Long-distance contributions to the decay $K^+ \to \pi^+ \nu\bar{\nu}$

D. Rein and L. M. Sehgal
III. Physikalisches Institut, Technische Hochschule Aachen, Aachen, West Germany
(Received 6 February 1989)

We calculate two distinct long-distance contributions to the decay $K^+ \to \pi^+ \nu\bar{\nu}$: (i) that arising from the intermediate states πl and Dl and (ii) that associated with hadronic contributions to the $K\pi Z$ vertex. A combined branching ratio of 0.7×10^{-13} is obtained (for three neutrino species), to be compared with the short-distance estimate $(1-8)\times 10^{-10}$. A remark is made on the energy spectrum of the pion.

According to modern wisdom,[1] the rare decays of the K meson such as $K^0 \to \mu^+\mu^-$, $K \to \pi e^+ e^-$, and $K \to \pi\nu\bar{\nu}$ are determined by two types of interaction: (a) a short-distance (SD) interaction between free quarks and leptons, producing an effective local four-fermion coupling $(\bar{s}\Gamma_\mu d)(\bar{l}\Gamma'_\mu l)$; and (b) a long-distance (LD) interaction involving hadronic (rather than quark) degrees of freedom in intermediate states. The SD contribution is sensitive to the existence of heavy quarks occurring as virtual particles in the transition $s\bar{d} \to l\bar{l}$. By contrast, the LD effects are largely governed by low-mass hadronic states, virtual heavy systems being damped by energy denominators and form factors. (For a recent review of rare K decays, see Ref. 2.)

It turns out that of the processes mentioned above, the decay $K_L \to \mu^+\mu^-$ has an experimental rate that is consistent with expectations[3] based on the measured rate of $K_L \to 2\gamma$. Likewise, the experimental rate of $K^+ \to \pi^+ e^+ e^-$ is equally compatible with the short-distance estimate (based on the electromagnetic penguin diagram) and the long-distance estimate based on hadronic models of the $K^+\pi^+\gamma$ vertex.[4] These channels therefore offer no direct evidence of a specific SD mechanism and do not place a significant constraint on the parameters of heavy quarks that might mediate these transitions.

A more interesting situation seems to be offered by the decay $K^+ \to \pi^+\nu\bar{\nu}$. Calculations based on the quark diagrams shown in Fig. 1 yield an effective Lagrangian[5]

$$\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{\pi \sin^2\theta_w} (\bar{s}_L \gamma_\mu d_L) \times \sum_{i=e,\mu,\tau} \bar{\nu}_{iL}\gamma_\mu \nu_{iL} \sum_{j=c,t} V_{js}^* V_{jd} D(x_j) , \quad (1)$$

where V_{ij} are elements of the quark mixing matrix and

$$D(x) = \frac{1}{8}\left[1 + \frac{3}{(1-x)^2} - \frac{(4-x)^2}{(1-x)^2}\right] x \ln x + \frac{x}{4} - \frac{3}{4}\frac{x}{1-x} \quad (2)$$

with $x_i = m_i^2/m_W^2$. [Charged-lepton masses are neglected in Eq. (1).] Using empirical information on the quark mixing parameters, and a range of top-quark masses 50 GeV $< m_t <$ 200 GeV, Ellis *et al.*[6] obtained for the branching ratio of $K^+ \to \pi^+\nu\bar{\nu}$ (summed over three neutrino species):

$$B(K^+ \to \pi^+\nu\bar{\nu}) = (1-8)\times 10^{-10} . \quad (3)$$

This is a sizable rate, amenable to detection by ongoing experiments, and would be important evidence for the SD mechanisms depicted in Fig. 1, provided the LD effects can be shown to be negligible. The authors of Ref. 6 remarked that the LD contributions in the present case are very small. In this paper, we carry out an explicit calculation in order to quantify this statement.

