Nearest-neighbor interactions and the physical content of Fritzsch mass matrices

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We show that in the standard model with fewer than five generations, starting with arbitrary Yukawa couplings, it is always possible to find a weak basis where the quark mass matrices have the nearest-neighbor interaction form. Therefore, for three or four generations, the zeros of the Fritzsch mass matrices are not a contrived feature of the Fritzsch *Ansatz*, but just a special choice of weak basis.

The understanding of the pattern of quark masses and mixings is one of the major unsolved problems in particle physics. In the standard electroweak model[1] the Yukawa couplings which generate the up- and down-quark mass matrices M_u and M_d are arbitrary, and as a result the quark masses and the mixing angles are free parameters. In the past, there have been various attempts to relate the quark masses and mixing angles through the imposition of spectral *Ansätze* for M_u and M_d. One of the most popular *Ansätze,* suggested by Fritzsch,[2] has the following two distinctive features.

(i) Hermiticity: The matrices M_u and M_d are assumed to be Hermitian.

(ii) Nearest-neighbor interactions (NNI): It is assumed that the "light" quarks acquire their masses through an interaction with their nearest neighbors. For n generations this implies the following form for M_u and M_d:

$$M_u,M_d=\begin{bmatrix} 0 & x & 0 & 0 & \cdots & 0 & 0 \\ x & 0 & x & 0 & & 0 & 0 \\ 0 & x & 0 & x & & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & & 0 & x \\ 0 & 0 & 0 & 0 & \cdots & x & x \end{bmatrix}, \tag{1}$$

where x stands for a generic nonvanishing matrix element.

It is well known that in the standard model one can always choose, without loss of generality, the quark mass matrices M_u and M_d to be Hermitian. This results from the fact that in the standard model if two sets of quark mass matrices (M_u,M_d) and (M'_u,M'_d) are related by $M'_u=U^\dagger M_u V_u, M'_d=U^\dagger M_d V_d$, with U, V_u, and V_d some unitary matrices, then they give rise to the same quark masses and charged-current flavor mixings. One can use this freedom to choose M_u and M_d to be Hermitian. Therefore, feature (i) of the Fritzsch *Ansatz* does not have, by itself, any physical content.

In this paper we will address the question whether the NNI hypothesis has by itself any physical content. This hypothesis is the most striking feature of the Fritzsch *Ansatz* and naively one may consider it to be rather contrived, since it requires a set of mass matrix elements to vanish, at least in the tree approximation.

It turns out that the answer to the above question depends on the number of generations. We will show that for fewer than five generations one can always choose a weak basis where the mass matrices M_u and M_d have the form of Eq. (1), and therefore, the NNI hypothesis has no content by itself. This has important consequences for the Fritzsch *Ansatz,* since it implies that *for three or four generations the zeros of the Fritzsch mass matrices are not a contrived feature of that Ansatz, but just a special choice of weak basis.* On the contrary, for five or more generations it is no longer possible to choose, without loss of generality, a weak basis where M_u and M_d have the form of Eq. (1), and therefore the Fritzsch *Ansatz* becomes less appealing, and the NNI hypothesis has physical content by itself, implying correlations between the quark masses and mixings.

Next we will prove the above result for the case of three generations, and then we generalize it to four generations.

Theorem. Given any two 3×3 quark mass matrices M_u, M_d, there is always a weak-basis transformation such that the new mass matrices M'_u,M'_d have vanishing matrix elements (1,1), (1,3), (3,1), and (2,2).

Proof. Let us consider the Hermitian matrices $H_u\equiv M_u M_u^\dagger$ and $H_d\equiv M_d M_d^\dagger$, which have simple transformation properties under a weak-basis transformation:

$$H'_{u,d}=U^\dagger H_{u,d}U\ , \tag{2}$$

with unitary U. We will first show that given two arbitrary Hermitian matrices H_u,H_d, it is always possible to make a weak-basis transformation such that in the new basis

$$(H'_u)_{12}=(H'_d)_{12}=0\ . \tag{3}$$

We will then prove that if the 3×3 Hermitian matrices H'_u,H'_d satisfy the condition (3) it is always possible to find V_u,V_d such that M'_u,M'_d have vanishing matrix elements (1,1), (1;3), (3,1), and (2,2). In order to prove our first assertion, we will show that, given two arbitrary 3×3 Hermitian matrices H_u, H_d, one can always find an unitary matrix U such that

