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Loop calculus for lattice gauge theories

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Hamiltonian calculations are performed using a loop-labeled basis where the full set of identities for the SU(N) gauge models has been incorporated. The loops are classified as clusterlike structures and the eigenvalue problem leads to a linear set of finite-difference equations easily amenable to numerical treatment. Encouraging results are reported for SU(2) at spatial dimension 2.

I. INTRODUCTION

The Euclidean path-integral formalism with a lattice regulator[1] has been widely used to obtain relevant nonperturbative information about gauge theories. However, in spite of considerable efforts, the most important questions remain unanswered.[2] This is, for instance, the case when one tries to identify the true weak-coupling behavior predicted by asymptotic freedom. In spite of its elegance and its nature, easily adaptable to numerical implementations, this method is probably not the most economical device to understand the real dynamics of the system. The highly symmetrical character of the action produces a host of irrelevant degrees of freedom which constitute a substantial difficulty for efficient numerical calculations.

A first natural step toward the identification of the real degrees of freedom of the problem is to work within the Hamiltonian formalism of Kogut and Susskind.[3] However one must still get rid of the residual time-independent gauge symmetry. This may be accomplished by working in the subspace defined by Gauss's law which plays the role of the primary constraints of the theory. This is the main purpose of working within the C representation. In fact, the introduction of a loop-labeled basis is the easiest way to parametrize the subspace of solutions of Gauss's constraints. Although this approach is recently becoming more popular there are still many technical issues to be explored concerning the correct manipulations of loop-dependent functionals.[4]

In this paper one presents a general computational method based on loop techniques. The Hamiltonian formalism is formulated in a space of states labeled by elements of the group of loops.[5] A complete treatment of Mandelstam's identities is given since it is necessary to identify correctly the underlying gauge symmetry.

Although, from the numeric point of view, the final results are expressed by a finite-difference system with nice computational properties, the real problem is to identify nonequivalent loops in the lattice and to compute transitions induced by the Hamiltonian in the loop space. This requires the use of an efficient list-processing computer language which should be also well adapted to deal with standard artificial intelligence problems, such as pattern matching and shape recognition. In fact, a substantial part of the whole project has been the development of the routines that use PROLOG's powerful unification and matching inherent resources, as well as its design for full back-tracking search to find all the solutions to the several problems associated with loop generation and recognition.

The use of the C representation as a tool to extract dynamical information from a finite-difference Schrödinger equation was first introduced in Ref. 6. The present method uses these ideas as general references, but it is widely different in many respects, especially in the explicit construction of the loop-labeled basis.

Although the method is general in scope we have restricted ourselves in this paper to the SU(2) gauge model. Moreover the paper has been devoted essentially to the discussion of the main ideas involved. Therefore one has only included a few small- or medium-scale calculations in two spatial dimensions for illustrative purposes.

The relevant role played by linked clusters in the construction of the basis classifies somehow the method as a strong-coupling series approach with a built-in extrapolation mechanism toward the weak-coupling region. For this reason, the quoted numeric results should perhaps be compared with those available from cluster-expansion calculations.[7] Moreover, since the successive approximants to physical quantities are computed by truncating the basis, the method exhibits some variational features and it could also be interesting to compare it with the corresponding standard variational calculations.[8–10]

The paper is organized as follows. Section II is devoted to the technical aspects of the C representation. Although the present method uses a different basis in the

loop space, this material is included to provide a self-contained treatment of Mandelstam's identities which play an important role in the dynamical implementation discussed in this paper. The presentation in Ref. 11 is followed closely and should be consulted for additional information.

In Sec. III the main aspects of Hamiltonian dynamics in loop space are developed. Most of this section may be considered as the lattice version of the continuum dynamics given in Ref. 11. The necessary elements to compute the action of the Hamiltonian in the new basis, are given in this section.

Section IV contains the main ideas of the method including the proper identification of the states in the basis and the explicit calculations to compute physical quantities by solving the finite-difference Schrödinger system.

In Sec. V one introduces the idea of using collective variables to improve the general convergence properties of the method. The rest of the section is devoted to the discussion of the results obtained, using the whole method, in the case of SU(2) at spatial dimension 2. The paper concludes in Sec. VI with some overview and comments.