We first consider a contribution to $K^+ \to \pi^+\nu\bar{\nu}$ that is an LD analog of the process shown in Fig. 1(a), i.e., a transition involving two virtual W bosons and hence two semileptonic charged-current interactions. The process is shown in Fig. 2, where we have taken the intermediate states to be $\pi^0 e$ and $D^0 e$. The resulting amplitude is determined by the amplitudes for the transitions $K^+ \to \pi^0 e^+ \nu_e$, $\pi^+ \to \pi^0 e^+ \nu_e$, $D^0 \to K^- e^+ \nu_e$, and $D^0 \to \pi^- e^+ \nu_e$ and possesses Glashow-Iliopoulos-Maiani (GIM) symmetry in the sense of vanishing when $m_D = m_\pi$. [A model of this type was used by one of the authors (L.M.S.) in the precharm era to estimate the

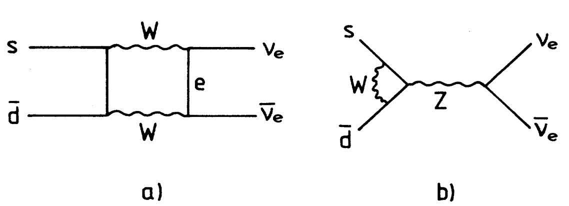

FIG. 1. Feynman diagrams for the transition $s\bar{d} \to \nu\bar{\nu}$.

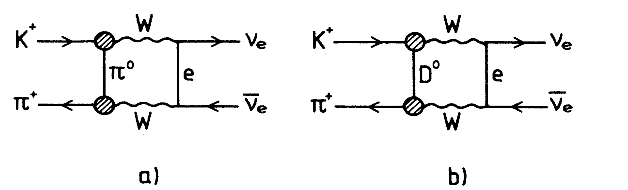

FIG. 2. Long-distance contribution to $K^+ \to \pi^+\nu\bar{\nu}$ involving two semileptonic charged-current interactions.

second-order weak amplitude of $K_2 \to \pi^0 e^+ e^-$ (Ref. 7).]

The invariant matrix element corresponding to the transition in Fig. 2 may be written as

$$M = F(w,\Delta)\bar{u}\not{Q}(1+\gamma_5)v(k') \; , \tag{4}$$

where the momenta are defined by $K(Q) \to \pi(p) + \nu(k) + \bar{\nu}(k')$ and the form factor $F(w,\Delta)$ is a function of the two independent invariants

$$w = t' + t, \quad \Delta = t' - t$$

with (5)

$$t = (Q-k)^2, \quad t' = (Q-k')^2 \; .$$

In particular, w is related to s, the invariant mass of the $\nu\bar{\nu}$ pair by

$$w = m_K^2 + m_\pi^2 - s \tag{6a}$$

and to the pion energy in the rest frame of the K meson by

$$w = 2m_K E_\pi \; . \tag{6b}$$

Here m_K and m_π denote the masses of the K and π mesons. The physical domain of the Dalitz plot is defined by

$$\begin{aligned} &2m_K m_\pi \le w \le m_K^2 + m_\pi^2 \; , \\ &-(w^2 - 4m_K^2 m_\pi^2)^{1/2} \le \Delta \le (w^2 - 4m_K^2 m_\pi^2)^{1/2} \; . \end{aligned} \tag{7}$$

Following the procedure in Ref. 7, we evaluate $F(w,\Delta)$ using a dispersion relation. The imaginary part of F (neglecting the electron mass) is given by

$$\begin{aligned} \mathrm{Im}F(w,\Delta) = -\frac{G^2}{8\pi}\cos\theta\sin\theta \times \Bigg[&\frac{(t-m_\pi^2)^2}{t}\Theta(t-m_\pi^2) I_\pi(t) \\ &- \frac{(t-m_D^2)^2}{t}\Theta(t-m_D^2) I_D(t) \Bigg] \; , \end{aligned} \tag{8}$$

