another treatment in order to avoid the choice of a gauge condition was performed by Bardeen.[4] Out of metric and matter perturbations he constructed gauge-invariant variables satisfying relatively simple equations which he could solve explicitly. However, these combinations are not interesting although, as Bardeen has emphasized, they have in consequence of their gauge invariance an inherent meaning. The limitation of the original quantities, especially of the relative density perturbation, in an admissible coordinate system is a criterion for stability; the limitation of the combination does not exclude a correlated increase of the original quantities.

In this paper we present a new method for studying the time development of perturbations. We choose a gauge in such a manner that all fluctuations of matter (i.e., perturbations in density, pressure, and velocity) vanish and only pure metric fluctuations remain. In that system (we call it in the following always the "pure-metric-fluctuation system") the problem simplifies so dramatically that we succeed in decoupling the system of differential equations describing the evolution of the perturbation and, for an equation of state which is simple enough, also in solving it. After that we transform this solution into a desired gauge.

The density contrast modes which arise thereby permit a physical interpretation, because their origin is made transparent. We will see that there are modes whose coefficients are connected directly with the "history of the universe" in question. The coefficients of other modes, on the contrary, can be chosen arbitrarily without changing that history. Those modes reflect merely the remaining freedom in the choice of the coordinate system within the imposed gauge.

The notation "history of the universe" is understood here as a class, consisting of all sets of perturbation quantities (as functions of space-time) which can be identified by means of infinitesimal coordinate transformations. Thus all sets belonging to one class describe the same perturbated universe relative to different coordinate systems. It is a result of our transformation method that we can enumerate without ambiguity all possible histories of the universe (i.e., all classes specified above) by means of two parameters.

It is a second purpose of this paper to supply density contrasts in some gauges. In part these results may be found already in the literature: in the case of the de Donder gauge, which has been proposed recently by Rose, Rahmstorf, and Dehnen[3] as "best adjusted to the physical problem under investigation" the solution is given for the first time. We will see that their solution is not the general one; they have merely treated the academic case of an unperturbed Friedmann universe in displaced coordinates.

The plan of this paper is as follows. In Sec. II we study the influence of infinitesimal coordinate transformations upon the perturbation quantities, Secs. III and IV present the mathematical framework of our transformation method, Sec. V deals with the classification of all possible histories of the Universe demonstrated by using a simple equation of state and Sec. VI presents density contrast modes for other equations of state in various gauges. Some conclusions complete this paper.

Notation. We use the same notation as that in the book of Weinberg.[5] Instead of $\partial f/\partial t$ we write $\dot f$ and $h_{,1}$ means $\partial h/\partial x^1$. Finally q in $\exp(iqx)$ is different from zero unless otherwise indicated.

II. INFINITESIMAL COORDINATE TRANSFORMATIONS

For simplicity we restrict ourselves to a Friedmann universe with vanishing spatial curvature as background. We choose the coordinates in the background such that we get for the metric

$$ {}^0g_{00}=-1,\quad {}^0g_{ij}=R^2(t)\delta_{ij} \tag{2.1}$$

(all other components vanish). $R(t)$ is the scale factor of the Universe and the index 0 refers to the background.

Now we study the influence of the infinitesimal coordinate transformation

$$x'^{\mu}=x^{\mu}-\epsilon^{\mu}(x^{\lambda}) \tag{2.2}$$

upon the perturbation quantities. This changes the correspondence between points in the background and points in the physical space-time. The properties of that transformation are well known and easily derived. For the fluctuation of the energy density ρ (that is, the coordinate scalar $T^{\mu\nu}U_{\mu}U_{\nu}$, where $T^{\mu\nu}$ is the energy-momentum tensor of the perfect fluid and U^{μ} is the four-velocity) defined by

$$\rho_1(x^{\lambda})\equiv\rho(x^{\lambda})-\rho_0(x^{\lambda})\ , \tag{2.3}$$

where $\rho(x^{\lambda})$ is the energy density at the space-time point x^{λ} in the perturbed universe and ρ_0 is the energy density at the space-time point x^{λ} in a fictitious Friedmann universe (the background), we obtain the transformation law

$$\rho_1'(x^{\lambda})=\rho_1(x^{\lambda})+\dot\rho_0(x^{\lambda})\epsilon^{t}(x^{\lambda})\ . \tag{2.4}$$

Analogously, we get for the fluctuation of the pressure p the transformation formula

$$p_1'(x^{\lambda})=p_1(x^{\lambda})+\dot p_0(x^{\lambda})\epsilon^{t}(x^{\lambda})\ . \tag{2.5}$$

Similarly, the fluctuation of the four-velocity defined by

$${}^1U^{\mu}(x^{\lambda})\equiv U^{\mu}(x^{\lambda})-{}^0U^{\mu}(x^{\lambda})\ , \tag{2.6}$$

where ${}^0U^{\mu}$ is the background four-velocity $(-1,0,0,0)$, transforms as

$${}^1U'^{\mu}(x^{\lambda})={}^1U^{\mu}(x^{\lambda})-\dot\epsilon^{\mu}(x^{\lambda})\ . \tag{2.7}$$