II. THE C REPRESENTATION

Hamiltonian gauge theories are formulated in a hypercubic lattice of d spatial dimensions. The quantum description is given in a linear space spanned by the vectors $|u\rangle$ which are labeled by a choice for every positive link of a matrix belonging to the gauge group under consideration. These are eigenstates of the link operators,

$$U_l|u\rangle = u_l|u\rangle \ , \tag{2.1}$$

where u_l is the matrix associated with the positive link l. Negative links are not independent since one demands

$$U_{-l} = U_l^\dagger \ . \tag{2.2}$$

The canonical conjugate degrees of freedom are represented by the electric operator

$$E_l = E_l^A X_A \ , \tag{2.3}$$

which satisfies the commutation relations

$$[E_l^A, E_{l'}^B] = \mathrm{i} f^{ABC} \delta_{ll'} E_l^C \ , \tag{2.4}$$

$$[E_l^A, U_{l'}] = -\delta_{ll'} X^A U_l \ , \tag{2.5}$$

where f^{ABC} are the structure constants

$$[X^A, X^B] = \mathrm{i} f^{ABC} X^C \ , \tag{2.6}$$

which are normalized by

$$\mathrm{Tr}(X^A X^B) = \delta^{AB} \tag{2.7}$$

In terms of these objects, the Hamiltonian is given by

$$H = E_{\mathrm{op}} + B_{\mathrm{op}} = \frac{g^2}{2} \sum_{l>0} E_l^A E_l^A + \frac{1}{g^2} \sum_{\square} \mathrm{Re}\,\mathrm{Tr} U_{\square} \ , \tag{2.8}$$

where $U_{\square}$ is the product of the four U_l operators along an elementary square of the lattice.

The u states are highly redundant due to local gauge transformations associated here with multiplication of all matrices coming out of a given vertex by a fixed element of the gauge group. The C representation is introduced to work directly with physical degrees of freedom. Thus, to every closed loop in the lattice, one associates a vector defined by giving its components in terms of the original basis

$$\langle u|C\rangle = \mathrm{Tr}\left[\prod_{l\in C} u_l\right] \equiv \mathrm{Tr} U(C) \ . \tag{2.9}$$

Then, by introducing the Wilson loop operator

$$W(C) \equiv \mathrm{Tr}\left[\prod_{l\in C} U_l\right] \tag{2.10}$$

and a null loop reference state $|0\rangle$ given by

$$\langle u|0\rangle = 1 \quad \forall |u\rangle \tag{2.11}$$

it is evident that the C states are given by

$$|C\rangle = W(C)|0\rangle \ . \tag{2.12}$$

The important question of whether these states form a full basis of the linear space may be answered in the affirmative if the knowledge of $\langle u|C\rangle$ for all C and fixed u allows one to uniquely reconstruct the configuration in $|u\rangle$. This important theorem was proved by Giles and was extended in Ref. 11 to incorporate the unitarity and unimodularity conditions of the SU(N) gauge formulation. The main lines in the reconstruction process may be summarized as follows.

Let $W(C)$ be some loop-valued operator candidate to be interpreted as the trace of a SU(N) representation of the group of loops. It must obviously satisfy the relations

$$W(C_1 C_2) = W(C_2 C_1) \ , \tag{2.13}$$

which are known as Mandelstam's identities of the first kind. Here C_1 and C_2 are loops with some common point which is used as the origin to perform the product $C_1 C_2$ which is basically the concatenation of the two contours plus a set of reduction rules to guarantee the existence of a group structure in loop space.[5]

Moreover, let P be a permutation of the symmetric group of N elements S_N and let us consider its decomposition into cycles

$$P=(n_{11}n_{12}\cdots n_{1N_1})(n_{21}n_{22}\cdots n_{2N_2})\cdots(n_{K1}n_{K2}\cdots n_{KN_k}) .$$

Then one may introduce the operator

$$M_N(C_1,C_2,\ldots,C_N)\equiv\frac{1}{N!}\sum_{P\in S_N}(-1)^{(N-K)}W(C_{n_{11}}C_{n_{12}}\cdots C_{n_{1N_1}})W(C_{n_{21}}C_{n_{22}}\cdots C_{n_{2N_2}})\cdots W(C_{n_{K1}})$$
$$\times W(C_{n_{K2}})\cdots W(C_{n_{KN_k}}) , \tag{2.14}$$

which satisfies the recursive building rule

$$(p+1)M_{p+1}(C_1,C_2,\ldots,C_{p+1})=W(C_{p+1})M_p(C_1,C_2,\ldots,C_p)-M_p(C_1C_{p+1},C_2,\ldots,C_p)$$
$$-\cdots-M_p(C_1,C_2,\ldots,C_pC_{p+1}),\quad M_1(C)\equiv W(C) . \tag{2.15}$$

In terms of these operators the second-kind Mandelstam identities are given simply by

$$M_{N+1}(C_1,C_2,\ldots,C_{N+1})=0 . \tag{2.16}$$

It was proven by Giles[12] that if $W(C)$ satisfies the Mandelstam identities then it may be written as the trace of some N-dimensional representation of the group of loops. Moreover, it was proven in Ref. 11 that if the supplementary condition

$$M_N(C_1C,C_2C,\ldots,C_NC)=M_N(C_1,C_2,\ldots,C_N)\quad\forall C \tag{2.17}$$

is also satisfied, the representation will be in terms of unimodular matrices. This representation will automatically be unitary if the scalar product among the $|C\rangle$ states is the one induced by the orthonormal standard product in the u representation corresponding to the SU(N) gauge group.

