Towards the Einstein-Hilbert action via conformal transformation

Kei-ichi Maeda*
Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan
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A conformal transformation is used to prove that a general theory with the action $S=\int d^Dx\sqrt{-g}\,[F(\phi,R)-(\epsilon/2)(\nabla\phi)^2]$, where $F(\phi,R)$ is an arbitrary function of a scalar ϕ and a scalar curvature R, is equivalent to a system described by the Einstein-Hilbert action plus scalar fields. This equivalence is a simple extension of those in R^2-gravity theory and the theory with nonminimal coupling. The case of $F=L(R)$, where $L(R)$ is an arbitrary polynomial of R, is discussed as an example.

General relativity describes gravity very well at least in the weak-field limit and Einstein gravity plus scalar fields is well studied in various situations. We know the so-called cosmic no-hair theorem for a Bianchi-type universe in the Einstein-Hilbert system.[1] We have also the positive-energy theorem in Einstein theory.[2] Because of quantum corrections in a strong gravitational field, however, we may expect some additional terms of the curvature, e.g., a nonminimal-coupling term $\xi\phi^2R$ or curvature-squared terms R^2, to the Einstein-Hilbert action.[3] Those additional terms may become important in the early Universe or near black holes. However, those "non-Einstein"-type theories are not well studied because of their complicated structures.

In the theory with a scalar curvature-squared (R^2) term and/or with a nonminimal coupled scalar, the equivalence between such a theory and a system described by the Einstein-Hilbert action plus scalar fields has been proved.[4–6] We have found rather simple basic equations, being able to analyze easily the dynamical behavior of fields in those models. Because of a simple coupling, the behavior of "scalar" fields can be seen just from the potential. We can apply the above cosmic no-hair theorem[5] or the positive-energy theorem[7] in our Einstein-Hilbert system. From the equivalence, we may easily find the dynamical behavior in the original "non-Einstein"-type theories also. Hence the description in the equivalent Einstein-Hilbert form has a big advantage. Recently the same equivalence is proved for the theory with a Lagrangian $L(R)$ (an arbitrary function of a scalar curvature R) in four dimensions.[8]

In this Brief Report we consider one of the most general models in which the above models are included and we prove that the same equivalence is also true in such an extended theory.

The action we consider here is

$$S=\int d^Dx\sqrt{-g}\left[F(\phi,R)-\frac{\epsilon}{2}(\nabla\phi)^2\right], \tag{1}$$

where $F(\phi,R)$ is an arbitrary function of a scalar field ϕ and of a scalar curvature R of a spacetime. D is the dimensionality of the spacetime. ϵ is usually unity (or zero if their is no scalar field), but we leave it as a free parameter, because ϵ is negative for the effective four-dimensional theories obtained from dimensional reduction in higher-dimensional theories.[9,7] Although the most general case, of course, may contain the Ricci tensor $R_{\mu\nu}$ and Weyl curvature $C_{\mu\nu\rho\sigma}$, we do not consider such a complicated system.

The basic equations are given from the action (1) by taking variations with respect to the metric $g_{\mu\nu}$ and ϕ as follows:

$$G_{\mu\nu}=\left[\frac{\partial F}{\partial R}\right]^{-1}\left\{\frac{\epsilon}{2}[\nabla_\mu\phi\nabla_\nu\phi-\tfrac{1}{2}g_{\mu\nu}(\nabla\phi)^2]+\tfrac{1}{2}g_{\mu\nu}\left[F-\frac{\partial F}{\partial R}R\right]+\left[\nabla_\mu\nabla_\nu\left[\frac{\partial F}{\partial R}\right]-\Box\left[\frac{\partial F}{\partial R}\right]g_{\mu\nu}\right]\right\}, \tag{2}$$

$$\epsilon\Box\phi=-\frac{\partial F}{\partial\phi}, \tag{3}$$

where $G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor.