Finally, the perturbation metric defined by

$$h_{\mu\nu}(x^{\lambda})\equiv g_{\mu\nu}(x^{\lambda})-{}^0g_{\mu\nu}(x^{\lambda})\ , \tag{2.8}$$

where ${}^0g_{\mu\nu}(x^{\lambda})$ is the Robertson-Walker metric (2.1), transforms as

$$h'_{\mu\nu}(x^{\lambda})=h_{\mu\nu}(x^{\lambda})+\epsilon_{\mu}(x^{\lambda})_{;\nu}+\epsilon_{\nu}(x^{\lambda})_{;\mu}\ . \tag{2.9}$$

Note that the formulas (2.4), (2.5), (2.7), and (2.9) are valid only in the first order.

III. TRANSFORMATION INTO THE "PURE-METRIC-FLUCTUATION SYSTEM"

We consider a perturbation which has two-dimensional symmetry planes; i.e., in suitable coordinates all perturbation quantities shall depend only on x $(=x^1)$ and t $(=x^0)$, but not on y $(=x^2)$ or z $(=x^3)$. Beyond that we demand that ${}^1U^2$ and ${}^1U^3$ both vanish in order to exclude rotational perturbations which would disturb the symmetry.

This symmetry allows the introduction of coordinates such that the metric adopts the form

$$d\tau^2=-g_{ab}(x,t)dx^a dx^b-f(x,t)(dy^2+dz^2)\ , \tag{3.1}$$

where

$$a,b=0 \text{ or } 1\ . \tag{3.2}$$

Note that g_{ab} and f depend on x and t only.

Comparison with the metric (2.1) shows

$$g_{00}(x,t)=-1+h_{00}(x,t)\ , \tag{3.3a}$$

$$g_{01}(x,t)=h_{10}(x,t)\ , \tag{3.3b}$$

$$g_{11}(x,t)=R^2(t)+h_{11}(x,t)\ , \tag{3.3c}$$

$$g_{22}(x,t)=g_{33}(x,t)=R^2(t)+h_{22}(x,t)\ . \tag{3.3d}$$

The thereby defined $h_{\mu\nu}(x,t)$ (all other components vanish) should be small compared with the corresponding background quantities.

We use the field equations for the perturbation quantities in first order in the form

$$\delta G^{\mu\nu}=-8\pi G\delta T^{\mu\nu}\ , \tag{3.4}$$

where $G^{\mu\nu}$ is the Einstein tensor and $T^{\mu\nu}$ is the energy-momentum tensor for a perfect fluid. We choose that form in order to obtain differential equations containing only first time derivatives (the 0μ constraints). They will help us later to solve the problem. From (3.4) we obtain after a simple and straightforward, but rather lengthy calculation the following equations.

00 component:

$$-6\left[\frac{\dot R}{R}\right]^2 h_{00}-\frac{\dot R}{R}\dot h^{1}{}_{1}-\frac{2\dot R}{R}\dot h^{2}{}_{2}+\frac{2\dot R}{R}h^{1}{}_{0,1}+\frac{h^{2}{}_{2,11}}{R^2} =-8\pi G(\rho_1+\rho_0 h_{00})\ . \tag{3.5}$$

01 component:

$$3\left[\frac{\dot R}{R}\right]^2 h^1{}_0-\frac{\dot R}{R}\frac{h_{00,1}}{R^2}-\frac{\dot h^2{}_{2,1}}{R^2} =-8\pi G[p_0h^1{}_0+(p_0+\rho_0)^1U^1] . \quad (3.6)$$

11 component:

$$\left[\frac{2\ddot R}{R}+\left[\frac{\dot R}{R}\right]^2\right]h_{00}+\frac{3\dot R}{R}\dot h^2{}_2+\frac{\dot R}{R}\dot h_{00}+\ddot h^2{}_2 =-8\pi G p_1 . \quad (3.7)$$

22 component = 33 component:

$$\left[\frac{2\ddot R}{R}+\left[\frac{\dot R}{R}\right]^2\right]h_{00}+\frac{3\dot R}{2R}\dot h^1{}_1+\frac{3}{2}\frac{\dot R}{R}\dot h^2{}_2+\frac{\dot R}{R}\dot h_{00}+\frac{\ddot h^2{}_2}{2} +\frac{\ddot h^1{}_1}{2}+\frac{h_{00,11}}{2R^2}-\frac{h^2{}_{2,11}}{2R^2}-\frac{3\dot R}{R}h^1{}_{0,1}-\dot h^1{}_{0,1} =-8\pi G p_1 . \quad (3.8)$$

Additionally, we get from the conservation of the energy-momentum $T^{\mu\nu}{}_{;\nu}=0$, in first order, conservation of energy

$$\dot\rho_1+\frac{3\dot R}{R}(\rho_1+p_1)+(\rho_0+p_0)[^1U^1{}_{,1}+\tfrac{1}{2}(\dot h^1{}_1+\dot h^2{}_2+\dot h^3{}_3)] =0 \quad (3.9)$$

and conservation of momentum

$$p_{1,1}+\frac{\rho_0+p_0}{2}h^0{}_{0,1} +\left[\frac{\partial}{\partial t}+\frac{3\dot R}{R}\right][(p_0+\rho_0)(^1U^1R^2+h_{01})]=0 . \quad (3.10)$$