A simple realization of these constraints may be produced by introducing the states

$$|C_1,C_2,\ldots,C_n\rangle\equiv M_N(C_1,C_2,\ldots,C_N)|0\rangle \tag{2.18}$$

and defining the action of the Wilson operator as

$$W(C)|C_1,C_2,\ldots,C_N\rangle$$
$$=|C_1C,C_2,\ldots,C_N\rangle+\cdots+|C_1,C_2,\ldots,C_NC\rangle ,$$

which obviously implies (2.16). It may thus be proven that the first kind Mandelstam identities are identically satisfied if $W(C)$ is realized as a commutative operator.[11]

At last, the unimodularity condition implies

$$|C_1C,C_2C,\ldots,C_NC\rangle=|C_1,C_2,\ldots,C_N\rangle . \tag{2.19}$$

This is a statement concerning transitional invariance in the loop space and may be incorporated simply by working with states containing $N-1$ relative loops.

The Hamiltonian dynamics in the C representation may easily be derived from the commutation relations between the electric operator and the M_N operators together with the invariance of the null loop state under gauge rotations:

$$E_l^A|0\rangle=0 . \tag{2.20}$$

The resulting dynamical formulation was discussed in full detail in Ref. 11. In this paper one will work in a dynamical framework based on the introduction of a slightly different representation as discussed in the next section. Nevertheless, it will become apparent that the implementation of Mandelstam's identities discussed above plays an important role in establishing the action of the Hamiltonian operator in the new basis of loop states.

III. HAMILTONIAN DYNAMICS

Let us consider in this section a different set of loop states defined by

$$|C_1,C_2,\ldots,C_M\rangle\equiv W(C_1)W(C_2)\cdots W(C_M)|0\rangle , \tag{3.1}$$

which, using (2.15), may easily be expressed as linear combinations of C representation states. In this basis the magnetic part of the Hamiltonian acts in a trivial way. In fact, each W term in the sum defining $B_{\rm op}$, acts on simply by appending a plaquette to the list of loops labeling the state.

To find the action of the electric part one starts from the commutator

$$[E_{\rm op},W(C)]=\frac{g^2}{2}\sum_{l,l'\in C}\sum\delta_{ll'}\left[W(C_O^y)W(C_y^O)-\frac{1}{N}W(C)\right]-g^2\sum_{l\in C}W_l(C) , \tag{3.2}$$

where O is the origin of the loop, C_O^y is the portion of C from the origin to the double point y implied by the Kronecker delta

$$\delta_{ll'}=\begin{cases}1 \text{ for } l=(y,i) \text{ and } l'=(y,i),\\ -1 \text{ for } l=(y,i) \text{ and } l'=(y+i,-i)\ ,\\ 0 \text{ otherwise,}\end{cases} \tag{3.3}$$

and $W(C)$ is defined as

$$W_l(C)\equiv \mathrm{Tr}[U(C_{xx})E_l]\ , \tag{3.4}$$

where x is the origin of link l.

This operator satisfies the commutation relations

$$[W(C),W_l(C')] = \sum_{l'\in C}\delta_{ll'}\left[W(C_{xx}C'_{xx})-\frac{1}{N}W(C)W(C')\right]\ , \tag{3.5}$$

where x is the common origin of l,l'.

From (3.2) and (3.5) together with (2.20) it is possible to deduce the action of the electric part of the Hamiltonian on the loop states introduced in (3.1). This action may be classified into four well-differentiated effects. The first one is given by

$$E_L|C_1,C_2,\ldots,C_M\rangle=\frac{g^2N}{2}\sum_{i=1}^{M}L_i|C_1,C_2,\ldots,C_N\rangle\ , \tag{3.6}$$

where L_i is the number of links of the ith loop.

The action of the second term is also diagonal:

$$E_\Lambda|C_1,C_2,\ldots,C_M\rangle = -\frac{g^2}{2N}\sum_{i,j=1}^{M}\Lambda_{ij}|C_1,C_2,\ldots,C_M\rangle\ , \tag{3.7}$$

where

$$\Lambda_{ij}=\sum_{l\in C_i}\sum_{l'\in C_j}\delta_{ll'} \tag{3.8}$$

is a quadratic measure of the overlap between couples of loops in the list. It will be sometimes referred to as the quadratic length.

The third term produces fusion effects among colliding loops in the list. It is given by

$$E_F|C_1,C_2,\ldots,C_M\rangle=\alpha g^2\sum_{\substack{i,j=1\\ i<j}}^{M}\sum_{l\in C_i}\sum_{l'\in C_j}\delta_{ll'}|C_i,\ldots,C_{i-1},C_{ixx}C_{jxx},C_{i+1},\ldots,C_{j-1},C_{j+1},\ldots,C_M\rangle\ , \tag{3.9}$$

where x is the origin of the contact link between the two loops.

