Analytic solution of a chaotic inflaton

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(Received 3 February 1989)

We directly solve the Langevin equation of motion for a scalar field in the slow-roll limit for a $\lambda\phi^4/4$ potential. The probability distribution for the inflaton and several moments of the distribution are also calculated. An upper bound to the percentage deviation of the mean of the distribution from the classical value is also derived. For the case of an initially uniform field, non-Gaussian features only appear when the inflaton deviates from the classical value by more than $\sim\lambda^{-1/2}$ standard deviations. We also show that in the non-Gaussian regime the slow-roll Langevin equation of motion is no longer adequate.

I. INTRODUCTION

A number of works have explored the effect of quantum fluctuations on the inflaton. A fruitful approach is via stochastic methods, developed by Starobinsky.[1] For a recent review of these methods, see Ref. 2. Graziani and Olynyk[3] numerically studied the "Mexican hat" Higgs potential, with the assumption that the scalar field was always Gaussian distributed. Linde analytically examined the inflaton probability distribution for chaotic inflation,[4] and demonstrated the possibility of eternal inflation.[5] Several potentials were also explored by Bardeen and Bublik,[6] where they numerically solved for the evolution of the probability distribution in the slow-roll limit. Further numerical work[7] evolved moments of the probability distribution for the $\lambda\phi^4$ potential, again with the assumption/approximation that the scalar field was always Gaussian distributed. The issue of whether or not nonlinear effects could transform an initially Gausian distribution into a non-Gaussian one was explored in Ref. 6. They examined the simple $m^2\phi^2/2$ potential for the case $m/m_{\rm Pl}=0.1$ (realistic values of m could not be studied due to a "prohibitive increase in CPU time"), and claimed that nonlinear effects would be unimportant for realistic theories in which m is chosen small enough so that the fluctuation amplitude is consistent with the observed isotropy of the microwave background.

It was also argued, however, that non-Gaussian fluctuations could arise in power-law inflation,[8] as a result of the curvature of an exponential potential. And more recently, there has been some analytic work, by the same group,[9] on the evolution of the probability distribution of the inflaton, in the slow-roll limit, for a wide class of potentials, where it is claimed that non-Gaussian distributions are a generic outcome of inflation. This is somewhat surprising, as the statistics of the nearly scale-invariant fluctuation spectrum[10] should be Gaussian to a good approximation, due to the extreme flatness of the potential. Non-Gaussian perturbations are certainly possible, but seem to require more complicated mechanisms, typically involving more than one field.[11] Here, we explore in detail the properties of a simple analytic solution of the Langevin equation of motion in the slow-roll limit for the $\lambda\phi^4$ potential. We quantify where non-Gaussian effects might appear, and also discuss the validity of the slow-roll Langevin equation in this regime.

II. STATISTICAL PROPERTIES OF THE $\lambda\phi^4$ INFLATON

In the popular "chaotic inflation" model of Linde,[4] a weakly coupled scalar field ϕ emerges from the Planck epoch with a "random" distribution (which has yet to be well defined). The idea is then that there must be, somewhere, a sufficiently smooth patch of field that is sufficiently displaced from the minimum so that it leads to an acceptable Universe (e.g., solving the horizon problem and diluting unwanted relics) via inflation. We briefly review a number of relevant features of this model, before we account for the effects of quantum fluctuations. For detailed reviews of inflation, see Ref. 12.

The potential we consider is $V(\phi)=\lambda\phi^4/4$. A smooth patch of field with $\phi=\phi_0$ will temporarily exit a radiation-dominated Friedmann-Robertson-Walker phase into a quasi–de Sitter phase when the potential energy $V(\phi_0)$ dominates the radiation density $\rho_{\rm rad}=3/32\pi G t^2$ (assuming curvature is unimportant when inflation starts), which occurs at $t_0\simeq(3m_{\rm Pl}^2/8\pi\lambda\phi_0^4)^{1/2}$, where $m_{\rm Pl}$ is the Planck mass. The evolution of the scalar field ϕ, with spatial gradients neglected, is given by

$$\ddot{\phi}+3H\dot{\phi}+dV/d\phi=0\ , \tag{2.1}$$

where the Hubble parameter H is

$$H^2=\frac{8\pi}{3m_{\rm Pl}^2}[\rho_{\rm rad}+\dot{\phi}^2/2+V(\phi)]\ . \tag{2.2}$$