where we have denoted the mass of the D meson by m_D. The two terms in this expression are the contributions of the $\pi^0 e$ and $D^0 e$ intermediate states, which appear with opposite signs. Note that the absorptive part associated with $\pi^0 e$ occurs for $t > m_\pi^2$ which overlaps with the physical domain $m_\pi^2 < t < m_K^2$ of the decay $K^+ \to \pi^+ \nu\bar{\nu}$. On the other hand, the absorptive part due to the $D^0 e$ state occurs only for values $t > m_D^2$ which are outside the physical region. The functions $I_\pi(t)$ and $I_D(t)$ depend on the form factors describing the matrix elements $\langle \pi^0 | V | K^+ \rangle$, $\langle \pi^0 | V | \pi^+ \rangle$, $\langle K^+ | V | D^0 \rangle$, and $\langle \pi^+ | V | D^0 \rangle$. In the case that these form factors are taken to be constant, $I_\pi(t) = I_D(t) = 1$. For our calculation, we have represented the form factors by pole terms, i.e.,

$$\begin{aligned} &f(\pi^+ \to \pi^0) = (1 + m_\rho^2/Q^2)^{-1} \; , \\ &f(K^+ \to \pi^0) = (1 + m_{K^*}^2/Q^2)^{-1} \; , \\ &f(D^0 \to \pi^-) = (1 + m_{D^*}^2/Q^2)^{-1} \; , \\ &f(D^0 \to K^-) = (1 + m_{F^*}^2/Q^2)^{-1} \end{aligned} \tag{9}$$

and have neglected all renormalization effects at $Q^2 = 0$. The functions $I_\pi(t)$ and $I_D(t)$ are then expressible as integrals which are given in the Appendix. The real part of $F(w,\Delta)$ is obtained by means of a dispersion relation in w, holding Δ fixed: i.e.,

$$\mathrm{Re}F(w,\Delta) = \frac{1}{\pi} P \int \frac{\mathrm{Im}F(w',\Delta)dw'}{w' - w} \tag{10}$$

(remembering that $t = w - \Delta$). The differential decay rate is[8]

$$\frac{d\Gamma}{dw\,d\Delta} = \frac{1}{2^9 \pi^3 m_K^2}(w^2 - 4m_K^2 m_\pi^2 - \Delta^2) \times [|\mathrm{Re}F(w,\Delta)|^2 + |\mathrm{Im}F(w,\Delta)|^2] \; . \tag{11}$$

We now summarize our results. Writing the total decay width of $K^+ \to \pi^+ \nu_e \bar{\nu}_e$ as

$$\Gamma(K^+ \to \pi^+ \nu_e \bar{\nu}_e) = \Gamma_{\mathrm{abs}} + \Gamma_{\mathrm{disp}}^{\pi} + \Gamma_{\mathrm{disp}}^{D} - \Gamma_{\mathrm{disp}}^{\mathrm{int}} \; , \tag{12}$$

where Γ_{abs} is the absorptive contribution of the $\pi^0 e$ graph, $\Gamma_{\mathrm{disp}}^{\pi}$ ($\Gamma_{\mathrm{disp}}^{D}$) the dispersive contributions of the $\pi^0 e$ ($D^0 e$) intermediate states, and $\Gamma_{\mathrm{disp}}^{\mathrm{int}}$ the interference term arising from the GIM cancellation of the $\pi^0 e$ and $D^0 e$ amplitudes, we find

$$\begin{aligned} &\Gamma_{\mathrm{abs}} = 0.02 \times 10^{-6}\ \mathrm{s}^{-1} \; , \\ &\Gamma_{\mathrm{disp}}^{\pi} = 3.75 \times 10^{-6}\ \mathrm{s}^{-1} \; , \\ &\Gamma_{\mathrm{disp}}^{D} = 0.27 \times 10^{-6}\ \mathrm{s}^{-1} \; , \\ &\Gamma_{\mathrm{disp}}^{\mathrm{int}} = 2.00 \times 10^{-6}\ \mathrm{s}^{-1} \; , \\ &\Gamma(K^+ \to \pi^+ \nu_e \bar{\nu}_e) = 2.02 \times 10^{-6}\ \mathrm{s}^{-1} \; . \end{aligned} \tag{13}$$