Now we want to perform the transformation into the pure-metric-fluctuation system. For that purpose we choose the transformation

$$\epsilon^t(x,t)=-\rho_1(x,t)/\dot\rho_0(t) , \quad (3.11)$$

$$\epsilon^1(x,t)=\int^t[^1U^1(x,t')dt'] , \quad (3.12)$$

$$\epsilon^2=\epsilon^3=0 . \quad (3.13)$$

In the new coordinate system the following relations obviously hold:

$$\rho_1'={}^1U'^1=0 . \quad (3.14)$$

We now investigate the important question of whether or not the fluctuation of the pressure p_1 also vanishes in that coordinate system. With the help of (2.4) and (2.5) one can very easily show that the following criterion holds: p_1 and ρ_1 can be transformed away simultaneously if and only if

$$p_1/\rho_1=\dot p_0/\dot\rho_0 . \quad (3.15)$$

Equation (3.15) is satisfied if the equation of state under consideration is valid not only for the background quantities but also for those in the perturbed universe and if the pressure in that equation can be expressed by the energy density ρ exclusively or vice versa. Namely, in this case we have $p_1=(dp/d\rho)_0\rho_1$ (0 refers to the unperturbed background) and $\dot p_0=(dp/d\rho)_0\dot\rho_0$ and this leads to (3.15). For example, $p=w\rho$ ($w=$ const) and $p_0=w\rho_0$ satisfies our criterion and we obtain a gauge-invariant expression for the ratio p_1/ρ_1:

$$p_1/\rho_1=w . \quad (3.16)$$

We now assume that (3.15) is satisfied and perform the transformation (3.11)–(3.13) in order to remove all matter fluctuations (those of energy, density, pressure, and four-velocity).

In the new coordinate system we obtain, from (3.5), (3.6), (3.9), and (3.10) [we have used the conservation equations instead of the dynamical equations (3.7) and (3.8)],

$$\left[8\pi G\rho_0-6\left[\frac{\dot R}{R}\right]^2\right]-\frac{\dot R}{R}\dot h^1{}_1-\frac{2\dot R}{R}\dot h^2{}_2 +\frac{2\dot R}{R}h^1{}_{0,1}+\frac{h^2{}_{2,11}}{R^2}=0 , \quad (3.17)$$

$$\left[3\left[\frac{\dot R}{R}\right]^2+8\pi G\rho_0\right]h^1{}_0-\frac{\dot R}{R}\frac{h_{00,1}}{R^2}-\frac{\dot h^2{}_{2,1}}{R^2}=0 , \quad (3.18)$$

$$\dot h^1{}_1+2\dot h^2{}_2=0 \quad (3.19)$$

$$\dot h^1{}_0-\frac{h_{00,1}}{2R^2}+\left[5\frac{\dot R}{R}+\frac{\dot p_0+\dot\rho_0}{p_0+\rho_0}\right]h^1{}_0=0 . \quad (3.20)$$

We are now in the position to decouple this system of differential equations. We express $h^2{}_{2,11}$ by means of Eq. (3.17), use (3.19) and differentiate with respect to the time. Next we form the expression for the same quantity using (3.18) and differentiating with respect to x. Equating both expressions and differentiating with respect to x again we arrive at

$$(5\dot R^2+2R\ddot R+8\pi G\rho_0R^2)h^1{}_{0,11}-\frac{\dot R}{R}h_{00,111}=(6\dot R^2-8\pi G\rho_0R^2)^{\cdot}h_{00,1}+(6\dot R^2-8\pi G\rho_0R^2)\dot h_{00,1}-2R\dot R\dot h^1{}_{0,11} . \quad (3.21)$$

We remove $h_{00,111}$, $h_{00,1}$, and $\dot h_{00,1}$ with the help of (3.20) and obtain finally a differential equation for $h^1{}_0$:

$$\left[2R\ddot{R}-5\dot{R}^{2}+8\pi G p_0 R^2-2R\dot{R}\frac{\dot{p}_0+\dot{\rho}_0}{p_0+\rho_0}\right]h^1{}_{0,11}-2R^2(6\dot{R}^2-8\pi G\rho_0 R^2)\ddot{h}^1{}_0$$

$$=\left[2R^2(6\dot{R}^2-8\pi G\rho_0R^2)^{\cdot}+(6\dot{R}^2-8\pi G\rho_0 R^2)\left[14R\dot{R}+2R^2\frac{\dot{p}_0+\dot{\rho}_0}{p_0+\rho_0}\right]\right]\dot{h}^1{}_0$$

$$+\left[(6\dot{R}^2-8\pi G\rho_0R^2)\left[10R\dot{R}+2R^2\frac{\dot{p}_6+\dot{\rho}_0}{p_0+\rho_0}\right]\right]^{\cdot}h^1{}_0 \ . \tag{3.22}$$

With that we have succeeded in decoupling the coupled system of differential equations describing the development of the perturbation regardless the form of the equation of state provided that our criterion (3.15) is satisfied.