The constant α has been introduced to allow for a general normalization of the W operators which we write now as

$$W(C)\equiv\alpha\mathrm{Tr}[U(C)]\ . \tag{3.10}$$

At last there is a fission effect which enlarges the list by splitting loops in it containing multiple links:

$$E_f|C_1,C_2,\ldots,C_M\rangle=\frac{g^2}{2\alpha}\sum_{i=1}^{M}\sum_{l,l'\in C_i}\delta_{ll'}|C_1,\ldots,C_{i-1},C_{ix}^y,C_{iy}^x,C_{i+1},\ldots,C_M\rangle\ , \tag{3.11}$$

where x,y denote the double points where the breaking takes place.

These expressions provide the implementation for the dynamics of the SU(N) model in the basis (3.1). However it is important to realize that not all these states are independent. In fact the constraints (2.17) impose relations between them. To see this in detail, let us consider the case $N=2$ where these expressions may also be cast in the more convenient form

$$W(C_1C_2)+W(C_1\bar{C}_2)=\frac{1}{\alpha}W(C_1)W(C_2)\ , \tag{3.12}$$

which is easily seen to be equivalent to (2.17). Now let us consider some loop C containing a double link. Since loops appear as arguments of the trace operator $W(C)$, they should be considered invariants under cyclic permutations of their links. Therefore, it will always be possible to express C as $lAlB$, where A and B must be closed parts of C. Therefore, one has

$$W(C)=W(lAlB)=\frac{1}{\alpha}W(lA)W(lB)-W(A\bar{B})\ , \tag{3.13}$$

which expresses $W(C)$ in terms of traces of loops where l appear with lowered multiplicity. Therefore, it is clear that a list containing a loop with multiple links, may be expressed as a linear combination of lists where the links have reduced multiplicity. By inductively repeating this argument it follows that, in forming the basis, one has only to consider lists with members with no multiple links. Notice however that links in the same spatial location and opposite orientations are not ruled out by this discussion.

This characterization of the states in the basis provides an economic way of implementing (3.12) or (2.17). In fact, one may consider this construction as an effective integration of these constraints. This is by far the main advantage of working with the basis introduced in (3.1).

IV. THE NUMBER REPRESENTATION

Following the discussion in last section, we will consider a basis of states labeled by unordered lists of simple

loops. In the SU(2) case simple loops are defined as closed contours without repeated links. A given list may contain several sublists located wide apart in the lattice. Therefore it is useful to define a cluster as a list of loops confined in a finite spatial region. Thus a general list is formed by a set of clusters separated by formally infinite interdistances.

One will be interested mainly in the description of the quantum ground state of the gauge system. Poincaré invariance of this state means that it cannot depend on the location and the orientation of clusters. Therefore, it is useful to think of a cluster as a class of equivalent loops differing by spatial translations or rigid transformations of the lattice. Within this view, it is only necessary to specify the number of occupation of each nonequivalent cluster. Therefore, by taking an arbitrary numbering of clusters, a general list will be specified as

$$|n_1,n_2,\ldots,n_M,\ldots\rangle\ , \tag{4.1}$$

where n_i denotes the number of times that cluster i appears in the list.

The next point is to introduce an ordering among clusters. This may be done in a recursive way. The null loop is order 0 and a single plaquette is order one. The Nth order is defined to consist of these loops obtained by the action of the Hamiltonian operator on clusters of order $N-1$. This will include these loops obtained by appending single plaquettes to clusters of order $N-1$, since this is the definition of the magnetic part of the Hamiltonian. The electric part will also generate clusters of this order through the fusion and fission terms (3.9) and (3.11).

Within this classification, clusters grow with the order, mainly as lists of linked plaquettes. However, loops of large area will eventually be produced by electric fusion and disconnected loops will also appear as the result of the fission term acting on plaquettes sharing links with opposite directions. In fact, it is not difficult to convince oneself that every possible loop will be obtained within this scheme by combination of the magnetic plaquette linking process and the electric fusion and fission effects.

It is natural now to propose an approximation procedure by considering only clusters of order less than or equal to a given number N. Accordingly the basis is assumed to consist of states

$$|n_1,n_2,\ldots,n_M\rangle\ , \tag{4.2}$$

where M is the last cluster of order less than or equal to the last order retained in the given approximation.

