# Exact solution of a class of two-state periodic Schrödinger problems

P. S. Krstić, R. K. Janev, and T. P. Grozdanov
*Institute of Physics, P.O. Box 57, 11001 Belgrade, Yugoslavia*
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A neutral system coupled to an external periodic magnetic field is considered. The proposed form of the applied field allows us to solve the resulting coupled-channel two-state problem exactly. The transition probability is discussed in detail in a wide range of parameters of the system. The optimal parameters are identified exactly, thus providing maximum control of the experimental conditions in the degaussing problem. Special attention is paid to the problem of neutron-antineutron oscillations.

A neutral system can couple to an external magnetic field due to the magnetic moments of the constituents of the system. The increasing interest in this problem comes in recent years in connection with the experimental possibility of detecting baryon-nonconserving transitions. A prominent example is neutron-antineutron oscillations.[1] The coupling of the anomalous magnetic moment of a neutron to Earth's magnetic field reduces the $n \rightleftarrows \bar{n}$ transitions by many orders of magnitude. A natural idea to restore these oscillations is with an applied oscillating magnetic field, driving the system for many periods of the field at its fundamental frequency. The frequency of the magnetic field pulses is then of decisive importance for the effect rather than their shape and intensity. Several authors[2-5] have tried to solve the problem either analytically or numerically, starting from the system of coupled differential equations[2] ($\hbar = c = 1$)

$$d\alpha/dt = -i\omega_B(t)\alpha(t) - i\omega_m\beta(t) , \tag{1a}$$

$$d\beta/dt = -i\omega_m\alpha(t) + i\omega_B(t)\beta(t) , \tag{1b}$$

where $\omega_B(t)$ is the time-dependent external magnetic field coupling energy. In the $n \rightleftarrows \bar{n}$ example[1] $\omega_m$ is the energy characterizing the fundamental barion mixing force $\omega_m \simeq 10^{-4}$ s$^{-1}$, while Earth's static magnetic field coupling energy $\omega_0 \simeq 10^4$ s$^{-1}$ defines the order of magnitude of $\omega_B(t)$ [$\omega_B(t) = \omega_0 + f_B(t)$, where $f_B(t)$ is an applied periodic magnetic field]. In this case $\alpha$ and $\beta$ are the neutron and antineutron wave functions, respectively.

If $f_B(t)$ is taken to be a simple harmonic function, the system of coupled equations (1) is equivalent to a Hill equation[6] and can be solved only approximately.[7] But when the applied periodic magnetic field is shaped in the form of rectangular pulses (Fig. 1), the resulting transition probabilities can be calculated exactly and these show some peculiar features that have not been stressed in the approximate treatments.[2-5] Therefore, we take, for the coupling energy,

$$\omega_B(t) = \begin{cases} W_1 = W_0 + R, & t \in (kT, T_1 + kT] , \\ W_2 = W_0 - R, & t \in (T_1 + kT, (k+1)T] , \end{cases} \tag{2}$$

where

$$W_0 = \omega_0 + (R_1 - R_2)/2 , \tag{3a}$$

$$R = (R_1 + R_2)/2 . \tag{3b}$$

The $T$ is a period of the oscillations, $R_1$ and $R_2$ are the amplitudes of the $f_B(t)$ in the two parts of the period with the duration $T_1$ and $T_2$, respectively (Fig. 1), and $k$ is a non-negative integer ($R_1 = R_2 = 0$ for $t < 0$). At small amplitudes (such as Earth's magnetic field), the rectangular oscillations can be produced with no technical difficulties.

The evolution matrix $\underline{\tau}$ of the system, after a period $T$ is defined as

$$\begin{bmatrix} \alpha(T) \\ \beta(T) \end{bmatrix} = \underline{\tau}(T) \begin{bmatrix} \alpha(0) \\ \beta(0) \end{bmatrix} .$$

It has the form

$$\tau_{11}(T) = \tfrac{1}{2}\{ A_+ \cos(\theta_+) + A_- \cos(\theta_-) - i[B_+ \sin(\theta_+) + B_- \sin(\theta_-)]\} , \tag{4a}$$