During inflation, the potential energy dominates the Hubble parameter, any curvature becomes irrelevant, and the motion of ϕ is friction-dominated. (If ϕ is very large, $\phi\gtrsim m_{\rm Pl}/\lambda^{1/6}$, the motion of ϕ is quantum fluctuation dominated.[5] However, scales in the observable Universe correspond to $\phi\lesssim 5m_{\rm Pl}$.) In these approximations, Eq. (2.1) can be solved:

$$\phi_{\rm cl}=\phi_0\exp[-\alpha(t/t_0-1)]\ , \tag{2.3}$$

where $\alpha\equiv m_{\rm Pl}^2/4\pi\phi_0^2$, and the subscript cl refers to a classical, deterministic solution. The inflationary period begins to fizzle out when the kinetic energy $\dot{\phi}^2/2$ becomes comparable to the potential energy, which occurs when $\phi_{\rm end}/m_{\rm Pl} \simeq 1/\sqrt{3\pi}$ and $t_{\rm end}/t_0 \simeq 1+0.5\alpha^{-1}\ln\alpha^{-1}$. To obtain sufficient inflation, the number of e-folds $N \equiv \int H\,dt \gtrsim 60$, which translates to the constraint $\phi_0 \gtrsim 4.4 m_{\rm Pl}$. To get the correct amplitude of the inflation-predicted nearly Zel'dovich spectrum of density perturbations, fluctuations at the time of horizon crossing should be of order 10^{-4}–10^{-5}, which leads to values of λ of order 10^{-14}.

Now, we account for the effects of quantum fluctuations on the evolution of the inflaton. A noise term $\eta(t)$ is added, in the standard way,[1,3,6–9] to the equation of motion Eq. (2.1). In the slow-roll limit, the equation is

$$\dot{\phi}+(3H)^{-1}dV/d\phi=\eta(t)H^{3/2}/2\pi\ , \tag{2.4}$$

where η is assumed to be Gaussian distributed, with $\langle \eta(t)\rangle=0$ and $\langle \eta(t)\eta(t')\rangle=\delta(t-t')$. Higher moments can be obtained using the moment theorem. It is readily verified that the solution of the above equation with constant potential and Hubble parameter,

$$\phi-\phi_0=\frac{H^{3/2}}{2\pi}\int_{t_0}^{t}\eta(t)dt \tag{2.5}$$

leads to fluctuations $\langle(\phi-\langle\phi\rangle)^2\rangle$ that match the de Sitter result $H^3(t-t_0)/4\pi^2$ for a massless scalar field.[13] The equation of motion (2.4) is interpreted as describing the coarse-grained evolution of ϕ inside a given Hubble volume $\sim H^{-3}$. Fields in different Hubble volumes are assumed to evolve independently of one another, with field gradients ignored.

By trying potentials of the form $V \propto \phi^n$ (and using $H \propto \phi^{n/2}$), one finds that Eq. (2.4) can be integrated if $n=4$. The solution is

