Interpreting four-quark interactions in finite-temperature, SU(2), lattice gauge theory

Joe Kiskis and Rajamani Narayanan
Department of Physics, University of California, Davis, California 95616
(Received 12 December 1988)

Associated with the finite-temperature lattice gauge theory is an effective theory of Wilson lines. The strength of the four-line interaction has been measured by Monte Carlo simulation in earlier work. This paper adds perspective to the numerical work in two ways. First, the physical interpretation of the four-line coupling is given. Second, the relationship with critical behavior in three-dimensional scalar theories is established. On this basis, a complete postdiction of the numerical results can be given.

I. INTRODUCTION

This paper discusses properties of the correlation function of four Wilson lines in finite-temperature, SU(2), lattice gauge theory. It is a continuation of the work in Ref. 1 (I).

The order parameter of the deconfinement-confinement phase transition in pure gauge theory is the Wilson line. An effective theory for the lines can be obtained by integrating out all other degrees of freedom in finite-temperature gauge theory. The effective line theory is three dimensional and has a global $Z(2)$ symmetry. It has an order-disorder phase transition that is associated with the deconfinement-confinement phase transition. All indications are that the effective theory has short-range interactions,[2] that the transition is second order, and that it has the non-Gaussian critical behavior of the universality class that includes the three-dimensional Ising model and $(\phi^4)_3$ theory. In this way, the critical region of the gauge theory defines a three-dimensional, interacting field theory of lines.[3]

Thus one expects that the one-particle-irreducible (1PI), four-line correlation function should be different from zero and should scale appropriately relative to the diverging correlation length. In I, this expectation was confirmed in a Monte Carlo simulation.

The purpose of this paper is to give a more complete perspective on the numerical results in I. The quantity measured there was called G_4. It is the ratio of the renormalized, 1PI, zero-momentum four-line function to the renormalized mass and is an appropriate dimensionless measure of the interaction strength. Section II of this paper discusses the physical interpretation of G_4. At large separation, the four-quark free energy is dominated by contributions arising from the pairwise connection of the quarks by noninteracting strings. As the separations decrease, the interactions of the strings become relatively more important. The leading corrections are proportional to G_4. It is worth noting that the distance scale at which these interactions become large is $T/\sigma(T)$. For $T\rightarrow T_c$, this scale is much larger than the other natural scales $\sigma(0)^{-1/2}$ or $\sigma(T)^{-1/2}$. The larger scale $T/\sigma(T)$ is due to finite-temperature, transverse fluctuations of the string.[4]

In the context of $(\lambda/4!)\phi^4$ theory in three dimensions, the analog of G_4 is $g\equiv\lambda/m$. The lattice version of $(\phi^4)_3$ theory and other theories in this universality class have been studied extensively.[5] Most of the results that we need can be found in Ref. 6. Section III reviews some important properties of the behavior of g on the critical surface. These results are reformulated a bit so as to be more convenient for our purposes.

Section IV reviews the relationship between the finite-temperature gauge theory and three-dimensional scalar theories. With this, Sec. V can establish the correspondence of quantities and results in the two arenas. In particular, the properties of g and its relation to G_4 are used to give a complete explanation of the numerical results for G_4 in I. This includes both the qualitative behavior and the numerical value as $\xi\rightarrow\infty$.

In summary, this paper adds perspective to the numerical work in I in two ways: First, by completing the physical interpretation of G_4, and second, by exploiting the connection to the known critical behavior of g in three dimensions.

II. FOUR-QUARK FREE ENERGY

In this section we establish the relationship between the quantity G_4 defined in I and the four-quark free energy in the confined phase.

