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Quantum jumps, geodesics, and the topological phase

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The final state of a quantum jump due to a filtering measurement is obtained by parallel transporting the initial state along a geodesic. It is shown that a topological phase arises in any cyclic series of measurements. Unitary time evolution is not important for it to appear. The Aharonov-Anandan phase is derived as a limiting case.

Berry's discovery[1] that a quantum state acquires an extra phase when the Hamiltonian of the system is transported adiabatically along a closed curve in a parameter space has attracted much interest in the last few years.[2-6] Simon[2] expressed Berry's phase as the holonomy of a canonical connection on the eigenstate bundle of the Hamiltonian. The non-Abelian extension has also been studied.[3] Aharonov and Anandan have given a generalization[4] of Berry's phase. They have shown that a topological phase emerges in any cyclic evolution which is governed by the time-dependent Schrödinger equation. The phase they have discovered is independent of the actual form of the Hamiltonian or of the assumption of adiabatic evolution.

Recently a number of experimental works[7-12] have reported the observation of the topological phase. In some of these experiments,[11,12] however, the cyclic process is a series of "quantum jumps." In those experiments the measured topological phase has been compared with the Aharonov-Anandan phase arising from parallel transport along a geodesic ploygon. The polygon in question is spanned by points representing the various quantum states the system has been set in by the experimental apparatus during the cyclic process. It has been emphasized[12] that using the geodesic polygon in interpreting the experiments requires further theoretical justification. It has also been recently suggested[10,11,13] that, in some sense, the Aharonov-Anandan phase is a continuous analogue of the one discovered by Pancharatnam in the 1950s.[14] Moreover, the experimental observation of Pancharatnam's phase has brought to light yet another interesting generalization,[11] namely, that the unitary time evolution is not essential for the appearance of the Aharonov-Anandan phase. Our purpose here is to clarify the above-mentioned questions by reformulating some basic facts of quantum measurement theory in geometric terms.

It is clear that the phase acquired by the wave function in a cyclic process consisting of filtering measurements is obtained by multiplying the appropriate projection operators. This elementary fact is our starting point in the present investigation. We will show that the result of the "quantum jump" due to a filtering measurement is the same as if the wave function had been parallel transported along the shortest geodesic connecting the corresponding two points in the projective Hilbert space. The Aharonov-Anandan phase will be derived as a consequence of our observation, without any reference to the Schrödinger equation. We shall also give an explanation of the use of the geodesics in interpreting the experiments.[11,12]

For simplicity, let us consider a quantum system whose state space $\mathcal{H}$ is of finite dimension, $\mathcal{H}\cong\mathbb{C}^{N+1}$. $\mathcal{H}$ carries the usual Hermitian scalar product $\langle\,\rangle$. As a real vector space $\mathcal{H}\cong\mathbb{R}^{2N+2}$ and the real part of $\langle\,\rangle$, $\langle\,\rangle_{\mathbb{R}}$, is a Euclidean scalar product on it. The unit sphere

$$S^{2N+1}=\{|A\rangle\in\mathcal{H}\;|\;\langle A|A\rangle=1\} \tag{1}$$

is a principal U(1) fiber bundle (the Hopf bundle[15,4]) over $\mathbb{C}P^N$, the projective Hilbert space of the system. The tangent space $T_{|A\rangle}(S^{2N+1})$ consists of vectors $|v\rangle\in\mathcal{H}$ which satisfy $\langle v|A\rangle_{\mathbb{R}}=0$. There is a canonical connection on the bundle $S^{2N+1}\xrightarrow{\pi}\mathbb{C}P^N$ defined by the decomposition $T_{|A\rangle}(S^{2N+1})=V_{|A\rangle}\oplus H_{|A\rangle}$

$$V_{|A\rangle}=\{i\alpha|A\rangle\;|\;\alpha\in\mathbb{R}\}\,, \tag{2a}$$

$$H_{|A\rangle}=\{|u\rangle\in\mathcal{H}\;|\;\langle u|A\rangle=0\}\,. \tag{2b}$$