The magnetic part of the Hamiltonian will act on this basis by appending single plaquettes to the list in all possible ways. Some of these plaquettes will join the ith cluster and produce the jth one. This will induce the magnetic transition

$$|n_1,\ldots,n_i,\ldots,n_j,\ldots n_M\rangle\ ,$$
$$|n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,N_M\rangle\ ,$$

between states recognized within the given approximation. Some plaquettes will link to clusters of order N producing, by definition clusters of order $N+1$ which have not been considered. Such transitions cannot be processed exactly within this approximation. A possible way to deal with them could be to write down the transition

$$|n_1,\ldots,n_i,\ldots,n_M\rangle\rightarrow|n_1,\ldots,n_i-1,\ldots,n_M\rangle\ , \tag{4.3}$$

where i is the cluster hit by the plaquette. This ignores the creation of the new cluster since it is outside the range of the considered list. All other palquettes will be considered to add uncorrelated single-plaquette clusters, thus producing the transitions

$$|n_1,n_2,\ldots,n_M\rangle\rightarrow|n_1+1,n_2,\ldots,n_M\rangle\ , \tag{4.4}$$

The diagonal parts of the electric Hamiltonian will simply multiply the states by the corresponding lengths. The fusion term will, in principle, produce clusters with multiple links. However, the reduction mechanism (3.13), discussed in the last section, will express these loops as combinations of clusters without repeated links. This, as well as the action of the fission effects, will induce electric transitions which are treated with the same conventions as the magnetic ones.

To see all these ideas work together, a simple example will be worked out in full detail. Let us consider the SU(2) gauge theory in a lattice of spatial dimension 2. For convenience we will work with

$$H'\equiv\frac{g^2}{2}H \tag{4.5}$$

and introduce

$$\mu\equiv\frac{g^4}{2}\ , \tag{4.6}$$

so one has

$$H'=-\sum_{\square}W_{\square}+\mu\mathbb{E}\ , \tag{4.7}$$

where $W_{\square}$ is now normalized as in (3.12) with $\alpha=\frac{1}{2}$ and $\mathbb{E}$ is the corresponding multiple of $E_{\rm op}$ in (3.2).

Let us consider the proposed approximation procedure with $N=2$. One must then include cluster 1 which is the single plaquette, cluster 2 which consists of a list containing 2 times the same plaquette,

$$W(p)W(p)|0\rangle\ , \tag{4.8}$$

and cluster 3 which is formed with two plaquettes with a single common link. The basis is then restricted to the states

$$|n_1,n_2,n_3\rangle\ . \tag{4.9}$$

The action of the magnetic part of the Hamiltonian produces the count

$$-n_1|n_1-1,n_2+1,n_3\rangle-4n_1|n_1-1,n_2,n_3+1\rangle-5n_2|n_1,n_2-1,n_3\rangle-8n_3|n_1,n_2,n_3-1\rangle$$
$$-(P-5n_1-5n_2-8n_3)|n_1+1,n_2,n_3\rangle\ , \quad (4.10)$$

where P is the total number of plaquettes.

The diagonal parts of the electric operator produce

$$(3\mu n_1+4\mu n_2+\tfrac{13}{2}\mu n_3)|n_1,n_2,n_3\rangle\ . \quad (4.11)$$

The fusion term on cluster 2 will produce

$$\tfrac{1}{2}\times 4\times W(p^2)=2[2W(p)W(p)-W(0)]$$
$$=4W(p)W(p)-2\ , \quad (4.12)$$

where the reduction identity (3.13) has been used. Therefore one has the term

$$4\mu n_2|n_1,n_2,n_3\rangle-2\mu n_2|n_1,n_2-1,n_3\rangle\ . \quad (4.13)$$

The fusion term on cluster 3 produces a rectangle which is order 3 by definition. The corresponding electric transition is then partially counted as

$$-\frac{\mu}{2}n_3|n_1,n_2,n_3-1\rangle\ . \quad (4.14)$$

Now one may consider the Schrödinger equation

$$\langle\Psi_0|H'|n_1,n_2,n_3\rangle=E_0'\langle\Psi_0|n_1,n_2,n_3\rangle \quad (4.15)$$

and obtain the finite-difference equation

$$n_1[5\Psi_0(n_1+1,n_2,n_3)-\Psi_0(n_1-1,n_2+1,n_3)-4\Psi_0(n_1-1,n_2,n_3+1)+3\mu\Psi_0(n_1,n_2,n_3)]$$
$$+n_2[5\Psi_0(n_1+1,n_2,n_3)-5\Psi_0(n_1,n_2-1,n_3)+8\mu\Psi_0(n_1,n_2,n_3)-2\mu\Psi_0(n_1,n_2-1,n_3)]$$
$$+n_3\left[8\Psi_0(n_1+1,n_2,n_3)-8\Psi_0(n_1,n_2,n_3-1)+\frac{13\mu}{2}\Psi_0(n_1,n_2,n_3)-\frac{\mu}{2}\Psi_0(n_1,n_2,n_3-1)\right]$$
$$=P[\Psi_0(n_1+1,n_2,n_3)-\epsilon_0\Psi_0(n_1,n_2,n_3)]\ , \quad (4.16)$$