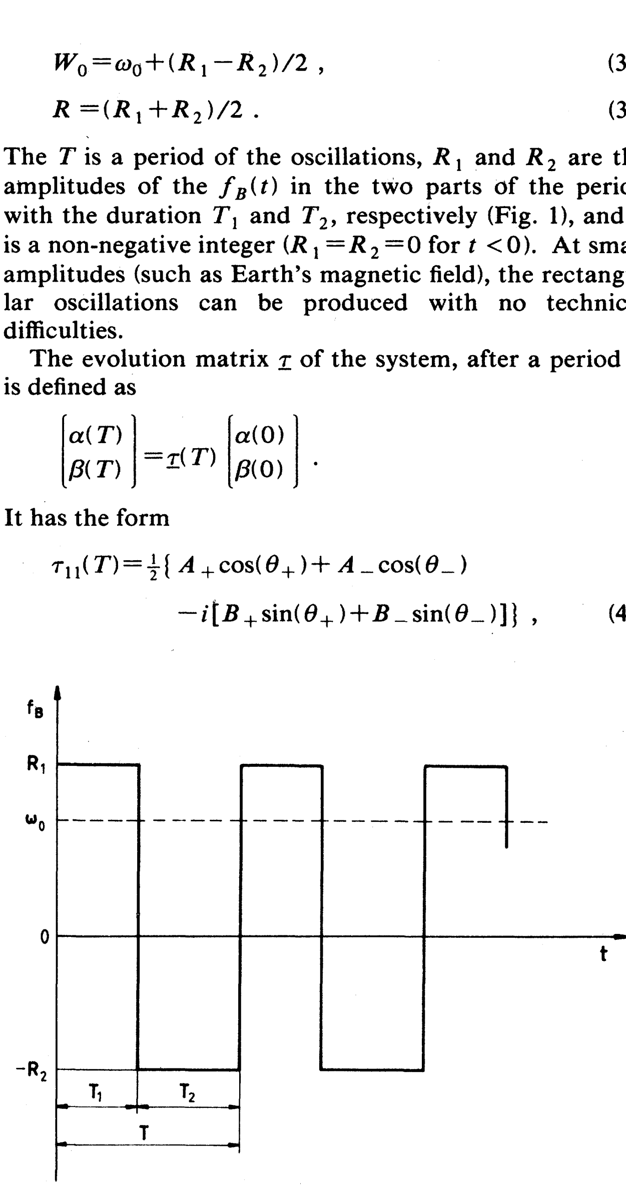


FIG. 1. The applied periodic magnetic field $f_B(t)$ of the period $T = T_1 + T_2$; $\omega_0$ is the external static magnetic field.

 

$$\tau_{21}(T)=\tfrac{1}{2}\{C[\cos(\theta_+)-\cos(\theta_-)] + i[D_+\sin(\theta_+)-D_-\sin(\theta_-)]\} , \tag{4b}$$

$$\tau_{12}(T)=-\tau_{21}^*(T), \quad \tau_{22}(T)=\tau_{11}^*(T) , \tag{4c}$$

where

$$A_\pm=1\pm(\delta^2+1-\kappa^2)/(\epsilon_1\epsilon_2) , \quad D_\pm=\delta(\epsilon_1+\epsilon_2)/(\epsilon_1\epsilon_2) , \tag{5a}$$

$$B=[(1+\kappa)\epsilon_2\pm(1-\kappa)\epsilon_1]/(\epsilon_1\epsilon_2), \quad C=2\kappa\delta/(\epsilon_1\epsilon_2) , \tag{5b}$$

$$\theta_\pm=W_0(\epsilon_1T_1\pm\epsilon_2T_2) , \quad \epsilon_{1,2}=[(1\pm\kappa)^2+\delta^2]^{1/2} , \tag{5c}$$

and

$$\delta=\omega_m/W_0 , \quad \kappa=R/W_0 . \tag{5d}$$

The diagonalization of the $\underline{\tau}$ yields the eigenvalues

$$\lambda_{1,2}=\exp(\pm i\psi) , \tag{6}$$

where

$$\cos(\psi)=\mathrm{Re}(\tau_{11}) . \tag{7}$$

Because of the periodicity of the $\omega_B(t)$, the Flocquet theorem[6] applies and the evolution matrix $\underline{\zeta}$, after a time $t=NT$, is obtained in the form

$$\zeta_{11}(t)=\cos(\gamma t)+i\Omega\sin(\gamma t)/\sin(\gamma T) , \tag{8a}$$

$$\zeta_{21}(t)=\tau_{21}(T)\sin(\gamma t)/\sin(\gamma T) , \tag{8b}$$

$$\zeta_{12}(t)=-\zeta_{21}^*(t), \quad \zeta_{22}(t)=\zeta_{11}^*(t) , \tag{8c}$$