Assuming the three neutrino flavors to be equally probable (i.e., neglecting the effects of the charged-lepton mass in the intermediate state) we obtain the long-distance contribution to the branching ratio of $K^+ \to \pi^+ \nu\bar{\nu}$ to be

$$B(K^+ \to \pi^+ \nu\bar{\nu})|_{\mathrm{LD}} = 0.25 \times 10^{-13} \; . \tag{14}$$

This is distinctly smaller than the short-distance estimate given in Eq. (3).

It should be stressed that in the absence of form factors, the diagrams in Fig. 2 are each quadratically divergent. The GIM cancellation removes the quadratic divergence, but a logarithmic divergence persists. With the inclusion of the pole-type form factors, both diagrams converge separately, with the noncharmed intermediate state $\pi^0 e$ giving the dominant contribution. (It may be remarked that in the analogous calculation of $K_2 \to \pi^0 e^+ e^-$ described in Ref. 7, a significantly more convergent amplitude is obtained because of CP invariance, which dictates a cancellation between $\pi^+ \nu$ and $\pi^- \bar{\nu}$ intermediate states occurring in the t and t' channels, the resulting rate for $K_2 \to \pi^0 \nu_e \bar{\nu}_e$ being 2 orders of magnitude lower than for $K^+ \to \pi^+ \nu_e \bar{\nu}_e$.)

We turn now to a second LD contribution, depicted in Fig. 3, which is the hadronic analogue of the quark diagram in Fig. 1(b). The amplitude of $K^+ \to \pi^+ \nu\bar{\nu}$ in this case may be written as

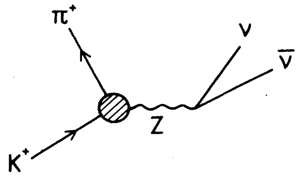

FIG. 3. Long-distance contribution to $K^+ \to \pi^+ \nu\bar{\nu}$ involving a nonleptonic $\Delta S = 1$ interaction together with a virtual Z.

$$M = \frac{G}{\sqrt{2}} M_\mu \bar{u}(k')\gamma_\mu(1+\gamma_5)v(k) , \quad (15)$$

where

$$M_\mu = \int \langle \pi^+(p) | T(H_w(0) J_\mu^{\rm NC}(x)) | K^+(Q) \rangle e^{iq\cdot x} d^4x$$

with

$$J_\mu^{\rm NC} = V_\mu^3 + A_\mu^3 - 2\sin^2\theta_W V_\mu^{\rm em} . \quad (16)$$

V_μ^3 and A_μ^3 being the third components of the vector and axial-vector isospin currents and $V_\mu^{\rm em}$ the electromagnetic current.[9] The operator H_w is the nonleptonic charged-current Hamiltonian, which we assume to be a product of left-handed currents. The matrix element M_μ can be parametrized as

$$M_\mu = g_+(q^2)(Q+p)_\mu + g_-(q^2)(Q-p)_\mu , \quad (17)$$

where $q = Q - p$. The form factor g_- makes no contribution to the amplitude M (for massless neutrinos) and so our task is reduced to determining $g_+(q^2)$.

We first attempt to determine g_+ at $q^2 = 0$ using current-algebra arguments. From Eq. (16), we have (after an integration by parts)

$$iq_\mu M_\mu = -\int d^4x\, e^{iq\cdot x} \langle \pi^+ | \partial_\mu T(H_w(0) J_\mu^{\rm NC}(x)) | K^+ \rangle . \quad (18)$$