where we have introduced the vacuum energy density

$$\epsilon_0\equiv -E_0'/P=-\frac{g^2}{2}E_0/V\ . \quad (4.17)$$

These equations have solutions of the form

$$\Psi_0(n_1,n_2,n_3)=x_1^{n_1}x_2^{n_2}x_3^{n_3}\ . \quad (4.18)$$

The term proportional to P must vanish separately and produces the dispersion relation

$$\epsilon_0=x_1\ , \quad (4.19)$$

while the terms proportional to n_i produce

$$(5x_1+3\mu)x_1-x_2-4x_3=0\ ,$$
$$(5x_1+8\mu)x_2-2\mu-5=0\ , \quad (4.20)$$
$$\left[8x_1+\frac{13\mu}{2}\right]x_3-\frac{\mu}{2}-8=0\ ,$$

from which all three variables may be determined.

Elementary excitations may also easily be dealt with by looking for solutions of the form

$$\left[1+\frac{a_1n_1+a_2n_2+a_3n_3}{P}\right]x_1^{n_1}x_2^{n_2}x_3^{n_3} \quad (4.21)$$

from which the first excited eigenstate may be obtained and the mass gap calculated.

These considerations are readily extended to the general case where all clusters up to a given order are recognized. The Schrödinger equation is processed as a finite-difference system which is integrated with exponential solutions as in (4.18). The dispersion relation follows immediately from the vanishing of terms proportional to the total number of plaquettes. Vanishing of the terms proportional to n_i give the necessary relations

$$F_i(x_1,x_2,\ldots,x_M)=0,\ i=1,2,\ldots,M \quad (4.22)$$

to determine all the variables. This system contains $M-1$ linear equations depending on x_1. Using the solution of this linear system in the first equation produces a nonlinear equation which determines this variable.

The mass gap and, in principle, the whole glueball spectrum may be estimated by looking to excited solutions formed by polynomic modulations of the ground-state wave function. Following this procedure one finds that the mass gap is obtained as the lowest eigenvalue of the proper value problem:

$$\left[\frac{\partial F_i}{\partial x_j}\right]u_j=Mu_i\ . \quad (4.23)$$

From a computational point of view, the problem has three main operational stages. In the first one, one must identify and store all the relevant clusters up to the considered order of approximation. In the second one, the magnetic and electric transitions must be recognized and the corresponding weights computed. From these, one must construct the system. Finally, one runs into a numerical problem which involves a linear problem with a large and sparse matrix and a single nonlinear equation.

For the two first steps it is almost unavoidable to use some efficient list processing computation language that is well suited also to deal with pattern-recognition prob-

lems. All the results described in this paper have been obtained using a small size compiled version of PROLOG running on a personal computer.

V. COLLECTIVE VARIABLES

The eigenstates of the electric operator are closely related to the states in the C representation in the $\mu\to\infty$ limit. Therefore, calculations in this basis are well suited for strong-coupling developments. In fact, the approximation algorithm discussed in the last section, working up to some given order, determines an estimation of the energy density which exhibits a number of exact coefficients in its series in terms of reciprocal powers of μ. Hence one could consider the whole method as a rather elaborate extrapolation procedure to the weak-coupling region.

The behavior $\mu=0$ may be greatly improved if one uses collective variables. For such we understand some additive quantity Q taking values on loops in the lattice. For a general state (4.1) in the number representation, Q will take the value

$$Q=q_1n_1+q_2n_2+\cdots+q_in_i+\cdots\ , \tag{5.1}$$

where q_i is the value of Q at cluster i. It is permissible to exchange some variable, say n_1 by this linear combination and consider states of the form

$$|Q-q_2n_2-\cdots-q_in_i-\cdots,n_2,\ldots,n_i,\ldots\rangle\ , \tag{5.2}$$

where we have normalized Q such that $q_1=1$. While this is a simple redefinition of the whole basis, it has important consequences when one truncates the basis up to a given order of approximation.

The wave function will be rewritten as

$$\begin{aligned} &x_1^{n_1}x_2^{n_2}\cdots x_M^{n_M}=x_1^{Q}y_2^{n_2}\cdots y_M^{n_M}\ ,\\ &y_i\equiv x_ix_1^{-q_i}\ . \end{aligned} \tag{5.3}$$

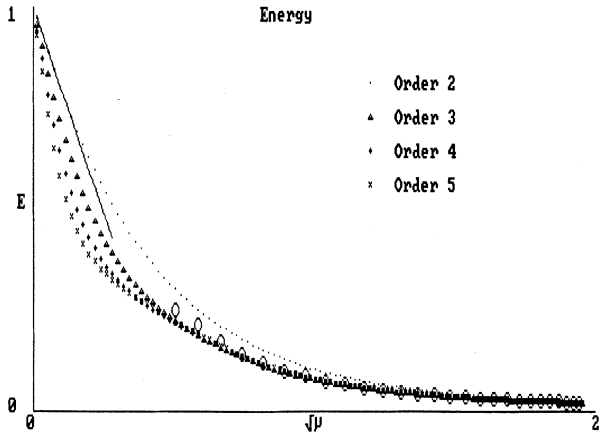

FIG. 1. Vacuum energy density vs square root of the coupling parameter for several orders. Ovals represent the strong-coupling series of Ref. 7 and the solid line is the weak-coupling estimate taken from Ref. 9.