First, using a conformal transformation, we prove that this model is equivalent to a system described by the Einstein-Hilbert action plus scalar fields. Let us consider the conformal transformation

$$\hat{g}_{\mu\nu}(x)=e^{2\omega(x)}g_{\mu\nu}(x), \tag{4}$$

where $\omega(x)$ is an unknown function, which will be determined later. From Eqs. (2)–(4), the Einstein tensor and the equation of the scalar field ϕ in the $\hat{g}_{\mu\nu}$ system are written as

$$\hat{G}_{\mu\nu}=\left[\frac{\partial F}{\partial R}\right]^{-1}\left\{\frac{\epsilon}{2}[\nabla_\mu\phi\nabla_\nu\phi-\tfrac{1}{2}g_{\mu\nu}(\nabla\phi)^2]+\tfrac{1}{2}g_{\mu\nu}\left[F-\frac{\partial F}{\partial R}R\right]+\left[\nabla_\mu\nabla_\nu\left[\frac{\partial F}{\partial R}\right]-\Box\left[\frac{\partial F}{\partial R}\right]g_{\mu\nu}\right]\right\}$$
$$-(D-2)(\nabla_\mu\nabla_\nu\omega-\Box\omega g_{\mu\nu})+(D-2)[\nabla_\mu\omega\nabla_\nu\omega+\tfrac{1}{2}(D-3)(\nabla\omega)^2g_{\mu\nu}]\ , \qquad (5)$$

$$\epsilon[\hat{\Box}\phi-(D-2)\hat{\nabla}\omega\cdot\hat{\nabla}\phi]=-e^{-2\omega}\frac{\partial F}{\partial\phi}\ , \qquad (6)$$

where variables with a caret denote those with respect to $\hat{g}_{\mu\nu}$. It is easy to show that the higher-derivative terms such as $\nabla_\mu\nabla_\nu(\partial F/\partial R)$ and $\nabla_\mu\nabla_\nu\omega$ cancel each other if we set

$$\omega=\frac{1}{D-2}\ln\left[2\kappa^2\left|\frac{\partial F}{\partial R}\right|\right]+\text{const}\ , \qquad (7)$$

where $\kappa^2=8\pi G_N$. By setting (7), it turns out that ω behaves like a scalar field in (5). We consider the following two cases separately: Case (A), F is a linear function of R, i.e., $F(\phi,R)=f(\phi)R-V(\phi)$; Case (B) $\partial F/\partial R$ depends on a scalar curvature R.

Case (A). ω is a function of ϕ and is not therefore a new dynamical variable. This is consistent with the fact that the system has no higher derivative and then there is no new degree of freedom even if we describe our system in the Einstein-Hilbert form.

Redefining a new scalar field Φ in order for the kinetic term to be canonical as

$$\kappa\Phi\equiv\int d\phi\left[\frac{\epsilon(D-2)f(\phi)+2(D-1)[df(\phi)/d\phi]^2}{2(D-2)f^2(\phi)}\right]^{1/2}, \qquad (8)$$

we find that the basic equations (5) and (6) are obtained from the Einstein-Hilbert system with the scalar field Φ, which action is given by

$$S=(\text{sign})\int d^Dx\,\mathcal{L}\ ,$$
$$\mathcal{L}=\sqrt{-\hat{g}}\left[\frac{1}{2\kappa^2}R(\hat{g})-\tfrac{1}{2}(\hat{\nabla}\Phi)^2-U(\Phi)\right]\ , \qquad (9)$$

where

$$U(\Phi)\equiv(\text{sign})[2\kappa^2|f(\phi)|]^{-D/(D-2)}V(\phi) \qquad (10)$$

and $(\text{sign})\equiv f(\phi)/|f(\phi)|$. Here we have assumed that the integrand of Eq. (8) is real, otherwise the scalar Φ has the kinetic term with a wrong sign; hence the system becomes unstable.

