Comment on "Exact ground state, mass gap, and string tension in lattice gauge theory"

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It is shown that the lattice Hamiltonian recently proposed by Guo, Zheng, and Liu does not have the correct continuum limit.

INTRODUCTION

Recently, Guo, Zheng, and Liu proposed a new Hamiltonian for lattice gauge theory which had striking features. First, it was supposed to have the same continuum limit as the usual Kogut-Susskind Hamiltonian, and, second, its ground state was exactly known. This would be a truly significant advance in lattice gauge theory. Unfortunately, we shall show that the proposed Hamiltonian does not have the correct continuum limit.

The Hamiltonian H of Guo, Zheng, and Liu[1] differs from the usual Kogut-Susskind Hamiltonian H_{KS} by a term ΔH. Guo, Zheng, and Liu show that in the limit of small lattice spacing a, ΔH is proportional to terms which have an explicit power of a multiplied by integrals of known functions of the Yang-Mills field tensors. They then assert that therefore ΔH tends to zero in the continuum limit. This proof assumes that neighboring plaquettes are correlated so that different terms such as

$$A_\mu(x+\hat{n}a)-A_\mu(x)$$

vanish linearly as $a \to 0$ ($\hat{n}$ is one of the periods in the lattice). But the ground state of their modified Hamiltonian is a product of independent plaquettes. What amounts to a self-consistency condition on the ground state, which would ensure the correlation of neighboring plaquettes, is violated. The net result is that the expectation of ΔH in their putative ground state does not vanish.

We can illustrate this point by an explicit calculation in (2+1)-dimensional QCD. With the Hamiltonian that they propose, and its exact ground state which they have found, we can calculate the expectation value of ΔH on the lattice and show that its continuum limit does not vanish. A simplified calculation along these lines is presented below.

This indicates a subtlety in the difference between Lagrangian path-integral formalisms and Hamiltonian formalisms. In the former, neglect of terms with explicit powers of a appear to be justified.[2] We show that in the Hamiltonian formalism neglect of these terms is not justified.

Guo and Zheng[3] argue that this nonclassical limit of ΔH merely affects the high-energy $O(1/a)$ behavior of the system and not its long-distance behavior. This seems implausible on the following physical grounds. The limit of the lattice theory is the physical limit only if the correlation length in units of the lattice spacing diverges as the lattice space vanishes. But the[1] wave function has no long-range correlations. It is an independent-plaquette wave function. Such a wave function does yield an area law indicative of confinement, but there is no reason to trust its estimates about masses.

CALCULATION

Following Ref. 1, with the gauge group SU(2), let E_l^a be conjugate to the link variable U_l:

$$[E_l^a, U_{l'}]=\frac{\sigma^a}{2}U_{l'}\delta_{ll'} . \tag{1}$$

Define

$$R=\frac{4}{3g^4}\sum_{\text{plaquettes}}\text{Tr}(U_P) . \tag{2}$$

Reference 1 claims that, in 3+1 dimensions,

$$\Delta H \equiv -\frac{g^2}{2a}\sum_{\text{links}}[E_l^a,R][E_l^a,R] \tag{3}$$

vanishes in the continuum limit, whereas in 2+1 dimensions it has a finite continuum limit

$$\frac{2}{9e^4}\int d^2x\,(\partial_j F_{ij}^a \partial_k F_{ik}^a) . \tag{4}$$

In 2+1 dimensions, g is related to the invariant gauge coupling e by

$$g^2=ae^2 . \tag{5}$$

We show instead that in 2+1 dimensions the expectation value of ΔH blows up like $1/a^4$ in the state suggested in Ref. 1:

$$|\psi\rangle=\exp(R)|0\rangle , \tag{6}$$

where $|0\rangle$ is the strong-coupling vacuum defined by

$$E_l^a|0\rangle=0 . \tag{7}$$

If we define

$$\langle A\rangle=\frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} \tag{8}$$

then $\langle \Delta H \rangle$ can be written as

$$\langle \Delta H \rangle = \frac{-8}{9ag^6} \sum_{\substack{P,P' \\ \text{links}}} \langle [E_l^a, \mathrm{Tr}(U_P)][E_l^a, \mathrm{Tr}(U_{P'})] \rangle$$

$$= \frac{-2N_l}{9ag^6} \langle [\mathrm{Tr}(\sigma^a U_P - U_Q \sigma^a)]^2 \rangle \ , \quad (9)$$

where P and Q are any two adjacent plaquettes sharing a common link. N_l is the number of links, related to the "volume" of the lattice V by

$$N_l a^2 = 2V \ . \quad (10)$$

Because the wave function (6) is a product of single-plaquette traces, the cross terms in (9) vanish and $\langle \Delta H \rangle$ becomes

$$\frac{-8V}{9a^3 g^6} \langle \mathrm{Tr}(\sigma^a U_P)^2 \rangle = \frac{32V}{9a^3 g^6} \langle 1 - \tfrac{1}{4}\mathrm{Tr}^2(U_P) \rangle \ . \quad (11)$$

In the state (6), for $g \to 0$, it is easy to show that

$$\langle 1 - \tfrac{1}{4}\mathrm{Tr}^2(U_P) \rangle \to \tfrac{9}{16} g^4 + \cdots \quad (12)$$

so that, recalling (5), $\langle \Delta H \rangle$ blows up like V/a^4.

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[1]S. Guo, W. Zheng, and J. Liu, Phys. Rev. D **38**, 2591 (1988).
[2]See, for example, H. S. Sharatchandra, Phys. Rev. D **18**, 2042 (1978).
[3]S. Guo and W. Zheng, Phys. Rev. D **39**, 3144 (1989).