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Gauge-invariant correlation functions

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A formalism is discussed that gives correlation functions, the poles of which define the physical excitation spectrum in gauge theories. Fermionic excitations in QED are used as an example. The fermion spectrum in a high-temperature plasma is studied as an illustrative application of the formalism. The connection to kinetic theory is made.

In nongauge theories the poles of propagators give the spectrum of the elementary excitations. In gauge theories this is generally not true due to the appearance of nonphysical (ghost) states. However, suppose that we choose a gauge condition that both fixes the gauge completely and admits a time-independent Hamiltonian. Because of complete gauge fixing there are no unphysical states in the Hilbert space spanned by the field operators. By using a complete set of eigenstates of the Hamiltonian, the propagator can then be cast in the spectral representation as in nongauge theories.[1,2] We show, however, that the requirement of complete gauge fixing can be relaxed, so that it is enough if we have a gauge condition which leads to a propagator that is invariant under the residual gauge freedom. This is because then the "physical" propagator, which we can obtain by deleting the nonphysical states by complete gauge fixing, must have this same invariant expression.

To be specific, consider fermionic excitations in a vacuum or in a homogeneous medium with the interactions described by QED. If we study the correlation of the fermion fields at an arbitrary point x and a point shifted by a displacement y, the corresponding propagator

$$S(y)=i\langle\psi(x+y)\bar{\psi}(x)\rangle\,, \tag{1}$$

which depends only on y, is manifestly gauge dependent in the sense that different incomplete gauge fixings give different results. Now let us choose a gauge condition so that the photon field vanishes *in the direction of the displacement*:

$$y\cdot A(x)=0,\quad \forall x\,. \tag{2}$$

For sake of simplicity assume for a while that the displacement y is in the three-direction, $y=(0,0,0,y_3)$. The independent gauge fields can be chosen to be A_1 and A_2. A_0 is not a dynamical field but it can be expressed by A_1 and A_2 and their conjugate momenta. There is a residual gauge freedom; we are still allowed to perform gauge transformations which are independent of x_3. To fix the gauge completely, we have to specify the value of A_0 and its derivative with respect to x_3 at an arbitrarily chosen reference point $x_3^{(0)}$ (see, for example, Refs. 3 and 4):

$$\begin{aligned}A_0(x_0,x_1,x_2,x_3^{(0)})&=P(x_0,x_1,x_2)\,,\\ \partial_3A_0(x_0,x_1,x_2,x_3^{(0)})&=Q(x_0,x_1,x_2)\,.\end{aligned} \tag{3}$$

Within the gauge condition (2) the fermion propagator does not depend on this residual freedom. To see that, perform a gauge transformation $\alpha(x)$. The propagator changes then to

$$S(y)\rightarrow S'(y)=i\langle\psi(x+y)e^{i[\alpha(x+y)-\alpha(x)]}\bar{\psi}(x)\rangle\,|_{y\cdot A=0}\,. \tag{4}$$

But the allowed residual transformations were those that do not depend on the coordinate in the displacement direction (three-direction) and hence $\alpha(x+y)=\alpha(x)$ establishing the invariance of $S(y)$ under the residual gauge transformations. We can imagine that we had fixed the gauge completely (by specifying the functions P and Q). The physical propagator obtained in that way is then necessarily this residually invariant propagator. Hence, we find that the unphysical states decouple in the fermion propagator when the gauge is chosen so that the photon field vanishes in the direction of propagation.

However, the gauge condition (2) is not the best choice for practical calculations. We would like to write the physical correlation function

$$\begin{aligned}C(y)&=i\langle\psi(x+y)\bar{\psi}(x)\rangle\,|_{y\cdot A=0}\\ &=i\langle[e^{y\cdot\partial_x}\psi(x)]\bar{\psi}(x)\rangle\,|_{y\cdot A=0}\end{aligned} \tag{5}$$

in a manifestly gauge-invariant form. This can be done obviously by replacing the partial derivative by the covariant one. So, we have the correlation function

$$C(y)=i\langle[e^{y\cdot D_x}\psi(x)]\bar{\psi}(x)\rangle\,, \tag{6}$$

which is gauge invariant and reduces to the fermion propagator, if we fix the gauge completely by the conditions (2) and (3). Consequently, the physical excitation spectrum can be obtained from the poles of this correlation function.

It should be noted that the procedure applied here is very similar to the calculation of the partition function in gauge theories. The physical, gauge-invariant, partition function[4,5]

$$Z\sim\int[dA]\Delta_F[A]\delta(F[A])\exp\left[\int d^4x\,\mathcal{L}[A]\right]\,, \tag{7}$$

agrees with the naive definition $\mathrm{Tr}e^{-\beta H}$ in a completely fixed gauge. In an arbitrary gauge, $\mathrm{Tr}e^{-\beta H}$ has no physical meaning due to the appearance of the unphysical states. As a matter of fact, the analogy is even closer.

$\mathrm{Tr}e^{-\beta H}$ in the axial gauge $A_3=0$ is independent of the residual gauge freedom,[4] and hence the unphysical states do not contribute to it.