Case (B). ω describes a dynamical freedom in the Einstein-Hilbert system and behaves like a scalar field. This "new" freedom appears when we describe our *higher-order derivative* theory in the second-order form, i.e., in terms of the Einstein-Hilbert action. In order to make the kinetic term canonical, we introduce a new "scalar" field ψ by

$$\kappa\psi\equiv\sqrt{(D-1)(D-2)}\,\omega$$
$$=\left[\frac{D-1}{D-2}\right]^{1/2}\ln\left[2\kappa\left|\frac{\partial F}{\partial R}\right|\right]\ . \qquad (11)$$

As the result, we obtain the equivalent action to (1):

$$S=(\text{sign})\int d^Dx\,\mathcal{L}\ ,$$
$$\mathcal{L}=\sqrt{-\hat{g}}\left\{\frac{1}{2\kappa^2}R(\hat{g})-\tfrac{1}{2}(\hat{\nabla}\psi)^2-\frac{\epsilon}{2}(\text{sign})\exp\left[-\left[\frac{D-2}{D-1}\right]^{1/2}\kappa\psi\right](\hat{\nabla}\phi)^2-U(\psi,\phi)\right\}\ , \qquad (12)$$

where

$$U(\psi,\phi)\equiv(\text{sign})\left[2\kappa^2\left|\frac{\partial F}{\partial R}\right|\right]^{-D/(D-2)}\left[R(\phi,\psi)\frac{\partial F}{\partial R}-F(\phi,\psi)\right]$$
$$=(\text{sign})\exp\left[-\frac{D}{\sqrt{(D-1)(D-2)}}\kappa\psi\right]\left\{\frac{(\text{sign})}{2\kappa^2}R(\phi,\psi)\exp\left[\left[\frac{D-2}{D-1}\right]^{1/2}\kappa\psi\right]-F(\phi,\psi)\right\} \qquad (13)$$

and

$$(\text{sign})\equiv\frac{\partial F}{\partial R}\Big/\left|\frac{\partial F}{\partial R}\right|\ .$$

In Eq (13), $R(\phi,\psi)$ denotes R in terms of ϕ and ψ which is obtained through the relation

$$\frac{\partial F}{\partial R}(\phi,R)=\text{a given function of } \phi \text{ and } R$$
$$=\text{a known function of } \psi \text{ [Eq. (11)]} \qquad (14)$$

and $F(\phi,\psi)\equiv F(\phi,R(\phi,\psi))$. As in the scalar curvature-squared theory, the trace of (2) gives the equation for R. We can easily check that the equation for ψ obtained from (12) is just the same as the trace of (2). Therefore, this system describes Einstein gravity with two scalar fields ψ and ϕ of nonlinear σ-model type.

We have proved that a most general model described by (1) is equivalent to the Einstein-Hilbert system with either one scalar Φ [case (A)] or with two scalar fields ϕ,ψ [case (B)]. There are two interesting theories which are included in our general model (1): (i) The Einstein action with quantum correction terms, i.e., $\epsilon=1$ and

$$F(\phi,R)=\frac{1}{2\kappa^2}R(g)-V(\phi)+aR^2-\tfrac{1}{2}\xi\phi^2R \ . \tag{15}$$

(ii) The higher-derivative terms, i.e., $\epsilon=0$ (no scalar field) and

$$F(\phi,R)=L(R)\equiv\sum_{n=0}^{n_0}a_nR^n \ . \tag{16}$$

Here we discuss only case (ii) as an example.[10] Case (i) has been discussed in Ref. 6.

From the equivalence theorem, we find that this system is equivalent to an Einstein system plus a scalar field ψ with the potential

$$U(\psi)=\left[2\kappa^2\sum_{n=0}^{n_0}na_nR^{n-1}\right]^{-D/(D-2)}\sum_{n=0}^{n_0}(n-1)a_nR^n \ ; \tag{17}$$

(sign) is assumed to be positive. Although full dynamics should be discussed by using the above potential, we shall restrict our present consideration into the large-R region where the $n=n_0$ term becomes dominant. (See Ref. 10 for the details in four dimensions.) The asymptotic behavior of this potential for $\psi(\text{or}R)\to\infty$ is

$$U(\psi)\sim R^{(D-2n_0)/(D-2)}\sim\exp\left[\frac{(D-2n_0)\kappa\psi}{(n_0-1)\sqrt{(D-1)(D-2)}}\right] . \tag{18}$$