IV. TRANSFORMATION INTO SOME GAUGES

Before performing such transformations we study the influence of the remaining freedom in choosing our coordinate system within the pure-metric-fluctuation gauge. The requirement $\rho_1'=\rho_1=0$ leads to the conclusion that ϵ^t must vanish. Similarly, the vanishing ${}^1U^1$ and ${}^1U'^1$ allow a $\epsilon^1(x^\lambda)$ with $\dot{\epsilon}^1(x^\lambda)=0$. Equation (2.9) shows this remaining freedom of performing transformations does not change h_{00} and $h^1{}_0$ but might add to $h^1{}_1$ and $h^2{}_2$ or $h^3{}_3$ a function which does not depend on time. We restrict ourselves to such coordinate systems which reveal the symmetry properties of the problem; i.e., all perturbation quantities shall depend only on x and t. Thus that function can depend only on x. The integration of (3.19) yields

$$h^i{}_i(x^\lambda)=\tau(x) \ . \tag{4.1}$$

We now see that $\tau(x)$ has no deeper meaning but reflects merely the remaining freedom of performing transformations within the pure-metric-fluctuation system.

From now on we assume a dependence of all perturbation quantities on x proportional to $\exp(iqx)$. Realistic cases must be treated by Fourier composition of the thereby obtained solutions.

A. Transformation into the synchronous gauge

This gauge is characterized by the condition

$$h_{\mu 0}\equiv 0 \ . \tag{4.2}$$

It distinguishes a set of coordinate systems corresponding to an observer who moves free-falling in Newtonian approximation. This can be seen easily by considering the equation describing a geodesic in the limit of a slowly moving particle; we obtain, in the lowest approximation,

$$\frac{d^2x^i}{dt^2}+\frac{1}{2}\frac{dx^i}{dt}\dot{h}_{00}=\frac{1}{2R^2}\frac{\partial h_{00}}{\partial x^i}-\frac{1}{R^2}\dot{h}_{i0} \ . \tag{4.3}$$

With the help of (4.2) this equation reduces to $d^2x^i/dt^2=0$; i.e., the observer does not realize a gravitational potential. With the help of (2.9) we obtain

$$0=h'_{00}=h_{00}-2\dot{\epsilon}^t \ . \tag{4.4}$$

This yields

$$\epsilon^t=\int^t\frac{h_{00}(x_1t')}{2}dt' \ . \tag{4.5}$$

Note that the prime in (4.4) indicates the synchronous system in this context. Similarly, $h_{01}'=0$ implies $h'^1{}_0=0$ and thus we have, with the help of (2.9),

$$0=h'^1{}_0=h^1{}_0+\frac{\epsilon_{t,1}}{R^2}+\dot{\epsilon}^1 \ . \tag{4.6}$$

The solution for ϵ^1 reads

$$\epsilon^1=\int^t dt'\left[iq\int^{t''}\frac{h_{00}(x,t'')dt''}{2R^2}-h^1{}_0(x,t')\right] \ . \tag{4.7}$$

The density contrast is defined by

$$\delta=\rho_1/(\rho_0+p_0) \ . \tag{4.8}$$

With the help of (2.4) we get, for the density contrast,

$$\delta'=\frac{\dot{\rho}_0\epsilon^t}{\rho_0+p_0} \ , \tag{4.9}$$

where ϵ^t is the temporal component of the transformation from the pure-metric-fluctuation system into the considered gauge; in the case of the synchronous gauge ϵ^t has to be replaced by (4.5).

B. Transformation into the Lagrangian gauge

Here we demand

$${}^1U'^1=h'^1{}_{0,1}=0 \ . \tag{4.10}$$

This set of coordinate systems corresponds to an observer who is comoving with the fluid. With the help of (2.7) we obtain

$$0={}^1U'^1={}^1U^1-\dot{\epsilon}^1=-\dot{\epsilon}^1 \ . \tag{4.11}$$

Of course, the prime here indicates that the quantity refers to the Lagrangian gauge. Equation (4.11) yields

$$\epsilon^1=E\exp(iqx) \ , \tag{4.12}$$

where E is a constant.