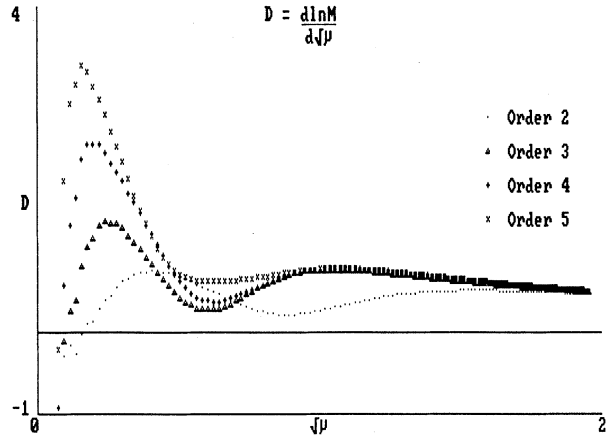

FIG. 2. Approach to weak coupling. Logarithmic derivative of the mass gap vs the square root of the coupling parameter for several orders of approximation.

Now a transition to a cluster outside the order will be counted partially through the term in Q by keeping track of the value of this quantity for the reached cluster. This means that transitions in (4.3) will not only be counted by reducing the number of i clusters but by the introduction of a corrective power of x_1 to maintain the correct count of the collective variable.

Let us now assume that the x_i variables behave as

$$x_i=1-a_i\sqrt{\mu}-b_i\mu-\cdots \tag{5.4}$$

in the weak-coupling regime. Thus, according to (5.3)

$$y_i=1-(a_i-a_1q_i)\sqrt{\mu}+O(\mu)\ . \tag{5.5}$$

Therefore if Q is chosen such that $q_i=a_1/a_1$, the non-analytic behavior disappears at the lowest order from y_i. Reversing the argument, it follows that the introduction of the appropriate collective variable will induce a square-root behavior on quantities that otherwise would always be analytic in the coupling parameter. In fact, this is a well-known problem in most strong-coupling cal-

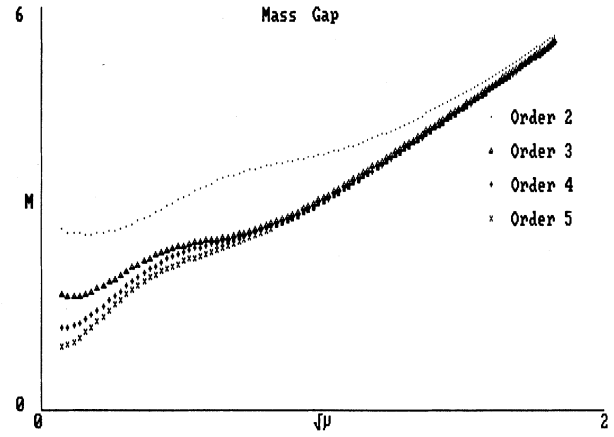

FIG. 3. Mass gap vs square root of the coupling constant for several orders of approximation.

culations where the asymptotic series in μ must be carried to the weak-coupling region, using analytic extrapolants, and it can never match the exact weak-coupling square-root law. By explicit examination of the finite-difference system produced by the present method using a collective variable, it is possible to prove that a square-root behavior will indeed be present if the count is maintained on a collective variable that satisfies

$$\langle \Delta Q \rangle = q_1 = 1, \quad \langle \Delta Q^2 \rangle \neq q_1^2 = 1 , \tag{5.6}$$

for each cluster. Here $\langle \Delta Q \rangle$ means the average of change of the collective variable over all magnetic and electric transitions of a given cluster.

It is found that the quadratic length Λ (3.8) matches this criterion. In fact any arbitrary linear combination of Λ with a trivially additive variable such as, for instance, the total number of links L, will satisfy (5.6). It is important to choose properly the relative weight between these two variables. Otherwise, the exponents of x_1 generated by the collective compensations using the quadratic length, grow too quickly and suppress any other effect in the involved transitions.

It should be noticed that the maximum value of Λ for clusters of order N is $4N^2$ reached at the cluster consisting of N identical plaquettes. To prevent Q from excessive growth we have normalized it by prescribing Q to 1 for such a cluster. This uniquely fixes Q to

$$Q = \frac{(N+1)L - \Lambda}{4N} , \tag{5.7}$$

which has been our choice to obtain the results discussed below.