Using the identity

$$\partial_\mu T(H_w(0) J_\mu^{\rm NC}(x)) = T(H_w(0) \partial_\mu J_\mu^{\rm NC}(x)) - [H_w(0), J_0^{\rm NC}(\mathbf{x},0)] \quad (19)$$

and taking the limit $q \to 0$, we obtain

$$\lim_{q\to 0} iq_\mu M_\mu = \langle \pi^+ | [H_w(0), Q^{\rm NC}(0)] | K^+ \rangle - \int d^4x \langle \pi^+ | T(H_w(0) \partial_\mu J_\mu^{\rm NC}(x)) | K^+ \rangle . \quad (20)$$

The first term involves an equal-time commutator of H_w with the weak-neutral-current charge

$$Q^{\rm NC} = F_3 + F_3^5 - 2\sin^2\theta_W Q^{\rm em} , \quad (21)$$

where F_3 and F_3^5 are the charges associated with the currents V_μ^3 and A_μ^3. Using the fact that

$$[H_w(0), F_3] = [H_w(0), F_3^5] \quad (22)$$

(which is a consequence of H_w containing left-handed currents only), the commutator term reduces to

$$\begin{aligned} \langle \pi^+ | [H_w(0), Q^{\rm NC}(0)] | K^+ \rangle &= 2\langle \pi^+ | [H_w(0), F_3] | K^+ \rangle \\ &= -\langle \pi^+ | H_w(0) | K^+ \rangle \\ &= \frac{F_\pi}{2\sqrt{2}} \langle \pi^+\pi^- | H_w | K_s \rangle , \end{aligned} \quad (23)$$

where the last step follows from chiral-symmetry arguments.[10] The second term in Eq. (20) can be converted into a matrix element for $K^+ \to \pi^+ + \text{soft } \pi^0$ using PCAC (partial conservation of axial-vector current):

$$\int d^4x \langle \pi^+ | T(H_w(0) \partial_\mu J_\mu^{\rm NC}(x)) | K^+ \rangle = \frac{F_\pi}{\sqrt{2}} \langle \pi^+ \pi^0(\text{soft}) | H_w | K^+ \rangle . \quad (24)$$

While the physical amplitude of $K^+ \to \pi^+\pi^0$ is suppressed by the $\Delta I = \frac{1}{2}$ rule, a model is required to estimate the off-shell amplitude $K^+ \to \pi^+ + \text{soft } \pi^0$. As an example, the ansatz of Sakurai,[11] based on an exact $\Delta I = \frac{1}{2}$ Hamiltonian, yields

$$\frac{\langle \pi^+\pi^0(\text{soft}) | H_w | K^+ \rangle}{\langle \pi^+\pi^- | H_w | K_S^0 \rangle} = -i \frac{m_\pi^2}{2(m_K^2 - m_\pi^2)} . \quad (25)$$

Alternatively, a model in which the weak Hamiltonian contains both $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$ pieces, but is built from left-handed currents only, yields[12]

$$\frac{\langle \pi^+\pi^0(\text{soft}) | H_w | K^+ \rangle}{\langle \pi^+\pi^0 | H_w | K^+ \rangle |_{\rm physical}} = 1 - \frac{1}{3} \frac{m_\pi^2}{m_K^2 - m_\pi^2} \left[1 + \frac{1}{2} \frac{\tilde{C}}{C_4} \right] , \quad (26)$$

where $\tilde{C} \equiv C_1 + 2C_2 + 2C_3$ and C_4 are coefficients defined in Ref. 13 and estimated to be $\tilde{C} = 2.82$, $C_4 = 0.4$. In this case, the off-shell amplitude differs from the mass-shell value only by an amount of order m_π^2/m_K^2. In either of the above models, the term in Eq. (24) is small compared to the commutator term in Eq. (23), and will be neglected below.