Next, we get, from $h'^1{}_{0,1}=0$,

$$\epsilon^t=\frac{i}{q}R^2h^1{}_0 \ . \tag{4.13}$$

C. Transformation into the de Donder gauge

Here we demand that the following condition holds:

$$h'^\lambda{}_{\mu;\lambda}-\tfrac{1}{2}h'^\lambda{}_{\lambda;\mu}=0 \ . \tag{4.14}$$

The de Donder gauge corresponds to the point of view of an observer who is fixed on the background in the sense that he obtains the correct Newtonian limit. Arguments for it are given by Rose, Rahmstorf and Dehnen.[3]

With the help of (2.9) and noting the commutator relations for the convenient derivatives, we obtain, from (4.14),

$$\epsilon_\mu{}^{;\lambda}{}_{;\lambda}-\epsilon^\sigma R_{\sigma\mu}=-Q_\mu \ , \tag{4.15}$$

where $R_{\sigma\mu}$ is the Ricci tensor and

$$Q_\mu=h^\lambda{}_{\mu;\lambda}-\tfrac{1}{2}h^\lambda{}_{\lambda;\mu} \ . \tag{4.16}$$

For $\epsilon^2=\epsilon^3=0$

$$\frac{\epsilon_{t,11}}{R^2}-\ddot{\epsilon}_t+3\left[\frac{\dot R}{R}\right]^2\epsilon_t-\frac{3\dot R}{R}\dot\epsilon_t-\frac{2\dot R}{R^3}\epsilon_{1,1}+\frac{3\ddot R}{R}\epsilon_t=-Q_t \ , \tag{4.17}$$

$$\frac{\epsilon_{1,11}}{R^2}-\ddot\epsilon_1-\frac{\dot R}{R}\dot\epsilon_1+\frac{2(R\ddot R+2\dot R^2)}{R^2}\epsilon_1-\frac{2\dot R}{R}\epsilon_{t,1}=-Q_1 \ , \tag{4.18}$$

where

$$Q_0=h^1{}_{0,1}-\frac{3\dot R}{R}h_{00}-\frac{\dot R}{R}h^1{}_1-\frac{2\dot R}{R}h^2{}_2 -\frac{\dot h_{00}}{2}-\frac{\dot h^1{}_1}{2}-\dot h^2{}_2 \ , \tag{4.19}$$

$$Q_1=-R^2\dot h^1{}_0-5R\dot R h^1{}_0+\frac{h^1{}_{1,1}}{2}+\frac{h_{00,1}}{2}-h^2{}_{2,1} \ . \tag{4.20}$$

We have to solve this coupled system of differential equations in order to obtain ϵ^1 and ϵ^t; with the help of the latter and using (4.9) we get the density contrast in the de Donder gauge. The mathematical scheme of the transformation method is not completely presented; in the next section we will solve Eq. (3.22) for a simple equation of state and will study the development of perturbations in such a universe.

V. THE DUST UNIVERSE

Such a universe is characterized by a vanishing pressure. Although that kind of equation of state is not very realistic with regard to the formation of galaxies, the following calculation shows all important features of what we can learn about the treatment of a perturbed universe from our method. Therefore we consider it here in detail. The Friedmann equations imply

$$R(t)=Kt^{2/3}, \quad K=\text{const} \ , \tag{5.1}$$

$$\rho_0=(6\pi G t^2)^{-1} \ . \tag{5.2}$$

Inserting this in equation (3.22) we obtain

$$\ddot h^1{}_0+\frac{2}{t}\dot h^1{}_0-\frac{4}{9t^2}h^1{}_0=0 \ . \tag{5.3}$$

The general solution is

$$h^1{}_0=A_1(x)t^{1/3}+A_2(x)t^{-4/3} \ , \tag{5.4}$$

with arbitrary functions A_1 and A_2 depending only on x. We consider here only perturbations proportional to $\exp(iqx)$; thus we have

$$h^1{}_0=(A_1t^{1/3}+A_2t^{-4/3})\exp(iqx) \tag{5.5}$$

with two constants A_1 and A_2. With the help of (3.20) it follows that

$$h_{00}=-\frac{i10}{3}\frac{K^2t^{2/3}}{q}A_1\exp(iqx) \ . \tag{5.6}$$

(The appearance of the imaginary unit i means that $h^1{}_0$ is phase shifted by $\pi/2$ with respect to h_{00}.) By means of Eqs. (3.17), (3.18), and (4.1) we obtain [note that $\tau(x)=\tau\exp(iqx)$ with a constant τ]

$$h^1{}_1=\left[-\frac{i}{q}\frac{8}{3}K^2t^{2/3}A_1-\frac{i}{q}\frac{8}{3}\frac{K^2}{t}A_2-\frac{iK^4}{q^3}\frac{80}{9}A_1+\tau\right]\times\exp(iqx) \ , \tag{5.7}$$

$$h^2{}_2=\left[\frac{i}{q}\frac{4}{3}K^2t^{2/3}A_1+\frac{i}{q}\frac{4K^2}{3t}A_2+\frac{iK^4}{q^3}\frac{40}{9}A_1\right]\times\exp(iqx) \ . \tag{5.8}$$

We can check these solutions by inserting them into the dynamical field equations.

We are now in the position to perform the transformations into some gauges. These yield the perturbation [expressed by (5.5)–(5.8) in the pure-metric-fluctuation system] described from the point of view of these gauges.