Using the trick of Ref. 6 we can rewrite Eq. (6) in the form

$$C(y)=i\left\langle \psi(x)\exp\left[-ie\int_z^x dw^\mu A_\mu(w)\right]\bar{\psi}(x)\right\rangle, \quad (8)$$

where $z=x+y$ and the integration is along a straight line from z to x. Note that the correlation function given by Eq. (8) has been known for a long time to be gauge invariant (see, for example, Refs. 7-9). However, the point that we would like to make here is that its relation to the fermion propagator in gauge (2) together with the invariance of the fermion propagator under the residual gauge freedom in this gauge is the fact which gives the physical meaning.

It is known[8] that for any completely fixed axial gauge on the lattice there is a unique path $r(\tau)$ joining the end points x and z in Eq. (8) which gives the propagator in that gauge. Along this path

$$A_\mu(r)\frac{dr_\mu}{d\tau}=0. \quad (9)$$

This propagator then has the same form as Eq. (8) except that the integration contour is now the curve $r(\tau)$. Thus we have obtained another correlation function which does not necessarily coincide with the original one having the straight-line integration path. However, this new correlation function with arbitrary integration path $C_\gamma(y)$ is related to the gauge condition (9), and this gauge does not generally provide *space-time translationally invariant states*. These are an essential ingredient in the derivation of the Källén-Lehmann representation.[1,2] Hence, the relation of $C_\gamma(y)$ to the physical excitation spectrum is not known. Consequently, the correlation function defined by the straight-line integration path remains the only one whose physical meaning is established.

The integral in the exponent in Eq. (8) can be evaluated if the photon field is expressed in terms of its Fourier components:

$$-ie\int_z^x dw^\mu A_\mu(w)=e\int_p \frac{y\cdot A(p)}{y\cdot p}(e^{-ix\cdot p}-e^{-iz\cdot p}). \quad (10)$$

Using the Schwinger-Dyson equation the propagator can be written in terms of the proper self-energy. This is also true for the correlation function (6). Recall that it coincides with the fermion propagator in a particular gauge. However, when the fields are expressed in terms of their Fourier components, the self-energy still depends on the direction of propagation:

$$C(y)=\int_q \frac{e^{-iq\cdot y}}{\not{q}-\Sigma(q,q_\parallel)}, \quad (11)$$

where $q_\parallel=(q\cdot y/y^2)y$. Thus, Fourier transforming $C(y)$ to momentum space is not a trivial operation. In certain physical situations, though, this is not necessary. For example, if we consider screening of static charges in a medium, y is the spatial separation of the charges. Then $q_\parallel$ is the component of q along this spatial direction. If we study fermionic oscillation modes in a medium, we can do that physically by probing the medium with a plane-wave hit: $\sim\delta(t)e^{i\mathbf{k}\cdot\mathbf{y}}$. Then we study the system after a long time (in order to remove the contributions of transient modes). This means that we let $y_0\gg|\mathbf{y}|$ and hence $q_\parallel\to q_0$.

Let us now consider the self-energy in one-loop order. It has two contributions: first, the ordinary fermion self-energy and, second, the terms arising when the exponential in Eq. (8) is expanded up to order e^2. Both of these are gauge dependent. However, the gauge dependence disappears in the sum already before the integrations. The result is

$$\Sigma(q,q_\parallel)=\int_p \frac{ie^2}{(p^2-m^2)(q-p)^2}\left[4m-2\not{p}+\frac{\not{y}(\not{p}+m)(\not{q}-m)+(\not{q}-m)(\not{p}+m)\not{y}}{y\cdot(p-q)} -\frac{y^2}{[y\cdot(p-q)]^2}[(p^2-m^2)(\not{q}-m)-(\not{q}-m)(\not{p}+m)(\not{q}-m)]\right]. \quad (12)$$

This result can also, of course, be obtained directly in the axial gauge $y\cdot A=0$.

Let us study more closely fermionic collective modes at finite temperature. This is indeed an interesting case, because in a vacuum (by direct calculation to two-loop order) the poles of the propagator are gauge invariant,[10] but at finite temperature this is not so. Because of the appearance of the heat bath, the self-energy in Eq. (11) depends separately on q_0 and $|\mathbf{q}|$. Moreover, as discussed above, we let $y_0\to\infty$ and hence the gauge becomes formally the temporal axial gauge, so that $q_\parallel=q_0$. Furthermore, we have to consider the retarded correlation function, which is the analytic continuation of the corresponding thermal function. [Strictly speaking the thermal propagator in the axial gauge is not an analytic function but the real part of an analytic function. The correct way to obtain the retarded function in that case is to use the prescription by Leibbrandt[11] and let the axial-gauge poles $y\cdot(p-q)\pm i\epsilon$ approach the real axis after all the integrations have been completed.] For sake of simplicity consider only the massless case. To find the collective modes, we have to find the poles of the propagator, i.e., solve the equation

$$(q_0-\Sigma_0)^2=(\mathbf{q}-\mathbf{\Sigma})^2. \quad (13)$$

We seek the solution in the form

$$q_0=\omega(\mathbf{q})-i\gamma(\mathbf{q}), \quad (14)$$

where ω is the real frequency and γ is the damping constant. We expand the self-energy in the limit q_0, $|\mathbf{q}|\ll T$

up to order T. Assuming that the damping is small, $\gamma \ll \omega$, the leading term $\sim T^2/q_0$ gives the frequency of the oscillations.[12] Note that this term in the ordinary self-energy is gauge invariant and the terms arising from the expansion of the exponential factor in Eq. (8) do not contribute here. The term $\sim T$ (which turned out to be purely imaginary) then gives the damping constant γ.