In the low-temperature phase, the main contributions to the free energy $F(\mathbf{X})$, of a widely separated quark pair are the confining part of the potential $\sigma(T)|\mathbf{X}|$, a smaller $T\ln|\mathbf{X}|$ term in the potential, and self-energies, which can be expressed as $-T\ln Z$. Thus, the long-distance behavior of the correlation function for two Wilson lines is given by

$$\langle L(\mathbf{0})L(\mathbf{X})\rangle = e^{-F(\mathbf{X})/T} = \frac{Z}{|\mathbf{X}|}e^{-m|\mathbf{X}|}\ , \qquad (2.1)$$

where m is the inverse of the correlation length, and is related to the finite-temperature string tension $\sigma(T)$ by $\sigma(T)/T$. Z is the wave-function renormalization constant. The quantity G_4 defined in I is

$$G_4 = \frac{\tilde{\Gamma}^{(4)}(0,0,0,0)}{m} , \tag{2.2}$$

where $-\tilde{\Gamma}^{(4)}(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4)$ is the renormalized, 1PI, four-point function of Wilson lines in momentum space.[7] $\tilde{\Gamma}^{(4)}$ is related to $\tilde{\Gamma}_0^{(4)}$, the unrenormalized, 1PI, four-point function by

$$\tilde{\Gamma}^{(4)} = Z^2 \tilde{\Gamma}_0^{(4)} . \tag{2.3}$$

In order to study the four-quark free energy F, we look at the four-Wilson-line correlation function $\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle$. This is related to $F(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)$ by

$$\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle = e^{-F(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)/T} . \tag{2.4}$$

We can write the four-Wilson-line correlation function as

$$\begin{aligned}\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle &= \langle L(\mathbf{X}_1)L(\mathbf{X}_2)\rangle\langle L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle + \langle L(\mathbf{X}_1)L(\mathbf{X}_3)\rangle\langle L(\mathbf{X}_2)L(\mathbf{X}_4)\rangle \\ &\quad + \langle L(\mathbf{X}_1)L(\mathbf{X}_4)\rangle\langle L(\mathbf{X}_2)L(\mathbf{X}_3)\rangle + \langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle_c \\ &= \langle 1234\rangle_{\mathrm{dc}} + \langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle_c .\end{aligned} \tag{2.5}$$

The first three terms on the right side denoted by $\langle 1234\rangle_{\mathrm{dc}}$ are the disconnected pieces arising from pairs of noninteracting strings. The last term is the connected piece. It includes the interactions of the strings and other short-distance effects. For separations much larger than $1/m$, $\langle 1234\rangle_{\mathrm{dc}}$ is expected to dominate. This motivates us to write the free energy in the form

$$\frac{-1}{T}F(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4) = \ln(\langle 1234\rangle_{\mathrm{dc}}) + \ln[1+f(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)] , \tag{2.6}$$

where

$$f(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4) = \frac{\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle_c}{\langle 1234\rangle_{\mathrm{dc}}} . \tag{2.7}$$

f will be very small for large separations. We are interested in its behavior as the quark separation decreases. This gives the leading corrections to the noninteracting string contribution in the first term of (2.6).

To study f, we begin by writing

$$\begin{aligned}\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle_c = &-\int d^3Y_1 d^3Y_2 d^3Y_3 d^3Y_4 \langle L(\mathbf{X}_1)L(\mathbf{Y}_1)\rangle\langle L(\mathbf{X}_2)L(\mathbf{Y}_2)\rangle \\ &\times\langle L(\mathbf{X}_3)L(\mathbf{Y}_3)\rangle\langle L(\mathbf{X}_4)L(\mathbf{Y}_4)\rangle\Gamma_0^{(4)}(\mathbf{Y}_1,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4) .\end{aligned} \tag{2.8}$$

This is interpreted as the four Wilson lines at $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, and $\mathbf{X}_4$ propagating to $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, and $\mathbf{Y}_4$, respectively, and interacting with strength $\Gamma_0^{(4)}$. Because of the $L\to -L$ symmetry of the confined phase, $\Gamma_0^{(4)}$ is actually the 1PI, unrenormalized, coordinate-space, four-point function. Using Eqs. (2.1) and (2.3), we can write Eq. (2.8) in a form that is approximately correct for large quark separation:

$$\begin{aligned}\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle_c = &-Z^2\int d^3Y_1 d^3Y_2 d^3Y_3 d^3Y_4 \frac{e^{-m|\mathbf{X}_1-\mathbf{Y}_1|}}{|\mathbf{X}_1-\mathbf{Y}_1|}\frac{e^{-m|\mathbf{X}_2-\mathbf{Y}_2|}}{|\mathbf{X}_2-\mathbf{Y}_2|}\frac{e^{-m|\mathbf{X}_3-\mathbf{Y}_3|}}{|\mathbf{X}_3-\mathbf{Y}_3|} \\ &\times\frac{e^{-m|\mathbf{X}_4-\mathbf{Y}_4|}}{|\mathbf{X}_4-\mathbf{Y}_4|}\Gamma^{(4)}(\mathbf{Y}_1,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4) .\end{aligned} \tag{2.9}$$