A. Transformation into the synchronous gauge

From (4.5) and (4.7) we obtain, with the help of (5.5) and (5.6),

$$\epsilon^t=-\frac{iK^2}{q}t^{5/3}A_1\exp(iqx)+\lambda_{\text{syn}}\exp(iqx) \ , \tag{5.9}$$

$$\epsilon^1=3\left[A_2-\frac{iq\lambda_{\text{syn}}}{K^2}\right]t^{-1/3}\exp(iqx)+\Pi_{\text{syn}}\exp(iqx) \ , \tag{5.10}$$

where λ_{syn} and Π_{syn} are arbitrary constants which arise by integrations in (4.4) and (4.6). These constants have no deeper meaning; they merely reflect the remaining freedom in performing transformations within the synchronous gauge. Equations (5.9), (5.10), and the transformation formulas of Sec. II yield, for the perturbation in the synchronous gauge,

$$\delta_{\text{syn}}=\frac{2iK^2}{q}A_1\exp(iqx)t^{2/3}-\frac{2\lambda_{\text{syn}}}{t}\exp(iqx) \ , \tag{5.11}$$

$${}^1U^1{}_{\text{syn}}=\left[A_2-\frac{iq\lambda_{\text{syn}}}{K^2}\right]t^{-4/3}\exp(iqx) \ , \tag{5.12}$$

$$h_{\text{syn}}{}^j{}_j=\left[\tau+6\left[\frac{q^2\lambda_{\text{syn}}}{K^2}+iqA_2\right]t^{-1/3}+2\Pi_{\text{syn}}-\frac{4iK^2}{q}A_1t^{2/3}+\frac{4\lambda_{\text{syn}}}{t}\right]\exp(iqx) \ . \tag{5.13}$$

B. Transformation into the Lagrangian gauge

Equations (4.3) and (5.5) imply

$$\epsilon^t = -\frac{iK^2}{q} A_1 t^{5/3} \exp(iqx) - \frac{iK^2}{q} A_2 \exp(iqx) , \quad (5.14)$$

and inserting this into (4.9) we obtain for the density contrast in the Lagrangian gauge

$$\delta_L = \frac{2iK^2}{q} A_1 t^{2/3} \exp(iqx) + \frac{2iK^2}{q} A_2 t^{-1} \exp(iqx) . \quad (5.15)$$

Note that the remaining freedom of performing transformations within the Lagrangian gauge has no influence upon the density contrast. Sakai[2] has expressed this fact by stating that in the Lagrangian gauge, in opposition to the synchronous gauge, a fictitious density contrast mode does not exist. In agreement with him we find here a $t^{2/3}$ and a t^{-1} mode (we do not give the solutions for ${}^1U_L^1$ and h_{Lj}^j here).

A very similar result was obtained by us in the synchronous gauge, but the t^{-1} mode there has a different meaning. Namely, in the synchronous gauge the corresponding coefficient has no relation to the history of the universe in question, whereas in the Lagrangian gauge it has.

In order to elucidate this we consider the solution in the pure-metric-fluctuation system (5.5)–(5.8). We found at the beginning of Sec. IV that h_{00} and $h^1{}_0$ do not change within the pure-metric-fluctuation gauge; thus the constants A_1 and A_2 have an inherent meaning. Their values determine the history of the universe.

Let $(\rho_1, h_{\mu\nu}, {}^1U^j)$ and $(\rho_1', h'_{\mu\nu}, {}^1U'^j)$ be two representatives of the same class; i.e., one of these sets can be carried over into the other by virtue of an infinitesimal coordinate transformation. They both describe the same perturbed universe. Let ϵ^μ be the transformation from the first-coordinate system (to which our first representative refers) into the pure-metric-fluctuation system, ϵ'^μ the transformation from the second coordinate system into the latter, and $\hat{\epsilon}^\mu$ the transformation from the first coordinate system into the second. With the help of (2.4) and (2.7) we obtain

$$\epsilon^t = \hat{\epsilon}^t + \epsilon'^t , \quad (5.16)$$

$$\epsilon^i = \hat{\epsilon}^i + \epsilon'^i + c^i(x) , \quad (5.17)$$

where $c^i(x)$ are arbitrary functions depending only on x. We have seen at the beginning of Sec. IV that such functions $c^i(x)$ merely reflect the freedom of performing transformations within the pure-metric-fluctuation gauge and can therefore be disregarded. Thus we can write

$$\epsilon^\mu = \hat{\epsilon}^\mu + \epsilon'^\mu . \quad (5.18)$$

Let $h^{(p)}{}_{\mu\nu}$ be the metric obtained by transforming our first set $(\rho_1, {}^1U^j, h_{\mu\nu})$ into the pure-metric-fluctuation system and $\tilde{h}^{(p)}_{\mu\nu}$ the metric which we obtain by performing the same procedure starting with our second set $(\rho_1', {}^1U'^j, h'_{\mu\nu})$. We have

$$h^{(p)}{}_{\mu\nu} = h_{\mu\nu} + \epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} , \quad (5.19)$$

$$\tilde{h}^{(p)}{}_{\mu\nu} = h'_{\mu\nu} + \epsilon'_{\mu;\nu} + \epsilon'_{\nu;\mu} ; \quad (5.20)$$

(5.18) yields

$$h^{(p)}{}_{\mu\nu} = \tilde{h}^{(p)}{}_{\mu\nu} . \quad (5.21)$$

This shows that both sets are related to the same pair (A_1/A_2). Moreover, consider two sets with different pairs (A_1/A_2) obtained by transforming these sets into the pure-metric-fluctuation system. In this case a transformation $\hat{\epsilon}^\mu$ which transforms the first set into the second cannot exist, for then, according to the above argument, (5.21) would hold, which implies the equality of the two pairs.