where $\gamma=\psi/T$ is the Flocquet exponent of the problem,[6] and $\Omega=\mathrm{Im}(\tau_{11})$. The condition of fixing the time at $t=NT$ can be easily weakened by multiplication of the $\underline{\zeta}$ matrix with the matrix which describes the evolution of the system inside a period $T$. This would introduce oscillations of the amplitude $\simeq\delta$ and frequency $\simeq W_0$, superimposed to the amplitudes in Eqs. (8), significantly complicating the form of the expressions. We would rather assume $\delta$ small (in the $n\rightleftarrows\bar{n}$ problem $\delta\simeq10^{-8}$) and consider $t\,(>T)$ in Eqs. (8) as any moment of time $t$.

With the initial conditions $\alpha(0)=1$, $\beta(0)=0$, the transition probability can conveniently be written in the form

$$P_{21}(t)=\mathcal{P}_{21}\sin^2(\gamma t) , \tag{9a}$$

where

$$\mathcal{P}_{21}=\tau_{21}^2/(\tau_{21}^2+\Omega^2) . \tag{9b}$$

Numerical investigations of the transition probability [Eqs. (9)] in terms of the scaled time, the applied field amplitude $\kappa$, and frequency $\omega=2\pi/T$ yield the curves presented in Figs. 2–4, calculated for $\delta\in(10^{-8},10^{-3})$ and for $T_1=T_2=T/2$. An exact resonance is defined by the condition $\mathcal{P}_{21}=1$, which yields $\Omega(\omega=\omega_R)=0$. Introducing, for convenience, the dimensionless parameter $x=W_0/\omega$, it follows that resonances appear when $x_R=n/2[1+O(\delta^2)]$, where $n$ is a natural number which defines the order of a resonance. The Flocquet exponent

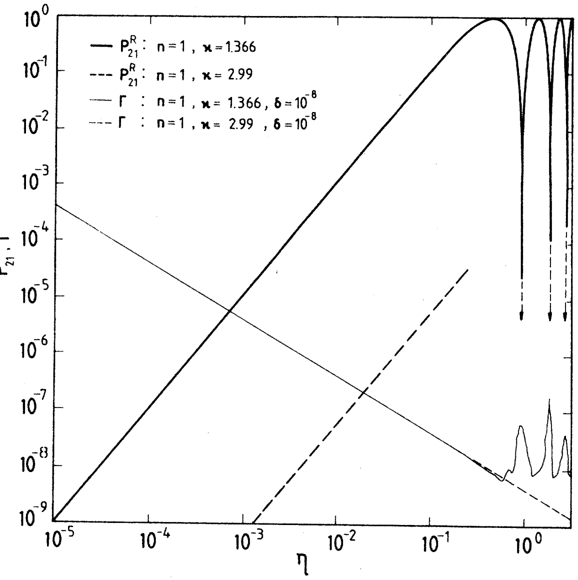


FIG. 2. The transition probability at the exact resonance and the width of a resonant peak vs the scaled time $\eta=\omega_mt/(2\pi)$.

$\gamma$, at $x=x_R$, is obtained in the form [up to $O(\delta^2)$]

$$\gamma_R=\begin{cases}\omega_m\sqrt{2}\kappa[1-(-1)^n\cos(\pi n\kappa)]/(n\pi|1-\kappa^2|) & \text{if } |\kappa-1|\gg\delta , \\ \omega_m/2 \quad \text{if } \kappa=1 . & \end{cases} \tag{10}$$