$$U_{i1}^*(H_u)_{ij}U_{j2}=0 \ , \tag{4a}$$

$$U_{i1}^*(H_d)_{ij}U_{j2}=0 \ , \tag{4b}$$

$$U_{i1}^*U_{i2}=0 \ . \tag{4c}$$

Equations (4) constrain the vector with components U_{i1}^* to be orthogonal to three other vectors. For this to be possible, U_{i2} should be chosen in such a way that those three vectors are linearly dependent: namely,

$$aU_{i2}+b(H_u)_{ij}U_{j2}+c(H_d)_{ij}U_{j2}=0 \ , \tag{5}$$

for some nonvanishing a, b, and c. This equation simply tells us that U_{i2} has to be a normalized eigenvector of the matrix $H_u+(c/b)H_d$, with eigenvalue $(-a/b)$. Once U_{i2} is found, one can explicitly construct a vector U_{i1} satisfying Eqs. (4): namely,

$$U_{i1}=N\epsilon_{ijk}U_{j2}^*U_{l2}^*(H_u)_{lk} \ , \tag{6}$$

where N is a normalizing factor. It follows from Eq. (6) that Eqs. (4a) and (4c) are satisfied, and Eq. (5) then implies that Eq. (4b) is also satisfied. It is obvious that once U_{i1} and U_{i2} are found, the full unitary matrix U can be constructed.[3]

We turn now to the second part of the proof, namely, that if a 3×3 Hermitian matrix $H=MM^\dagger$ has the element H_{12} vanishing, then one can always find a unitary matrix V such that $M'=MV$ has the elements M'_{11}, M'_{13}, M'_{31}, and M'_{22} all vanishing. We give the proof by explicitly displaying the matrix V:

$$V_{i1}=N_1\epsilon_{ijk}M_{1j}M_{3k} \ , \tag{7a}$$

$$V_{i2}=N_2M_{1i}^* \ , \tag{7b}$$

$$V_{i3}=N_3(M_{1i}^*H_{13}-M_{3i}^*H_{11}) \ . \tag{7c}$$

where the N_i are normalization factors. Using the definition $H=MM^\dagger$ and the fact that $H_{21}=0$ one may easily verify that

$$M_{1i}V_{i1}=M_{3i}V_{i1}=M_{1i}V_{i3}=M_{2i}V_{i2}=0 \ , \tag{8a}$$

$$V_{i1}V_{i3}^*=V_{i2}^*V_{i3}=V_{i1}V_{i2}^*=0 \ . \tag{8b}$$

QED.

More than three generations

Before extending our analysis to more than three generations, we will do a simple counting of parameters. For n generations, the number of physically meaningful parameters contained in M_u and M_d includes $2n$ quark masses and $(n-1)^2$ parameters characterizing the mixing matrix, making a total of (n^2+1) parameters. In order to count the number of parameters included in M_u and M_d when the NNI form is assumed, note that there are in this case $(2n-1)$ nonvanishing complex matrix elements in each one of the quark mass matrices. However, one can eliminate $2n$ phases through redefinitions of the right-handed quark fields u_R, d_R, and other $(n-1)$ phases through a common redefinition of the left-hand quark fields u_L, d_L. These redefinitions of the quark-field phases keep the charged current real, diagonal, and the NNI form for M_u and M_d. Therefore, there are altogether $(5n-3)$ real parameters in M_u and M_d when the NNI form is assumed. This number exceeds the number of physical parameters for $n=2$ or 3, equals it for $N=4$, and is less than it for $n\geq 5$. Thus, a parameter counting shows that for five or more generations the NNI hypothesis implies by itself some correlations between the quark masses and mixings. On the other hand, the counting of parameters leads us to suspect that for less than five generations the NNI hypothesis has no physical content by itself, being just a choice of weak basis. For three generations, we have shown that this is indeed the case. Next we will extend the proof to the case of four generations.

We first note that if a 4×4 matrix has the NNI form, then the Hermitian matrix $H=MM^\dagger$ has $H_{12}=H_{23}=H_{14}=0$. The converse is also true: i.e., if those elements of the matrix H vanish, one can always find an unitary matrix V such that $M'=MV$ has the NNI form. We show this by explicitly displaying the matrix V:

$$V_{i1}=N'_1\epsilon_{ijkl}M_{1j}M_{3k}M_{4l} \ , \tag{9a}$$

$$V_{i2}=N'_2M_{1i}^* \ , \tag{9b}$$

$$V_{i3}=N'_3[M_{1i}^*H_{13}H_{34}-M_{3i}^*H_{11}H_{34}+M_{4i}^*(H_{11}H_{33}-|H_{13}|^2)] \ , \tag{9c}$$

$$V_{i4}=N'_4(M_{1i}^*H_{13}-M_{3i}^*H_{11}) \ , \tag{9d}$$

where the N'_i are normalization factors. It is easily verified that with the above choice for V, the matrix M' has the NNI form, provided that H_{12}, H_{23}, and H_{14} vanish.