We have therefore shown that the pairs (A_1/A_2) classify all possible histories of the perturbed universe; the map between the history classes $[(\rho_1, {}^1U^j, h_{\mu\nu})]$ and the pairs (A_1/A_2) is bijective and does not depend on the particular representative of the class. The pair (0/0) corresponds to the unperturbed Friedmann universe, because in this case in the pure-metric-fluctuation system *all* fluctuations vanish (choose $\tau=0$). This complete spatial homogeneous and isotropic space-time must be the Friedmann universe.

We now see that the coefficients of some of the density contrast modes (as, for example, that of the $t^{2/3}$ mode in the synchronous gauge) are related to the history of the universe in question; the coefficients of other modes like that of the t^{-1} mode in the same gauge are not at all related to that history and can be chosen arbitrarily. Note that in the synchronous gauge A_2 does not appear, thus, and that is a lack of that gauge, δ does not reproduce the complete information about the history of the universe in question. The *different* worlds (0/0) and $(0/A_2)$ (where $A_2 \neq 0$) are described in this gauge by the *same* density contrast.

C. Transformation into the de Donder gauge

We consider here the case that the wavelength of the perturbation is large compared with the horizon. For such q we obtain, from (4.15) and (4.9),

$$\delta_{\rm dD} = \left[\frac{50}{51} \frac{iK^2}{q} A_1 t^{2/3} + \tfrac{3}{2} iq A_2 t^{-1/3} + (2\gamma_{\rm dD}^1 - \tau) + 2\gamma_{\rm dD}^2 t^{-7/3} + 2\lambda_{\rm dD}^1 t^{-3/2+\sqrt{11/12}} + 2\lambda_{\rm dD}^2 t^{-3/2-\sqrt{11/12}} \right] \exp(iqx) , \quad (5.22)$$

where $\gamma^1_{\rm dD}$, $\gamma^2_{\rm dD}$, as well as $\lambda^1_{\rm dD}$, $\lambda^2_{\rm dD}$, are arbitrary constants which were supplied by the solution of the homogeneous system of (4.14). We know that the corresponding modes merely reflect the remaining freedom in performing transformations within the de Donder gauge.

Rose, Rahmstorf, and Dehnen[3] have found only the two $\lambda_{\rm dD}$ modes. They have used for their derivation the approximation

$$ {}^1U^1{}_{,1} \ll \dot{h}^j{}_j \tag{5.23}$$

in the energy-conservation equation in order to decouple their system of differential equations. But this approximation is not allowed for nonvanishing q even if q is very small. This can be easily seen with the help of our transformation method by calculating ${}^1U^1$ and $\dot{h}^j{}_j$ in the de Donder gauge using the transformation formulas of Sec. II and the solution of (4.15). One finds that the expression for ${}^1U^1$ contains, compared with $\dot{h}^j{}_j$, one more q in the denominator, thus $q\,{}^1U^1$ is of the same order with respect to q as $\dot{h}^j{}_j$.

For $q=0$, however, all derivatives with respect to x vanish; in that case we can disregard ${}^1U^1{}_{,1}$ in the energy-conservation equation and perform the decoupling. In our formalism we obtain, for $q=0$, $Q_0=-2\tau/3t$, and inserting this in (4.14) we get the Rose result. But $q=0$ implies that the universe is not really perturbed; the Friedmann universe in displaced coordinates is then under consideration and in this case we cannot expect instabilities provided that the imposed gauge is "reasonable." For a really perturbed universe $[(A_1/A_2)\neq(0/0)$ and $q\neq 0]$ we obtain additional modes; if $A_1\neq 0$ we get also growing modes. Thereby we have disproved the statement by Rose, Rahmstorf, and Dehnen that the growing mode in the synchronous gauge is a pure coordinate effect for small q.

Note that the $q=0$ case cannot be obtained by considering $q\to 0$, for this limit does not exist [a sufficiently large x prevents $\exp(iqx)$ from being near to 1 regardless of the smallness of q $(q\neq 0)$].

VI. UNIVERSE WITH $p=w\rho$, $w=$const

Finally, we consider an equation of state which is more realistic (in the time long before the recombination era $w=\frac{1}{3}$ should be a fairly good approximation) than that of the previous section. The calculations are very similar to those of the dust-universe case, thus we present here only the final results.