Introducing the notation $C_n(\kappa)=2\pi\gamma_R/\omega_m$, as long as

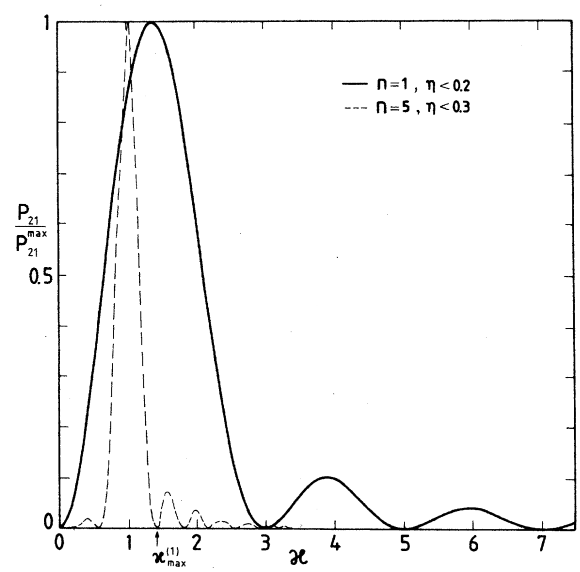


FIG. 3. The normalized transition probability as a function of the scaled field amplitude $\kappa=R/W_0$, at the exact resonance of the first order, $n=1$, and the fifth order, $n=5$.

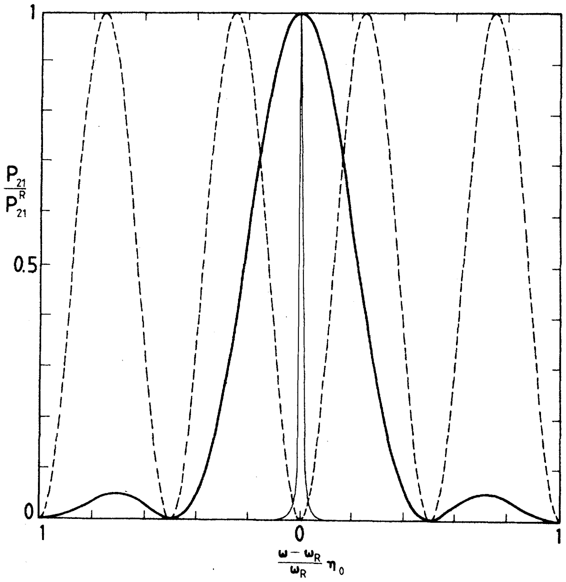


FIG. 4. The transition probability in a resonant region, normalized to its peak value, vs scaled frequency $\eta_0(\omega-\omega_R)/\omega_R$ (thick line); the sharply peaked factor $\mathcal{P}_{21}$ of Eq. (9) (thin line); $\sin^2(\gamma t)$ (dashed line).

$\gamma_R t \ll 1$, the probability at an exact resonance can be written in the form

$$P_{21}^R(t)=\sin^2(\gamma_R t)\simeq C_n^2(\kappa)\eta^2 \ , \tag{11}$$

where $\eta=\omega_m t/(2\pi)$. Equation (11) explains the straight-line portion of the $P_{21}^R$ curves at Fig. 2 and their independence on $\delta$. The oscillating part for $\kappa=1.366$ is due to the sine in Eq. (11) and it appears after the $C_n(\kappa)\eta$ reaches a value of $\sim\pi/2$ ($\eta\simeq0.2$). For the dashed line ($\kappa=2.99$) this happens much later, at $\eta\simeq23$. This is a consequence of the strong dependence of $C_n(\kappa)$ on both $\kappa$ and $n$. In view of the applications, since $\omega_m$ is assumed small, it is not very probable to have such a long time on disposal in a possible experiment to drive the resonant transition probability outside the straight-line portion of the curves of Fig. 2 (for example, the half-life time of a neutron is less than 10 min). Therefore, the time interval $(0,(\pi/2)/\gamma)$ is of most interest. Far enough from a resonance, the probability has the order of magnitude of $\delta^2$ and it oscillates with frequency $W_0$.

The curves of Fig. 3 are independent of $\delta$ and $t$ as long as the linear region of Fig. 2 is considered. Obviously, in this region these present the normalized $C_n(\kappa)$ curves and their structure is clear from Eq. (10). The probability is zero when $n\kappa$ and $n$ are integers of the same parity, i.e., when $n(\kappa+1)$ is an even integer (and, in addition, for $\kappa=0$). The exception is $\kappa\simeq1$, where this rule is not obeyed [Eq. (10)]. Thus, $P_{21}^R=0$ for $n=1$ at $\kappa=0,3,5,7,\ldots,(2l+1),\ldots$, for $n=2$ at $\kappa=0,2,3,4,\ldots,l,\ldots$, for $n=5$ at $\kappa=0,\frac{1}{5},\frac{3}{5},\frac{7}{5},\frac{9}{5},\ldots,(2l+1)/5,\ldots,$. The envelope of the local maxima (for $\kappa>\kappa_{\max}$) of Fig. 3 is a decreasing function of both $\kappa$ and $n$. Therefore, as far as the dependences on the field and the order of resonance are considered, the maximum probability is obtained for $\kappa_{\max}\simeq1$ and for the lowest $n$. Thus, $\kappa_{\max}=1.366$ at $n=1$ and it shifts toward 1 with an increase of $n$ ($\kappa_{\max}\simeq1.06$ at $n=5$). The linear portion of the thick line curve of Fig. 2 represents maximum probability of the system for a given $\eta$ as long as $\delta$ is small. For $\eta=10^{-2}$, $P_{21}^R\simeq1.18\times10^{-3}$ which is about the maximum probability that can be obtained for the neutron-antineutron transition, in the half-life time of a neutron.