With these arguments, we have, finally,

$$\begin{aligned} |\lim_{q\to 0} iq_\mu M_\mu| &= (m_K^2 - m_\pi^2) g_+(0) \\ &= \frac{F_\pi}{2\sqrt{2}} \langle \pi^+\pi^- | H_w | K_S^0 \rangle . \end{aligned} \quad (27)$$

Numerically,[14] $\langle \pi^+\pi^- | H_w | K_S \rangle = 7.85 \times 10^{-7} m_K$, $F_\pi = 135$ MeV, so that

$$g_+(0) = 8.1 \times 10^{-8} . \quad (28)$$

To relate the decay $K^+ \to \pi^+\nu\bar{\nu}$ to $K^+ \to \pi^0 e^+ \nu_e$, we define the amplitude for the latter as

$$M(K^+ \to \pi^0 e^+ \nu_e) = \frac{G}{\sqrt{2}} [f_+(Q+p)_\mu + f_-(Q-p)_\mu] \times \bar{e}\gamma^\mu(1+\gamma_5)\nu_e . \quad (29)$$

The relevant form factor f_+ is given by

$$f_+(0)=\frac{1}{\sqrt{2}}\sin\theta_C \ , \tag{30}$$

which is to be compared with the result $g_+(0)$ in Eq. (28). Assuming similar q^2 dependence for f_+ and g_+, we obtain

$$\frac{\sum_i \Gamma(K^+\to\pi^+\nu_i\bar{\nu}_i)}{\Gamma(K^+\to\pi^0 e^+\nu_e)}=3\left|\frac{g_+(0)}{f_+(0)}\right|^2=8.1\times10^{-13} \tag{31}$$

giving a branching ratio

$$B(K^+\to\pi^+\nu\bar{\nu})|_{\mathrm{LD}}=0.4\times10^{-13} \ . \tag{32}$$

This is very similar to the estimate (14) obtained from the LD contribution shown in Fig. 2. Adding the two incoherently we find the total LD contribution to the decay $K^+\to\pi^+\nu\bar{\nu}$ to be 0.7×10^{-13}, which is 3 orders of magnitude smaller than the short-distance estimate obtained in Ref. 6.

A final remark concerning the energy spectrum of the π^+ in $K^+\to\pi^+\nu\bar{\nu}$: the short-distance interaction [Eq. (1)] implies that this spectrum should be *identical* to the pion spectrum in $K^0\to\pi^- e^+\nu_e$. This follows from the fact that the Lagrangian (1) exhibits *lepton locality*,[15] which is also a feature of the charged-current interaction describing K_{l3} decay. If long-range contributions of the form shown in Fig. 2 were important, one would expect some distortion in the pion spectrum, connected with the fact that these contributions are nonlocal with respect to the two leptons. We have illustrated this in Fig. 4, which compares the pion energy spectrum following from the long-distance contribution of Fig. 2 with that of K_{l3} decay. If the decay $K^+\to\pi^+\nu\bar{\nu}$ is indeed observed with the branching ratio $\gtrsim 10^{-10}$ expected from short-distance effects, a measurement of the pion energy spectrum would provide important corroboration of this mechanism.

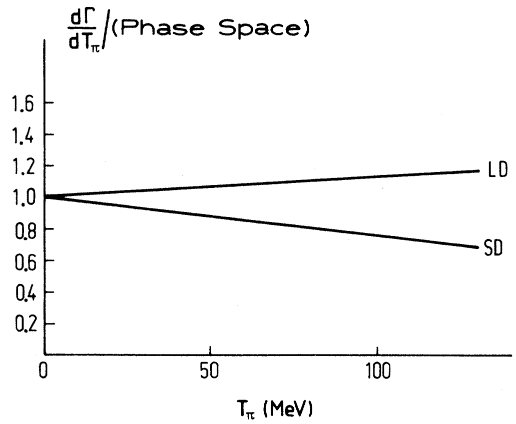

FIG. 4. Energy spectrum of pion (T_π=kinetic energy) expected from short-distance (SD) interaction, compared with that obtained from the long-distance (LD) interaction of Fig. 2.

This research has been supported by the Bundesministerium für Forschung und Technologie under Contract No. 05 4AC 96P.