It turns out that a flat plateau appears for $\psi\to\infty$ if $D=2n_0$. We find exponential inflation in this case. Applying Wald's cosmic no-hair theorem,[1] we can prove that inflation is a transient attractor for Bianchi models as in Ref. 5 and the initial anisotropy is isotropized in one Hubble expansion time. The power-law inflation in D-dimensional spacetime is possible for $2<D<D_{\max}$ with $D_{\max}\equiv 4n_0(n_0-1)+2$, although R evolves into infinity for $2<D<2n_0$. The same result was found for the case of $n_0=2$ in Ref. 5.

Finally we should mention one important remark. Since the conformal transformation considered here becomes singular where $\partial F/\partial R$ vanishes, we should restrict our transformation into one connected region of ϕ and R where $\partial F/\partial R$ is always positive or negative. A region where $\partial F/\partial R$ is positive (or negative) is just a part of the full region of (ϕ,R). The original space of (ϕ,R) may be divided into two or more regions, on which borders $\partial F/\partial R$ vanishes. In both regions of $\partial F/\partial R>0$ and $\partial F/\partial R<0$, we define conformal transformations. We have to introduce two or more disconnected conformal transformations; hence, we may wonder whether or not the spacetime point does cross over the border ($\partial F/\partial R=0$). When it crosses over the border, the value of scalar field, Φ or ψ, in two nearest-neighbor regions will change from $\mp\infty$ to $\pm\infty$ [see Eq. (8) or (11)]. In order to discuss such dynamics, we may need to go back to the original system, where the value of "scalar" does not diverge. If the spacetime point can cross over the border without a singularity, therefore, it may not be a better way to introduce conformal transformations.

Fortunately, however, if we consider an anisotropic homogeneous Bianchi-type model, we can show that a spacetime singularity usually appears on $\partial F/\partial R=0$ as we shall show it now.[11–13] This means that the spacetime point cannot cross over the border without appearing singularity. Each region is isolated by a singularity.

Let us consider a Bianchi model which metric is described by a diagonal matrix β as

$$ds^2=-dt^2+e^{-2\Omega}e^{2\beta_{ij}}\omega_i\omega_j \ , \tag{19}$$

where

$$\beta\equiv(\beta_{ij})=\text{diag}(\beta_++\sqrt{3}\beta_-,\beta_+-\sqrt{3}\beta_-,-2\beta_+) \tag{20}$$

and $\{\omega_i\}$ is the three-dimensional invariant basis. Writing down the basic equations (2), we find the equations for β_+,β_- as

$$\ddot\beta_+-3\dot\Omega\dot\beta_++\frac{\partial U}{\partial\beta_+}=-\dot\beta_+\frac{d}{dt}\ln\left|\frac{\partial F}{\partial R}\right| , \tag{21}$$

$$\ddot\beta_--3\dot\Omega\dot\beta_-+\frac{\partial U}{\partial\beta_-}=-\dot\beta_-\frac{d}{dt}\ln\left|\frac{\partial F}{\partial R}\right| , \tag{22}$$

where

$$U(\Omega,\beta_+,\beta_-)\equiv\tfrac{1}{2}e^{2\Omega}[V(\beta_+,\beta_-)-1] \tag{23}$$

with V being the potential appeared in the Bianchi model (see Ref. 14). Those equations lead to the equation for shear $\sigma^2\equiv6(\dot\beta_+^2+\dot\beta_-^2)$ as

$$\frac{d}{dt}\left[\left[\frac{\partial F}{\partial R}\right]^2(e^{-6\Omega}\sigma^2+6e^{-4\Omega}V)\right]=6\frac{d}{dt}\left[e^{-4\Omega}\left[\frac{\partial F}{\partial R}\right]^2\right]V. \tag{24}$$

For Bianchi type I, we can integrate (24) and find

$$\sigma^2=\sigma_0^2e^{6\Omega}\left[\frac{\partial F}{\partial R}\right]^{-2} , \tag{25}$$

where σ_0 is an integration constant. When $\partial F/\partial R$ vanishes, σ^2 always diverges and then the spacetime evolves into a singularity. We assume that the three-volume $e^{-3\Omega}$ does not diverge in a finite time, otherwise the space time manifold cannot be extended beyond that point.