As a first choice, we assume a point-type interaction given by

$$\Gamma^{(4)}(\mathbf{Y}_1,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4) = \lambda\int d^3Y\,\delta^3(\mathbf{Y}-\mathbf{Y}_1)\delta^3(\mathbf{Y}-\mathbf{Y}_2)\delta^3(\mathbf{Y}-\mathbf{Y}_3)\delta^3(\mathbf{Y}-\mathbf{Y}_4) . \tag{2.10}$$

This corresponds to an interaction given by $\lambda\delta^3(\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3+\mathbf{P}_4)$ in momentum space. It follows from the definition in (2.2) that $G_4=\lambda/m$. Substituting Eq. (2.10) into (2.9), we get

$$\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle_c = -Z^2\lambda\int d^3Y\frac{e^{-m(|\mathbf{X}_1-\mathbf{Y}|+|\mathbf{X}_2-\mathbf{Y}|+|\mathbf{X}_3-\mathbf{Y}|+|\mathbf{X}_4-\mathbf{Y}|)}}{|\mathbf{X}_1-\mathbf{Y}||\mathbf{X}_2-\mathbf{Y}||\mathbf{X}_3-\mathbf{Y}||\mathbf{X}_4-\mathbf{Y}|} . \tag{2.11}$$

Let

$$h(\mathbf{Y}) = |\mathbf{X}_1-\mathbf{Y}|+|\mathbf{X}_2-\mathbf{Y}|+|\mathbf{X}_3-\mathbf{Y}|+|\mathbf{X}_4-\mathbf{Y}| . \tag{2.12}$$

For the situation where $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, and $\mathbf{X}_4$ are all separated by distances larger than $1/m$, the major contribution to the integral comes from the $\mathbf{Y}$ region where $h(\mathbf{Y})$ is a minimum (Laplace's method). We note that the points $\mathbf{Y}=\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, or $\mathbf{X}_4$ cause no problems because the measure $|\mathbf{Y}|^2 d|\mathbf{Y}|d\Omega$ cancels the singularity present in the integrand. The point where $h(\mathbf{Y})$ is a minimum, $\mathbf{Y}_{\min}\equiv\mathbf{Z}$, is given by the equation

$$\frac{Z^i-X_1^i}{|\mathbf{Z}-\mathbf{X}_1|}+\frac{Z^i-X_2^i}{|\mathbf{Z}-\mathbf{X}_2|}+\frac{Z^i-X_3^i}{|\mathbf{Z}-\mathbf{X}_3|}+\frac{Z^i-X_4^i}{|\mathbf{Z}-\mathbf{X}_4|}=0, \quad i=1,2,3 , \tag{2.13}$$

where the superscript i denotes the components of the vector. The above equation has a nice geometric interpretation. If one draws unit vectors from the point of minimum to the four points, then the sum of the unit vectors is zero. For almost all configurations, this point of minimum will lie away from the four points and will be separated by a large distance from the four points. The special case of $\mathbf{Z}$ coinciding with one of the four points is measure zero and will be discussed later. We assume for the present that $\mathbf{Z}$ does not coincide with any of the four points. Let

$$a_{ij}=\left.\frac{\partial^2 h}{\partial Y^i \partial Y^j}\right|_{\mathbf{Y}=\mathbf{Z}}=\sum_{k=1}^{4}\left[\frac{\delta_{ij}}{|\mathbf{Z}-\mathbf{X}_k|}-\frac{(\mathbf{Z}-\mathbf{X}_k)^i(\mathbf{Z}-\mathbf{X}_k)^j}{|\mathbf{Z}-\mathbf{X}_k|^3}\right] . \tag{2.14}$$