We have now to show that given two arbitrary Hermitian matrices H_u, H_d, one can always find an unitary matrix U such that

$$U_{i1}^*(H_{u,d})_{ij}U_{j2}=0 \ , \tag{10a}$$

$$U_{i1}^*(H_{u,d})_{ij}U_{j4}=0 \ , \tag{10b}$$

$$U_{i2}^*(H_{u,d})_{ij}U_{j3}=0 \ . \tag{10c}$$

Before indicating how the matrix U can be constructed, it is convenient to make a counting of the free parameters and the constraint equations on U. First note that the overall phases of each one of the columns of U are irrelevant. Furthermore, if we impose normalization of columns, each one of the four columns of U contains at this stage three complex parameters, making a total of 12 complex variables in U. These complex variables have to satisfy the six complex equations (10a)–(10c), plus the six complex equations arising from orthogonality of the different columns of U. Therefore, as expected, we have an equal number of variables and constraint equations.

We show next how the matrix U can be constructed. From Eq. (10a) and the requirement that $U_{i1}^* U_{i2}=0$, it follows that the three vectors U_{i1}^*, $U_{j1}^*(H_u)_{ji}$, and $U_{j1}^*(H_d)_{ji}$ are all orthogonal to U_{i2}. On the other hand, it follows from Eq. (10b) together with $U_{i1}^* U_{i4}=0$, that the same three vectors are also orthogonal to U_{i4}. Since U_{i2} and U_{i4} cannot be collinear, one concludes that U_{i1}^*, $U_{j1}^*(H_u)_{ji}$, and $U_{j1}^*(H_d)_{ji}$, should span a space with less than three dimensions; i.e., they should be linearly dependent. It follows that U_{i1} satisfies

$$(H_u+\gamma H_d)_{ij} U_{j1}=\sigma U_{i1} , \tag{11}$$

which determines U_{i1}, apart from one complex degree of freedom in the choice of γ and a discrete fourfold ambiguity in the choice of the eigenvalue σ. In an entirely similar fashion, using Eqs. (10a) and (10c) one can show that $(H_u)_{ij} U_{j2}$, $(H_d)_{ij} U_{j2}$, and U_{i2} have to be orthogonal to both U_{i3}^* and U_{i1}^*. It follows that $(H_u)_{ij} U_{j2}$, $(H_d)_{ij} U_{j2}$, and U_{i2} are linearly dependent and therefore

$$(H_u+\gamma' H_d)_{ij} U_{j2}=\sigma' U_{i2} . \tag{12}$$

So far, we have the two columns U_{i1} and U_{i2} determined as solutions of Eqs. (11) and (12), and we still have two complex degrees of freedom, γ and γ'. We now use one of these degrees of freedom to impose the condition $U_{i1}^* U_{i2}=0$. Then from Eqs. (11) and (12) one obtains

$$U_{i2}^*(H_u)_{ij} U_{j1}+\gamma U_{i2}^*(H_d)_{ij} U_{j1}=0 , \tag{13a}$$

$$U_{i2}^*(H_u)_{ij} U_{j1}+\gamma'^* U_{i2}^*(H_d)_{ij} U_{j1}=0 . \tag{13b}$$

For $\gamma\neq\gamma'^*$, Eqs. (13) guarantee that Eq. (10a) is satisfied.

We now define the column U_{i3}^* as a normalized vector orthogonal to U_{i1}, U_{i2}, and $(H_u)_{ij} U_{j2}$, and the column U_{i4}^* as a normalized vector orthogonal to U_{i1}, U_{i2}, and $(H_u)_{ij} U_{j1}$. With these definitions and using Eqs. (11) and (12), one readily verifies that the four equations (10b) and (10c) are satisfied. However, at this stage U_{i3}^* and U_{i4} are not automatically orthogonal. We can now adjust the remaining degree of freedom in such a way that $U_{i3}^* U_{i4}=0$. We have thus constructed a unitary matrix U which satisfies Eqs. (10a)–(10c).

It is clear that the procedure just outlined for the construction of the unitary matrix U is much more involved for four generations than it was for three generations. However, the fact that it can be, at least in principle, carried out, proves that the NNI hypothesis has, by itself, no physical consequences for $n=4$.