Then it is not interesting for us to cross over the border $\partial F/\partial R=0$. Any other Bianchi models also behave as type I ($V=0$) except near the wall of the potential V which grows exponentially. We expect a singularity appears as $\partial F/\partial R \to 0$ unless the spacetime point just happens to approach the boundary of the potential at the same time. We shall show it more explicitly as follows.

Suppose that $\partial F/\partial R$ vanishes at a finite cosmic time t_0 in the evolution of the Universe. If a singularity appears at $t=t_0$, two spacetimes divided by the border $\partial F/\partial R=0$ are classically disconnected. In such a case, we can consider two spacetimes separately and then introduce a conformal transformation in each spacetime. Hence we shall investigate the possibility of $\partial F/\partial R=0$ without a singularity.

If a singularity does not appear when $t \to t_0$, $\beta_\pm$ remains finite, then $V(\beta_\pm)$ is also finite as $t \to t_0$. If $(d/dt)(\partial F/\partial R)$ does not diverge when $t \to t_0$,

$$\frac{d}{dt}\left[\left[\frac{\partial F}{\partial R}\right]^2 (e^{-6\Omega}\sigma^2+6e^{-4\Omega}V)\right]$$

vanishes [see Eq. (24)]; hence, $(\partial F/\partial R)^2(e^{-6\Omega}\sigma^2+6e^{-4\Omega}V)$ approaches some finite constant. Then we find that $(e^{-6\Omega}\sigma^2+6e^{-4\Omega}V)$ diverges. It corresponds to a singularity because σ^2 or V diverges. Therefore, if a singularity does not appear when $t \to t_0$, $(d/dt)(\partial F/\partial R)$ must diverge.

The condition that $(d/dt)(\partial F/\partial R) \to \infty$ as $t \to t_0$ is not generic. Because the equation for $\partial F/\partial R$ with the metric (19) is given as, from (2),

$$-\frac{d^2}{dt^2}\left[\frac{\partial F}{\partial R}\right]+3\dot{\Omega}\frac{d}{dt}\left[\frac{\partial F}{\partial R}\right]-\frac{(D-2)\epsilon}{2(D-1)}\dot{\phi}^2 -\frac{D}{2(D-1)}F+\frac{1}{D-1}F\frac{\partial F}{\partial R}=0 . \quad (26)$$

Since $\partial F/\partial R=0$ is not a singular point in this equation, we do not expect that $(d/dt)(\partial F/\partial R)$ always diverges when $\partial F/\partial R \to 0$.

We can conclude that the regions divided by the border $\partial F/\partial R=0$ in general do not connect each other smoothly at least for anisotropic Bianchi models, except for cases with elaborate fine-tuned initial conditions. Since the realistic Universe may have some anisotropic perturbations, each region is disconnected by a singularity. Therefore, it may be sufficient to consider only the single region which connects to our present Universe, that is the region of $\partial F/\partial R>0$ because $\partial F/\partial R\,|_{\text{present value}}=(2\kappa^2)^{-1}>0$.

In summary, we have proved that a general model (1) is equivalent to an Einstein-Hilbert system with scalar fields. The conformal technique discussed here gives us a simple method to analyze a general "non-Einstein" system (1). The description in terms of the Einstein-Hilbert action is useful because the cosmic no-hair theorem or the positive-energy theorem becomes available.

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*Present address: Dept. of Physics, Waseda University, Shinjuku-ku, Tokyo 169, Japan.

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