$$(\phi_{\rm cl}/\phi)^2=1-\sqrt{\lambda/12\pi^2\alpha}\,I(t/t_0)\ , \tag{2.6}$$

where

$$I(t/t_0)\equiv\int_1^{t/t_0}\exp[-2\alpha(s-1)]\eta(s)ds\ . \tag{2.7}$$

In obtaining this solution, $H^{3/2}$ in the equation of motion was replaced by an expression $\propto\phi^3$, which is only valid for $\phi\geq 0$. This is an appropriate solution, however; if we assume that ϕ is initially positive at the start of inflation then regions which evolve below $\phi_{\rm end}$ should be ignored—hence there is no need to also consider the $\phi<0$ solution [which can be obtained by changing the sign of I in Eq. (2.6)]. We also note that the solution (2.6) becomes ill-defined if $I>\sqrt{12\pi^2\alpha/\lambda}$. However, ϕ has to be infinite before this solution can become ill-defined, so the solution never enters such a regime for a physically meaningful theory. We also mention here that the equation of motion (2.4) could be reformulated so that the noise term has H dependence in the correlator $\langle\eta(t)\eta(t')\rangle$, and in such a case the simple solution (2.6) for ϕ would be replaced by an integral equation in ϕ. The inconsistency arises because H is treated as a constant in the normalization of the noise amplitude with the de Sitter result, but in the actual solution H is allowed to vary with ϕ.

Using a central-limit-theorem-type argument, the integral $I(t/t_0)$ should be Gaussian distributed. Since $\langle\phi^{-2}-\phi_{\rm cl}^{-2}\rangle=0$, and

$$w^2\equiv\langle(\phi^{-2}-\phi_{\rm cl}^{-2})^2\rangle=\lambda[(\phi_0/\phi_{\rm cl})^4-1]/3m_{\rm Pl}^4\ , \tag{2.8}$$

the probability distribution for ϕ is immediately deduced:

$$dP(\phi)/d\phi=2\phi^{-3}\exp[-(\phi^{-2}-\phi_{\rm cl}^{-2})^2/2w^2]/\sqrt{2\pi}w\ , \tag{2.9}$$

This is similar to the distribution derived in Ref. 9, except that we are missing one of their terms. No attempt has been made here to keep track of a small amount of probability that is lost in Eq. (2.9) due to the fact that it is only valid for $\phi\geq 0$ (more correctly, it is only valid for $\phi\geq\phi_{\rm end}$). Adequate approximations of the distribution, for potentials of the form $V\propto\phi^n$, are given in Ref. 2.

Moments of the distribution are easily calculated by taking advantage of the smallness of λ and performing an expansion of the solution (2.6):

$$\phi/\phi_{\rm cl}=1+\sqrt{\lambda/48\pi^2\alpha}\,I(t/t_0)+\lambda(32\pi^2\alpha)^{-1}I^2(t/t_0)+\cdots\ . \tag{2.10}$$

The deviation of the mean of the distribution $\langle\phi\rangle$ from the classical value $\phi_{\rm cl}$ is then given by

$$(\langle\phi\rangle-\phi_{\rm cl})/\phi_{\rm cl}=[1-(\phi_{\rm cl}/\phi_0)^4]\lambda\phi_0^4/8m_{\rm Pl}^4+O(\lambda^2)\ . \tag{2.11}$$

Therefore, given an initially uniform field, the deviation of the mean from the classical solution will be bounded by $\lambda\phi_0^4/8m_{\rm Pl}^4$, a very small number. As noted,[6,7] the effect of fluctuations on the inflaton puts the mean of the distribution slightly *ahead* of the classical value, since $\phi_{\rm cl}/\phi_0\leq 1$ by Eq. (2.3). The next moment is given by

$$\sigma^2\equiv\langle(\phi-\phi_{\rm cl})^2\rangle=[1-(\phi_{\rm cl}/\phi_0)^4]\lambda\phi_0^4\phi_{\rm cl}^2/12m_{\rm Pl}^4+O(\lambda^2). \tag{2.12}$$