We have only considered the SU(2) gauge model in spatial dimension 2, for order $N=2$, 3, 4, and 5. The corresponding number of clusters for each order is 3, 8, 23, and 80.

In Fig. 1 the different determinations of ϵ_0 are represented against $\sqrt{\mu}$. The curves behave linearly near $\mu=0$ thus exhibiting the expected square-root behavior. As mentioned this would also be the case for any other choice of the collective variable satisfying (5.6). However this does not mean that one has a sensible description of the weak-coupling region. As discussed earlier, this method is a disguised asymptotic strong-coupling series with a built-in extrapolation mechanism toward $\mu=0$. Hence, for every finite order of approximation, one should find a totally incorrect picture for sufficiently small values of μ.

This is clear, for instance, in Fig. 2 where the logarithmic derivative of the mass gap versus $\sqrt{\mu}$ is depicted. In this dimension it is known that no phase transition occurs. Nonetheless, our curves exhibit a neat peak that gets stronger and narrower when the order of approximation increases. A possible interpretation of these numerical results would be that our model mimics a smeared-out phase transition at finite values of μ. Fortunately, the "critical values" go deeper into the weak-coupling region as the approximation improves. In fact, these sets of values extrapolate nicely to 0 where the connection with the continuum physics should lie. Within the same spirit,

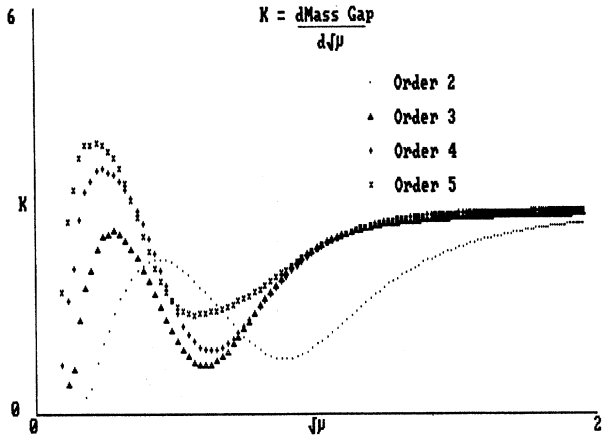

FIG. 4. Approach to scaling. Derivative of the mass gap vs the square root of the coupling parameter. The value at the maximum is taken as an estimate of K in the scaling relation.

the secondary bump near $\mu=1$, exhibited by these curves, could be interpreted as a borderline between the strong- and weak-coupling regimes. The two different nearly linear types of behavior exhibited by the mass gap in Fig. 3 could also be interpreted along the same lines.

In this superrenormalizable theory[13] the expected scaling behavior is simply

$$M = K\sqrt{\mu}, \quad \mu \simeq 0 . \tag{5.8}$$

To determine the value of K, the $\sqrt{\mu}$ derivative of the mass gap has been plotted in Fig. 4 against $\sqrt{\mu}$. These curves show a plateau which could be considered as an approximation to the region where the constant behavior (5.8) is established. The behavior to the left of this plateau is also to the left of the "transition point" and therefore must be considered as a spurious asymptotic noise of the method.

The values of K, for several orders of approximation, read from the value of the derivative of the mass gap at the maximum are

$$\begin{aligned} &K_2 = 2.28, \quad K_3 = 2.70, \\ &K_4 = 3.63, \quad K_5 = 4.03 , \end{aligned} \tag{5.9}$$

which extrapolate to a value in good agreement with the Monte Carlo calculations done in the Euclidean version of the model:[14]

$$K_\infty = K_{\text{Monte Carlo}} = 4.6 \pm 0.6 . \tag{5.10}$$

This result is also in good agreement with the Irving and Hamer estimate:[7]

$$K_{\text{IH}} = 4.4 \pm 0.5 , \tag{5.11}$$

which in turn has been more recently confirmed by the work of Irbäck and Peterson.[15]

VI. CONCLUSION

In this paper a computational procedure based on the geometry of the group of loops has been developed which allows the determination of physical quantities by solving a finite-difference version of the Schrödinger equation in the loop space.

The method heavily relies on the symbolic computer manipulations that allow the generation of the loop basis and the recognition of the electric and magnetic transitions.

Since the number of loops grows without measure when one increases the order of the approximation, it is important to have good general convergence behavior. We have shown that the introduction of collective variables may be very useful in this context as acceleration devices toward the weak-coupling region for a method which starts basically with strong-coupling information.

Our experience on this point has been that, in spite of the fact that the general trend is basically always qualitatively similar, the method is very sensitive to the cutoff procedure used to deal with transitions outside the order of approximation. This point has been only slightly touched upon in this paper. Undoubtedly the choice of the nature and number of the collective variables must be explored in subsequent work.

In fact, the same is true for many other aspects of this work which should be considered a first essay on a subject with many issues.

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