Consider a cyclic evolution of the wave function which projects to a closed curve C in $\mathbb{C}P^N$. The topological phase factor[4] acquired by the wave function is nothing but the holonomy of connection (2) along C. The projection $S^{2N+1}\xrightarrow{\pi}\mathbb{C}P^N$ gives rise to an isomorphism between $H_{|A\rangle}$ and $T_{|A\rangle\langle A|}(\mathbb{C}P^N)$, where $|A\rangle\langle A|$ is the projection operator representing the one-dimensional subspace spanned by $|A\rangle$ in $\mathcal{H}$. This isomorphism induces a scalar product on $T_{|A\rangle\langle A|}(\mathbb{C}P^N)$ from $\langle\,\rangle_{\mathbb{R}}$. In fact, this construction yields[4,15] the canonical (Fubini-Study) Riemannian metric on $\mathbb{C}P^N$.

Now consider two rays $|A\rangle\langle A|$ and $|B\rangle\langle B|$ which are not orthogonal to each other. Geometrically, this means that $|A\rangle\langle A|$ and $|B\rangle\langle B|$ are not antipodal points of the complex projective line they span in $\mathbb{C}P^N$. Suppose that the system's wave function is $|A\rangle$ and measure it by the "polarizer" $|B\rangle\langle B|$. This filtering will result in the state vector $|B\rangle\langle B|A\rangle$. We are going to recover the normalized vector $|B\rangle\langle B|A\rangle/|\langle B|A\rangle|$ by parallel transporting $|A\rangle$ along the shortest geodesic connecting $|A\rangle\langle A|$ and $|B\rangle\langle B|$. The geodesic in question is the shorter arc of the great circle passing through $|A\rangle\langle A|$ and $|B\rangle\langle B|$ on the copy of $\mathbb{C}P^1 \cong S^2$ they span. On the other hand, this geodesic γ in $\mathbb{C}P^N$ is the projection of a horizontal geodesic Γ lying in S^{2N+1}. Γ starts at $|A\rangle \in \pi^{-1}(|A\rangle\langle A|)$ and ends somewhere on the circle $\pi^{-1}(|B\rangle\langle B|)$, say at $|B\rangle\exp(i\phi_{AB})$. We are interested just in the end point of Γ. It is easy to see that all the horizontal geodesics of S^{2N+1} starting at $|A\rangle$ are of the form[15]

$$\Gamma(s) = |A\rangle\cos(s) + |u\rangle\sin(s) \tag{3}$$

for some $|u\rangle$ satisfying $\langle u|A\rangle = 0$, $\langle \mu/\mu\rangle = 1$. Therefore we have to solve, for $|B\rangle\exp(i\phi_{AB})$, the equation

$$|A\rangle\cos(s_0) + |u\rangle\sin(s_0) = |B\rangle\exp(i\phi_{AB}) . \tag{4}$$

First we multiply by $\langle A|$ and get

$$\cos(s_0) = \langle A|B\rangle\exp(i\phi_{AB}) . \tag{5}$$

It follows from $\Gamma(\pi) = -\Gamma(0)$ that $\gamma(\pi) = \gamma(0)$. This in turn implies that any copy of $\mathbb{C}P^1 \subset \mathbb{C}P^N$ is of radius $\frac{1}{2}$ and that for the shorter arc of the great circle passing through $|A\rangle\langle A|$ and $|B\rangle\langle B|$ we have $0 \le s \le s_0 < \pi/2$. Thus, $\cos(s_0) > 0$ and from (5) we obtain

$$\langle A|B\rangle = |\langle A|B\rangle| \exp(-i\phi_{AB}) . \tag{6}$$

Equation (6) yields the key formula of this paper:

$$|B\rangle\exp(i\phi_{AB}) = |B\rangle\frac{\langle B|A\rangle}{|\langle B|A\rangle|} . \tag{7}$$

This tells us that the result of the "quantum jump" (in terms of the normalized state vectors)

$$|A\rangle \to |B\rangle\frac{\langle B|A\rangle}{|\langle B|A\rangle|} \tag{8}$$

is the same as if the wave function $|A\rangle$ had been parallel transported along the shortest geodesic connecting $|A\rangle\langle A|$ with $|B\rangle\langle B|$ in the projective Hilbert space. (It seems to be a somewhat philosophical question whether the wave function actually follows the geodesic during a quantum jump.)