Left-right models

We will now extend our analysis to the case of left-right-symmetric (LRS) models.[4] A counting of parameters shows that in LRS models with more than two generations it is not possible to put both M_u and M_d in the NNI form, without loss of generality. In other words, starting with arbitrary quark mass matrices M_u and M_d, for more than two generations, it is not possible to find unitary matrices U,V such that both $M_u'=UM_uV$ and $M_d'=UM_dV$ have the NNI form. Let us first recall that the number of physical parameters contained in the mass matrices M_u and M_d, for n generations, includes $2n$ quark masses, $(n-1)^2$ parameters characterizing the left-handed mixing matrix, and n^2 parameters entering in the right-handed mixing matrix,[5] making a total of $(2n^2+1)$ physical parameters. On the other hand, each one of the matrices M_u and M_d in the NNI form have $(2n-1)$ nonvanishing complex matrix elements. The fact that weak-basis transformations of the type $M_u'=LM_uR$, $M_d'=LM_dR$, with L and R diagonal unitary matrices, do not alter the NNI form of M_u and M_d, allows one to remove $(2n-1)$ phases from M_u and M_d. It follows then that in LRS modes, the total number of parameters in M_u and M_d in the NNI form is $3(2n-1)$. For n larger than two this number is less than the number of physical parameters, and therefore in LRS models with more than two generations it is not possible to put both M_u and M_d in the NNI form, without loss of generality. Thus, in LRS models with three or more generations, the assumption of the NNI form for both M_u and M_d implies, by itself, correlations involving the quark masses, the left-handed mixing angles and the right-handed mixing angles.[6]

Naturalness

Naively, the most contrived aspect of the Fritzsch *Ansatz* is the assumption that various matrix elements of M_u and M_d vanish, at least at the tree level. However, our results show that, within the context of the standard model with less than five generations, that "assumption" is just a choice of a special weak basis, which we will designate by "NNI basis." For definiteness, let us consider the standard model with three generations. We have shown that, starting with completely arbitrary Yukawa couplings,

$$\mathcal{L}_Y=g_{ij}^d\overline{\Psi}_{iL} d_{jR}\Phi+g_{ij}^u\overline{\Psi}_{iL} u_{jR}\tilde{\Phi}+\text{H.c.} , \tag{14}$$

one can always make a weak-basis transformation into the NNI basis, where $g_{11}^{d,u}$, $g_{13}^{d,u}$, $g_{31}^{d,u}$, and $g_{22}^{d,u}$ all vanish. For three generations, the physical content of the Fritzsch *Ansatz* resides then in the assumption that in the NNI basis

$$|g_{12}^{d,u}|=|g_{21}^{d,u}|, \quad |g_{23}^{d,u}|=|g_{32}^{d,u}| . \tag{15}$$

Although the relations (15) are not natural in the technical sense,[7] we find them plausible.

The difficulty in implementing the Fritzsch *Ansatz* in a technically natural way originates from the need for a sort of left-right symmetry under which the term $(\overline{\Psi}_{1L} d_{2R}\Phi)$ would transform into $(\overline{\Psi}_{2L} d_{1R}\Phi)^\dagger$. Such a symmetry cannot, of course, exist in the standard model.

For simplicity, we have considered the standard model with only one Higgs doublet. For more than one Higgs doublet, one has to distinguish between models with Higgs natural flavor conservation (HNFC) and those without it. In models with HNFC such as the minimal supersymmetric model,[8] it is obvious that it is still possible, through a choice of weak basis, to make g^d, g^u, M_u, and M_d all have the NNI form. In models without HNFC, where more than one Higgs doublet gives mass to the quarks of a given charge, one can always find a weak basis where both M_u and M_d have the NNI form, but, in general, it is not possible to have all the Yukawa coupling matrices in the NNI form simultaneously.

It should be pointed out that, in models with several Higgs doublets, it is possible[9] to implement the NNI form for the quark mass matrices by means of *ad hoc* global symmetries. Our results imply that for fewer than five generations such models, although they lead to constraints on the Higgs interactions, do not enforce any relations among the quark masses and mixing parameters, contrary to what one might naively expect.

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[3]It should be pointed out that we did not use the complex degree of freedom of the choice of (c/b). It is easily found that, by using this degree of freedom, one may actually go beyond the NNI form, imposing one complex equation relating the nonvanishing matrix elements of M_u and M_d.

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[5]We adopt the conventional counting of parameters, where the rephasing freedom of the quark fields is used to eliminate a maximum of phases from the left-handed mixing matrix.

[6]However, we have verified explicitly that for $n=3$ there is always a weak basis in which the matrix elements (1,1), (1,3), and (3,1)—but not (2,2)—of both M_u and M_d vanish.

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