We now consider the conditions necessary for non-Gaussian behavior. First, it should be noted that if we could neglect the $O(\lambda)$ and higher-order terms in the expansion (2.10) of ϕ in terms of the noise $I(t/t_0)$, ϕ would be Gaussian distributed. This indicates that non-Gaussian features are related to the size of λ, and become less prominent as λ decreases. This statement can be made more quantitative by massaging the exponential portion of the probability distribution (2.9) into

$$\exp\left[-\frac{(\phi-\phi_{\rm cl})^2}{2\sigma^2}\frac{\phi_{\rm cl}^2(\phi_{\rm cl}+\phi)^2}{4\phi^4}\right]\ , \tag{2.13}$$

where we have used the fact that $w^2=4\sigma^2/\phi_{\rm cl}^6$. It is then clear than non-Gaussian behavior is expected when

$$|\phi-\phi_{\rm cl}|/\phi_{\rm cl}\gtrsim 1 \tag{2.14}$$

as one might have guessed. We assess the likelihood of entering such a regime by writing $\phi-\phi_{\rm cl}\equiv M\sigma$, and determining the required number of standard deviations M. The result is that

$$M \gtrsim \phi_{cl}/\sigma = \left[\frac{12 m_{Pl}^4}{\lambda \phi_0^4 [1-(\phi_{cl}/\phi_0)^4]} \right]^{1/2} . \tag{2.15}$$

The probability of finding a Hubble volume that has "wandered" into the non-Gaussian regime is well approximated by $P \simeq \sqrt{2/\pi}\exp(-M^2/2)/M$, which leads to $P \sim 10^{-10^{12}}$ for typical parameters ($\lambda \sim 5\times 10^{-14}$, $\phi_0/m_{Pl} \sim 5$), indicating that non-Gaussian behavior is highly improbable, to say the least.

We have been considering, so far, the statistics of the inflaton for an initially uniform field—and found that the distribution was highly Gaussian and strongly peaked about the classical value. The effects of expansion alone on the distribution are easy to analyze. Starting with Gaussian initial conditions, with mean A_0 and standard deviation S_0, it is easy to show, for the case that there are no fluctuations, that the distribution remains Gaussian with a time-evolved mean $=A_0\exp[-\alpha(t/t_0-1)]$ and standard deviation $=S_0\exp[-\alpha(t/t_0-1)]$.

III. CONCLUDING REMARKS

We analytically solved the Langevin equation of motion in the slow-roll limit for Linde's $\lambda\phi^4$ chaotic inflation model. The probability distribution was deduced, and it was shown that moments of the distribution could easily be calculated from an expansion of the inflaton solution in powers of λ. Finally, we showed, given Gaussian initial conditions, that non-Gaussian features in the evolved distribution only appear for extremely improbable values of the scalar field. In any case, we emphasize that in the regime where such features occur, the slow-roll Langevin equation of motion breaks down. For example, for the case with $\phi=\phi_0$ initially, non-Gaussian features appeared only when $|\phi-\phi_{cl}|/\phi_{cl} \gtrsim 1$. If $\phi < \phi_{cl}$ and $\phi_0 > 0$, then non-Gaussian features are encountered only after the end of the inflationary period (i.e., for $\phi < \phi_{end}$), where the kinetic energy term $\dot{\phi}^2/2$ can no longer be neglected in the Hubble term, and the $\ddot{\phi}$ term in the equation of motion can no longer be neglected. Another constraint arises if we consider the maximum allowed fluctuation $\Delta\phi$ of ϕ in a Hubble time consistent with the assumption of inflation: $H^2(\Delta\phi)^2 \lesssim V(\phi)$, or $\Delta\phi \lesssim 0.3 m_{Pl}$. The non-Gaussian region can therefore violate the assumption that the kinetic term is negligible in the Hubble parameter (2.2). Non-Gaussian features may very well appear in this regime, but kinetic terms must be included for a self-consistent treatment. Works[8,9] which have attempted to explore non-Gaussian features using the "standard" Langevin equation should be very cautious. Therefore, the best way to sum up this work is to say that the Langevin equation (2.4) yields a very adequate, self-consistent, description of the evolution of ϕ, primarily because $\lambda \ll 1$, which does not generate any significant, or self-consistent, non-Gaussian features.

ACKNOWLEDGMENTS

We thank Ed Bertschinger, George Blumenthal, and Joel Primack for useful discussions, and A. Linde for bringing a number of relevant references to our attention. This work was supported by NSF Grant No. PHY88-00801, at the University of California, Santa Cruz.

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