APPENDIX

In order to determine the integrals $I_\pi(t)$ and $I_D(t)$ of Eq. (8) we follow the lines of Ref. 7 starting from

$$\mathrm{Im}M=\mathrm{Im}(M^{(\pi)}+M^{(D)}) \ , \tag{A1}$$

where the first term reads

$$\mathrm{Im}M^{(\pi)}=\frac{-G^2}{32\pi^2}\cos\theta_C\sin\theta_C\int\frac{d^3\mathbf{q}}{q_0}\frac{d^3\mathbf{l}}{l_0}f^{(\pi)}((Q-q)^2)f^{(\pi)}((p-q)^2)\bar{u}(k)\Gamma^{(\pi)}(1+\gamma_5)v(k')\delta^4(Q-k-q-l) \tag{A2}$$

and an analogous expression holds for $\mathrm{Im}M^{(D)}$. The functions $\Gamma^{(i)}$ (the index i refers to π,D intermediate states) are given by

$$\Gamma^{(\pi,D)}=\not{Q}\not{l}(\not{p}+\not{q})-\not{l}\not{q}\not{p}-m^2_{\pi,D}\not{l}+(t-m^2_{\pi,D})(\not{q}+\not{p}) \tag{A3}$$

and the form factors $f^{(\pi)},f^{(D)}$ by ρ,D^* pole terms [cf. Eq. (9)]. (We take $m_{K^*}=m_\rho$, $m_{F^*}=m_{D^*}$.) The form factors give rise to momentum-dependent terms

$$(1-2kl/m_V^2)^{-1}(1-2k'l/m_V^2)^{-1} \quad (m_V=m_\rho,m_{D^*}) \ ,$$

which are transformed into integrals according to

$$(1-2kl/m_V^2)^{-1}(1-2k'l/m_V^2)^{-1}=\int_{-1/2}^{+1/2}\frac{dy}{(1+zl)^2} \tag{A4}$$

by choosing

$$z=\frac{1}{m_V^2}[k'-k-2y(k'+k)] \ . \tag{A5}$$

It is then straightforward to integrate terms such as

$$\int \frac{d^3\mathbf{q}}{q_0}\frac{d^3l}{l_0}\frac{\not{l}}{(1+zl)^2}\delta^4(P-q-l)$$

in a frame, where $P=Q-k$ has vanishing spatial components and consequently $P_0=\sqrt{(Q-k)^2}=\sqrt{t}$. As a result we obtain, in the case of the pion in the intermediate state,

$$\int_{-1/2}^{+1/2}dy\int\frac{d^3\mathbf{q}}{q_0}\frac{d^3l}{l_0}[\not{Q}\not{l}(\not{p}+\not{q})-\not{l}\not{q}\not{p}-m_\pi^2\not{l}+(t-m_\pi^2)(\not{p}+\not{q})]\frac{\delta^4(P-q-l)}{(1+z_\pi l)^2}$$
$$=2\not{Q}\int_{-1/2}^{+1/2}dy\frac{\{(t-m_\pi^2)[2(Pz_\pi)^2-tz_\pi^2]+2Pz_\pi t\}I_b^{(\pi)}(t)-2Pz_\pi tI_a^{(\pi)}(t)}{(Pz_\pi)^2-tz_\pi^2}\ , \quad \text{(A6)}$$

where

$$I_a^{(\pi)}(t)=\frac{2\pi}{[(Pz_\pi)^2-tz_\pi^2]^{1/2}}\ln\frac{2t+\{Pz_\pi+[(Pz_\pi)^2-tz_\pi^2]^{1/2}\}(t-m_\pi^2)}{2t+\{Pz_\pi-[(Pz_\pi)^2-tz_\pi^2]^{1/2}\}(t-m_\pi^2)} \quad \text{(A7a)}$$

and

$$I_b^{(\pi)}(t)=\frac{2\pi(t-m_\pi^2)}{t+Pz_\pi(t-m_\pi^2)+\frac{z_\pi^2}{4}(t-m_\pi^2)^2} \quad \text{(A7b)}$$

with

$$z_\pi=[k'-k-2y(k'+k)]/m_\rho^2\ . \quad \text{(A7c)}$$

The corresponding expressions $I_a^{(D)}(t)$, $I_b^{(D)}(t)$, and z_D are obtained changing m_π, m_ρ into m_D, m_{D^*} in (A7). If the square root in (A7a) turns out to be imaginary the logarithm should be replaced by an arctan function according to the usual rules of analytic continuation.