Let us concentrate here on the long-wavelength limit ($\mathbf{q} \to 0$). The equation giving the pole then reduces to

$$\omega - i\gamma = \frac{e^2T^2}{8\omega} - i\frac{e^2T}{8\pi} + i\gamma\frac{e^2T^2}{8\omega^2} + \cdots , \qquad (15)$$

where the last term arises when the real part of $\Sigma_0(q_0)$ is expanded in powers of γ/ω. The smallest frequency and the damping constant at this frequency are then

$$\omega_0 = \frac{eT}{2\sqrt{2}}, \quad \gamma(0) = \frac{e^2T}{16\pi} . \qquad (16)$$

It should be noted, however, that higher-loop corrections may change the damping constant in the way explained in Ref. 13. Anyway, Eq. (8) provides a gauge-invariant way to calculate those corrections.

Two remarks are in order. First, if we calculate the (retarded) self-energy using the covariant gauge, the imaginary part is

$$\mathrm{Im}\Sigma(\mathbf{q}=0) = (7-9\xi)\frac{e^2T}{16\pi} , \qquad (17)$$

where ξ is the gauge parameter. Then if we naively calculate the damping constant from Eq. (13) in the covariant gauge, we get a result which depends on the gauge parameter and is even negative in some gauges. However, that is not an indication of an instability, but merely arises from an incorrect treatment of the unphysical degrees of freedom in a gauge theory.

Second, consider the statistical density matrix of kinetic theory

$$\hat{\rho}(x_1,x_2) = \psi(x_1)^* \psi(x_2) . \qquad (18)$$

It has a physical meaning, and should then have the same value in any gauge. However, if the gauge is not completely fixed we get different results due to the unphysical states. Analogous to Eq. (2) we can choose the gauge so that the photon field vanishes in the direction of the relative coordinate:

$$(x_2 - x_1) \cdot A(x) = 0 . \qquad (19)$$

Then the density matrix is invariant under the residual gauge freedom. Analogously with Eq. (6) we can write the density matrix in a manifestly covariant form

$$\hat{\rho}(x,y) = [e^{-y\cdot D_x/2}\psi(x)^*]e^{y\cdot D_x/2}\psi(x) , \qquad (20)$$

where x and y are the center and relative coordinates, respectively. The Wigner function of kinetic theory can be obtained by Fourier transforming the density matrix with respect to the relative coordinate. The "physical" gauge condition (19) explains why in the Wigner function the ordinary derivative must be replaced by the covariant one,[6] or equivalently why a phase factor containing the line integral from x_1 to x_2 appears in the correct definition.[14] This is in addition to the observation[6] that the correct classical limit of the Wigner function is obtained if and only if the integration path in Eq. (8) is a straight line.

In summary, we have elaborated on a formalism that leads to gauge-invariant correlation functions, the poles of which correspond to the physical excitation spectrum. An essential ingredient of this formalism is the fact that the unphysical states decouple in the fermion propagator if the photon field in the direction of propagation is gauged away. The details of the calculation of the fermionic spectrum at finite temperature with the full momentum dependence of the damping constant will be published elsewhere.

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[1] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
[2] J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, Reading, MA, 1988).
[3] P. Ramond, *Field Theory* (Benjamin-Cummings, New York, 1981).
[4] J. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
[5] C. Bernard, Phys. Rev. D **9**, 3312 (1974).
[6] H.-Th. Elze, M. Gyulassy, and P. Vasak, Nucl. Phys. **B276**, 706 (1986).
[7] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[8] C. DeTar, J. E. King, and L. McLerran, Nucl. Phys. **B249**, 644 (1985).
[9] H. Yamagishi, Phys. Lett. B **173**, 465 (1986); Int. J. Mod. Phys. A **3**, 1155 (1988).
[10] R. Tarrach, Nucl. Phys. **B183**, 384 (1981).
[11] G. Leibbrandt, Rev. Mod. Phys. **59**, 1067 (1987).
[12] V. V. Klimov, Yad. Fiz. **33**, 1734 (1981) [Sov. J. Nucl. Phys. **33**, 934 (1981)].
[13] U. Heinz, K. Kajantie, and T. Toimela, Phys. Lett. B **183**, 96 (1987); see also, R. Pisarski, FERMILAB-PUB-87/140-T (unpublished).
[14] P. Carruthers and F. Zachariasen, Rev. Mod. Phys. **55**, 245 (1983); U. Heinz, Phys. Rev. Lett. **51**, 351 (1983).