Now we investigate the change of the phase of the state vector induced by a cyclic series of filtering measurements. Let $|A_i\rangle\langle A_i|$ $(i=1,\ldots,n+1)$ $|A_1\rangle\langle A_1| = |A_{n+1}\rangle\langle A_{n+1}|$ be the projection operators (with the condition $\langle A_i|A_{i+1}\rangle \neq 0$) associated with the sequence of measurements. Let us prepare the system to be in a pure state $|A_1\rangle \in \pi^{-1}(|A_1\rangle\langle A_1|)$ and then subject it to a series of filterings described by the projection operators $|A_k\rangle\langle A_k|$ $(k=2,\ldots,n+1)$. As a result of this cyclic process, the state vector will change as

$$|A_1\rangle \to |A_1\rangle\langle A_1|A_n\rangle\langle A_n|A_{n-1}\rangle \cdots \langle A_2|A_1\rangle . \tag{9}$$

In particular, it will pick up the phase factor

$$\exp(i\phi) = \frac{\langle A_1|A_n\rangle\langle A_n|A_{n-1}\rangle\cdots\langle A_2|A_1\rangle}{|\langle A_1|A_n\rangle\langle A_n|A_{n-1}\rangle\cdots\langle A_2|A_1\rangle|} . \tag{10}$$

Let γ_i be the shortest geodesic in $\mathbb{C}P^N$ between $|A_i\rangle\langle A_i|$ and $|A_{i+1}\rangle\langle A_{i+1}|$. Applying the previous arguments we conclude that the total phase change of the wave function $|A_1\rangle$ due to a cyclic measurement is the same as if $|A_1\rangle$ had been parallel transported along the geodesic polygon $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ (for $n=3$, see Fig. 1). Since a measurement is not a unitary process, in general, it is clear that unitarity is not essential for the topological phase to appear.

The considerations above immediately explain the use of geodesics in interpreting the experiment.[11] To understand why the use of geodesics in Ref. 12 yields the correct result we can argue as follows. Consider, following the authors of Ref. 12, a positive-helicity photon propagating into direction $\hat{\mathbf{k}}$. Denote this spin state by $|\hat{\mathbf{k}},+\rangle$. It satisfies $(\mathbf{s}\cdot\hat{\mathbf{k}})|\hat{\mathbf{k}},+\rangle = |\hat{\mathbf{k}},+\rangle$, where s_i $(i=1,2,3)$ are the spin-1 generators of SO(3). For $\hat{\mathbf{k}} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$, in the representation where s_3 is diagonal, we can choose

$$|\hat{\mathbf{k}},\pm\rangle = \left[\frac{1\pm\cos\beta}{2}e^{-i\alpha}, \pm\frac{\sin\beta}{\sqrt{2}}, \frac{1\mp\cos\beta}{2}e^{i\alpha}\right] . \tag{11}$$

As a result of applying an ideal mirror whose normal vector is $\hat{\mathbf{n}}$, the spin state of the photon changes as[16]

$$|\hat{\mathbf{k}},+\rangle \to e^{i\pi}|\hat{\mathbf{k}}',-\rangle\frac{\langle\hat{\mathbf{k}}',-|\hat{\mathbf{k}},+\rangle}{|\langle\hat{\mathbf{k}}',-|\hat{\mathbf{k}},+\rangle|} , \tag{12}$$