Inserting (A6) and (A7) into (A1) and (A2) and making use of definitions (4) and (8) we may easily extract the desired functions $I_\pi(t)$ and $I_D(t)$ thus finding

$$I_{\pi,D}(t)=\frac{t}{2\pi(t-m_{\pi,D}^2)^2}\int_{-1/2}^{+1/2}dy(\{(t-m_{\pi,D}^2)[2(Pz_{\pi,D})^2-tz_{\pi,D}^2]+2Pz_{\pi,D}t\}I_b^{(\pi,D)}(t)$$
$$-2Pz_{\pi,D}tI_a^{(\pi,D)}(t))/[(Pz_{\pi,D})^2-tz_{\pi,D}^2]\ . \quad \text{(A8)}$$

This result corrects Eq. (14) of Ref. 7 which neglected the logarithmic term $I_a(t)$. As stated in the text the functions $I_{\pi,D}(t)$ approach unity, if form-factor masses m_ρ, m_{D^*} are shifted towards infinity. This can be verified expanding (A8) in terms of Pz and tz^2 which are of order m_V^{-2} and m_V^{-4}, respectively.

---

[1]M. K. Gaillard and B. W. Lee, Phys. Rev. D **9**, 897 (1974); M. K. Gaillard, B. W. Lee, and R. E. Shrock, *ibid.* **13**, 267 (1976); R. E. Shrock and M. B. Voloshin, Phys. Lett. **87B**, 375 (1979); T. Inami and C. S. Lim, Prog. Theor. Phys. **65**, 297 (1981).

[2]Ling-Fong Li, Carnegie-Mellon Report No. CMU-HEP 88-11 (unpublished).

[3]L. M. Sehgal, Phys. Rev. **183**, 1511 (1969); Nuovo Cimento **45**, 785 (1966).

[4]A. Vainshtein, V. Zakharov, L. Okun, and M. Shifman, Yad. Fiz. **24**, 820 (1976) [Sov. J. Nucl. Phys. **24**, 427 (1976)], and references therein.

[5]Inami and Lim (Ref. 1).

[6]J. Ellis, J. S. Hagelin, and S. Rudaz, Phys. Lett. B **192**, 201 (1987); J. Ellis, J. S. Hagelin, S. Rudaz, and D.-D. Wu, Nucl. Phys. **B304**, 205 (1988); see also J. Hagelin and L. Littenberg, Prog. Part. Nucl. Phys. (to be published).

[7]L. M. Sehgal, Phys. Rev. D **6**, 5 (1972).

[8]This is smaller by a factor of 2 than the formula given in Eq. (16) of Ref. 7.

[9]In what follows, we identify weak isospin with strong isospin, neglecting the $s\bar{s}$ and $c\bar{c}$ components of the neutral current.

[10]J. A. Cronin, Phys. Rev. **161**, 1483 (1967); T. N. Pham and I. Dupont, Phys. Rev. D **29**, 1369 (1984).

[11]J. Sakurai, Phys. Rev. **156**, 1508 (1967).

[12]J. Donoghue, E. Golowich, W. Ponce, and B. Holstein, Phys. Rev. D **21**, 186 (1980).

[13]M. A. Shifman, A. I. Vainshtein, and V. J. Zakharov, Nucl. Phys. **B120**, 315 (1977).

[14]T. J. Devlin and J. O. Dickey, Rev. Mod. Phys. **51**, 237 (1979).

[15]A. Pais and S. Treiman, Phys. Rev. **176**, 1974 (1968); S. K. Singh and L. Wolfenstein, Nucl. Phys. **B24**, 77 (1970).