The thick line curve of Fig. 4 is independent of $\delta$, $\kappa$, $n$, and $t$ as long as the values of these parameters keep the probability on the linear portion of the curves of Fig. 2. This and the appearance of the satellites is the consequence of the superposition of the sharply peaked structure of the $\mathcal{P}_{21}$ in Eq. (9) (thin line of Fig. 4) and the fast oscillating nature of the Flocquet exponent $\gamma$ [the dashed line of Fig. 4 represents $\sin^2(\gamma t)$]. The latter always has minima [given by Eq. (10)] at the resonant frequences. The factor $\mathcal{P}_{21}$ in the transition probability picks up the values of these minima. On the other hand, the two nearest maxima of the sine term pick up the values of the $\mathcal{P}_{21}$ at the far wings, determining the behavior of the resonant peak at the wings. The satellites come from the next-nearest maxima of the sine term.

The width of the resonant peak is defined as $\Gamma=\Delta\omega/\omega_R$, where $\Delta\omega$ is the frequency range about the resonance where the transition probability is greater or equal to $\frac{1}{2}$ of its peak value. $\Gamma$ is found to be dependent only on $\eta_0=W_0 t/(2\pi)$. This dependence has the form

$$\Gamma=0.44/\eta_0 \ . \tag{12}$$

The calculated $\Gamma$ is presented in Fig. 2 (thin line) as a function of $\eta$, for $\delta=10^{-8}$. The value of $\eta$ when the resonant peak splits (and the results for $\Gamma$ are no longer valid giving the thin-line oscillatory structure) corresponds to the termination of the linear porion of the $P_{21}^R$ curve. Equation (12) can be explained by expanding the probability in $\Delta x\ll1$ about the resonant $x_R$ and neglecting the terms of order $\delta^2/\Delta x$. The defining equation for the half-width then simplifies to $\sin(y)/y=1/\sqrt{2}$, where $y=4\pi\eta_0\Delta x/n$. The solution of this equation is $y\simeq0.44\pi$ and taking into account both wings Eq. (12) follows.

For the $n\rightleftarrows\bar{n}$ system, for $\eta=10^{-2}$, we get $\Gamma=4.4\times10^{-7}$, which is at least an order in magnitude larger than the width which would be expected from the peaked factor $\mathcal{P}_{21}$. A remarkable result is that $\Gamma$ is significantly larger for shorter times. An additional possibility is to increase $\Gamma$ for a fixed $t$ by decreasing $W_0$, without changing the transition probability, which can be done by a proper choice of the asymmetry in the pulse amplitudes $R_1$ and $R_2$, Eq. (3a).

In conclusion, the proposed form of the applied periodic magnetic field, with the exact solution of the coupled-channel two-state problem allows us for the first time to identify the optimal parameters exactly, thus providing the maximum control of the experimental conditions in the degaussing problem of an electrically neutral system. In the vicinity of resonances the dependence of the transition probability on the $R/W_0$ is not critical (see the $n=1$

curve of Fig. 3) and therefore the uniformity of both amplitude of periodic ($\sim R$) and static ($\sim W_0$) magnetic fields is also not critical in a possible experiment. Moreover, the degaussing by an additional static magnetic field is not necessary, but if done has the desirable effect of widening the resonant peaks in the frequency domain. Although the main motivation for working on this problem is connected to the possibility of stimulated $n \rightleftarrows \bar{n}$ transitions, the derived results can be applied to any neutral system which dominantly interacts with an applied magnetic field. If the coupling parameter $\delta$ is of the order of unity or even larger, the present theory needs small modifications to allow for the oscillations of the probability within a period of the applied magnetic field.


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