where $\hat{\mathbf{k}}' = \hat{\mathbf{k}} - 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$. As the mirror flips the spin state of the photon, a circuit with an odd number of reflections would give a final-spin state which is opposite to the initial-spin state. In the symmetric arrangement of the experiment the effect of the additional phase factor $e^{i\pi}$, as well as the dynamical phase difference, related to the optical path length have been compensated. The unit vectors $\hat{\mathbf{k}}, -\hat{\mathbf{k}}', \ldots,$ associated with the pure spin-state density matrices $|\hat{\mathbf{k}},+\rangle\langle\hat{\mathbf{k}},+|$, $|\hat{\mathbf{k}}',-\rangle\langle\hat{\mathbf{k}}',-|,\ldots,$ the photon goes through during the experiment span a geodesic polygon on the sphere of spin directions. By inserting the corresponding projectors into Eq. (10), the resulting topological phase shift (without the dynamical phase, and the extra $e^{i\pi}$ factors) turns out to be the negative of the solid angle of this geodesic polygon. In fact, this is the phase which has been observed.[12] It is worth noting that the

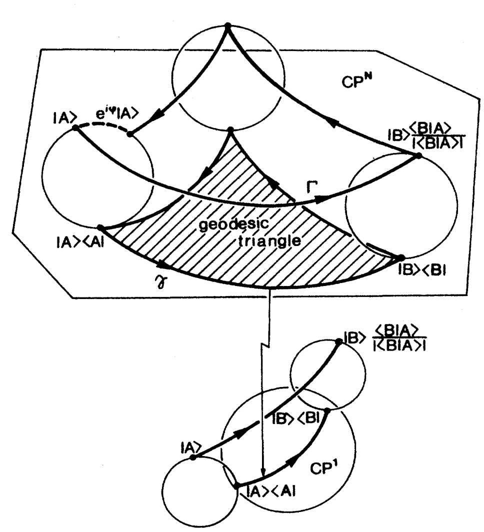

FIG. 1. Geometry of the quantum jump and the topological phase.

projectors $|\hat{\mathbf{k}},+\rangle\langle\hat{\mathbf{k}},+|$, $|\hat{\mathbf{k}}',-\rangle\langle\hat{\mathbf{k}}',-|,\ldots$, also span a geodesic polygon lying in $\mathbb{C}P^2$, the projective space of a three-component spin. From the general theory, the phase shift is also equal to the holonomy angle of connection (2) belonging to this latter geodesic polygon.

Finally, we show that the Aharonov-Anandan phase is the continuous limit of the phase induced by a cyclic sequence of measurements. Indeed, let us suppose that the history of the system is given by a smooth closed curve $C(\tau)[0\le\tau\le 1,\ C(0)=C(1)]$ in the projective Hilbert space. Physically this means that the system passes through the filtering devices described by the curve $C(\tau)=|A(\tau)\rangle\langle A(\tau)|$. Approximating $C(\tau)$ by geodesic polygons, we see at once that the state vector picks up a phase factor $e^{i\phi}$, which is entirely due to the "continuous measurement." $e^{i\phi}$ is the element of the holonomy group of the canonical connection (2) corresponding to the closed curve C. This is just the Aharonov-Anandan phase. It is clear that the picture given here is valid also for a quantum system whose state space is of infinite dimension. The connection (2), the geodesic equation (3), etc., all make sense[17] in that case as well.

The original derivation of the Aharonov-Anandan phase rests on the following fact: For any solution $|\psi(t)\rangle$ of the Schrödinger equation $i\partial_t|\psi(t)\rangle=H(t)|\psi(t)\rangle$ $[\langle\psi(t)|\psi(t)\rangle=1]$, the curve

$$|\tilde{\psi}(t)\rangle=\exp[i\langle\psi(t)|H(t)|\psi(t)\rangle]\,|\psi(t)\rangle$$

is the horizontal lift of $|\psi(t)\rangle\langle\psi(t)|$. On the other hand, we have derived the topological phase from elementary quantum-measurement theory. It is interesting to observe that there is a consistency between the measurement postulate and the time-dependent Schrödinger equation: They give rise to the same topological phase for any cyclic evolution.

Note added. After the manuscript had been submitted there appeared a paper by Samuel and Bhandari[18] which treats filtering measurements in terms of parallel transport. We have also received a report of the work of Anandan and Aharonov[19] containing very similar results. We thank J. Anandan for sending us this work before publication.

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