Using the Friedmann equations $p=w\rho$ implies

$$R(t)=Kt^{2/[3(w+1)]}\,, \tag{6.1}$$

$$\rho_0(t)=[6\pi G(w+1)^2t^2]^{-1}\,. \tag{6.2}$$

Inserting this in (3.22) we obtain

$$\frac{w}{R^2}h^1{}_{0,11}-\ddot{h}^1{}_0=\frac{2-4w}{1+w}\frac{\dot{h}^1{}_0}{t}-\frac{(1+9w)(4-6w)}{9(w+1)^2}\frac{h^1{}_0}{t^2}\,. \tag{6.3}$$

We put the spatial part of $h^1{}_0$ proportional to $\exp(iqx)$, then we obtain a differential equation of Bessel type which has the solution

$$h^1{}_0=\tilde{A}_1t^{-\alpha}J_p(\gamma t^\beta)\exp(iqx) + \tilde{A}_2t^{-\alpha}J_{-p}(\gamma t^\beta)\exp(iqx)\,, \tag{6.4}$$

where $J_{\pm p}$ is the Bessel function of order $\pm p$ and

$$\alpha=\frac{1-5w}{2+2w}\,, \tag{6.5}$$

$$\beta=\frac{3w+1}{3w+3}\,, \tag{6.6}$$

$$\gamma=\frac{3w+3}{3w+1}\sqrt{w}\frac{q}{K}\,, \tag{6.7}$$

$$p=\frac{3w+5}{6w+2}\,. \tag{6.8}$$

$\tilde{A}_1$ and $\tilde{A}_2$ are arbitrary constants. We consider here only the case $\gamma t^\beta \ll p$, i.e., a wavelength beyond the Jean's stability region (characterized by the opposite condition). Thus we can approximate (6.4) by

$$h^1{}_0=A_1t^{(9w+1)/(3w+3)}+A_2t^{(6w-4)/(3w+3)}\,. \tag{6.9}$$

We have included all constants in A_1 and A_2. An analogous calculation like that in the previous section supplies the other components of $h_{\mu\nu}$ and with the help of the transformation formulas of Sec. IV we obtain finally the following.

Synchronous gauge:

$$\delta_{\rm syn}=\left[\frac{6w+10}{(9w+5)(w+1)}\frac{iK^2}{q}A_1t^{(6w+2)/(3w+3)}+\frac{\lambda_{\rm syn}}{t}\right]\times\exp(iqx)\,. \tag{6.10}$$

Lagrangian gauge:

$$\delta_L=\left[\frac{iK^2A_1}{q}\frac{2}{1+w}t^{(6w+2)/(3w+3)}+\frac{iK^2A_2}{q}\frac{2}{1+w}t^{(w-1)/(w+1)}\right]\exp(iqx)\,. \tag{6.11}$$

de Donder gauge:

$$\delta_{\rm dD}=(\delta_1+\delta_2+\delta_3+\text{hom})\exp(iqx)\,, \tag{6.12}$$

where

$$\delta_1\propto\frac{iA_1}{q}t^{(6w+2)/(3w+3)}\,, \tag{6.13}$$

$$\delta_2\propto iqA_2t^{(9w-1)/(3w+3)}\,, \tag{6.14}$$

$$\delta_3\propto\tau\quad(\tau=\text{const, arbitrary})\,, \tag{6.15}$$

and where hom comprises all solutions of the homogeneous system of differential equations (4.14) which reflect the remaining freedom in performing transformations within the de Donder gauge.

Equations (6.10) and (6.11) are in agreement with Lifshitz[1] and Sakai,[2] respectively. Equations (6.12)–(6.15) are the complete solution in the de Donder gauge and are not in agreement with Rose, Rahmstorf, and Dehnen;[3] their solution is only valid for the unperturbed Friedmann universe for reasons mentioned in the previous section. We see that for all w and for $A_1 \neq 0$ one obtains growing modes in all three gauges *with the same exponent.* Again, the pair (A_1/A_2), or the pair $(\tilde{A}_1/\tilde{A}_2)$ if we drop the approximation leading from (6.4) to (6.9), determines the history of the universe. In the synchronous gauge A_2 does not appear; thus the history of the universe in question can only partially be gathered from the corresponding density contrast in that gauge.

VII. CONCLUSIONS

Rose, Rahmstorf, and Dehnen[3] have claimed that in the de Donder gauge for sufficiently large perturbation wavelength only decaying density contrast modes arise. We agree with them that the de Donder gauge is appropriate to examine the evolution of the perturbation and to answer the question whether or not it is stable; but we saw in this paper that always growing modes appear provided that the extension of the perturbation is large enough. Thus the calculation in the de Donder gauge does not lead, as Rose, Rahmstorf, and Dehnen have claimed, to an upper boundary for the spatial extension of unstable density perturbations; in any case not in such a simple manner.

For later theoretical examinations it may be advantageous to use a gauge-invariant classification of all possible histories of a perturbed universe. It is provided in this paper for some simple equations of state. Bearing in mind this classification one might formulate a necessary criterion for the qualification of a gauge. If we want to use the density contrast for description of the evolution of the perturbation it should supply the complete information about the history of the universe in question. Thus we require that the general expression for the density contrast involves A_1 as well as A_2, where the pair (A_1/A_2) determines that history. The Lagrangian and the de Donder gauge satisfy this criterion but the synchronous gauge does not.

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