Then,

$$h(\mathbf{Y})\simeq h(\mathbf{Z})+\tfrac{1}{2}a_{ij}(Y^i-Z^i)(Y^j-Z^j) . \tag{2.15}$$

We note that a_{ij} has dimensions of $[L^{-1}]$. Applying Laplace's method to Eq. (2.11) we get

$$\begin{aligned}\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle_c &\simeq -Z^2\lambda\frac{e^{-m(|\mathbf{X}_1-\mathbf{Z}|+|\mathbf{X}_2-\mathbf{Z}|+|\mathbf{X}_3-\mathbf{Z}|+|\mathbf{X}_4-\mathbf{Z}|)}}{|\mathbf{X}_1-\mathbf{Z}||\mathbf{X}_2-\mathbf{Z}||\mathbf{X}_3-\mathbf{Z}||\mathbf{X}_4-\mathbf{Z}|}\int d^3Y\, e^{-(m/2)a_{ij}(Y^i-Z^i)(Y^j-Z^j)} \\ &=-Z^2\lambda\frac{e^{-m(|\mathbf{X}_1-\mathbf{Z}|+|\mathbf{X}_2-\mathbf{Z}|+|\mathbf{X}_3-\mathbf{Z}|+|\mathbf{X}_4-\mathbf{Z}|)}}{|\mathbf{X}_1-\mathbf{Z}||\mathbf{X}_2-\mathbf{Z}||\mathbf{X}_3-\mathbf{Z}||\mathbf{X}_4-\mathbf{Z}|}\left[\frac{2\pi}{m}\right]^{3/2}\frac{1}{\sqrt{\det a}} .\end{aligned} \tag{2.16}$$

We note that $\det a$ has dimensions of $[L^{-3}]$. Combining Eqs. (2.16), (2.7), and (2.1) we can write

$$f(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)=\frac{-\lambda\dfrac{e^{-m(|\mathbf{X}_1-\mathbf{Z}|+|\mathbf{X}_2-\mathbf{Z}|+|\mathbf{X}_3-\mathbf{Z}|+|\mathbf{X}_4-\mathbf{Z}|)}}{|\mathbf{X}_1-\mathbf{Z}||\mathbf{X}_2-\mathbf{Z}||\mathbf{X}_3-\mathbf{Z}||\mathbf{X}_4-\mathbf{Z}|}\left[\dfrac{2\pi}{m}\right]^{3/2}\dfrac{1}{\sqrt{\det a}}}{\dfrac{e^{-m(|\mathbf{X}_1-\mathbf{X}_2|+|\mathbf{X}_3-\mathbf{X}_4|)}}{|\mathbf{X}_1-\mathbf{X}_2||\mathbf{X}_3-\mathbf{X}_4|}+\dfrac{e^{-m(|\mathbf{X}_1-\mathbf{X}_3|+|\mathbf{X}_2-\mathbf{X}_4|)}}{|\mathbf{X}_1-\mathbf{X}_3||\mathbf{X}_2-\mathbf{X}_4|}+\dfrac{e^{-m(|\mathbf{X}_1-\mathbf{X}_4|+|\mathbf{X}_2-\mathbf{X}_3|)}}{|\mathbf{X}_1-\mathbf{X}_4||\mathbf{X}_2-\mathbf{X}_3|}} . \tag{2.17}$$

We now estimate $f(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4)$ for the generic case when all distances are of the order $1/m$. Then all exponentials are of order 1 and $\det a$ will be of order m^3. We therefore obtain

$$\begin{aligned}f(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4) &\sim \frac{\left[-\lambda\Big/\left[\dfrac{1}{m}\right]^4\right]\dfrac{1}{m^{3/2}}\dfrac{1}{m^{3/2}}}{1/(1/m)^2} \\ &=-\frac{\lambda}{m}=-G_4 .\end{aligned} \tag{2.18}$$

Therefore, we have shown that this correction to free string behavior in the free energy of four quarks, changes from zero for infinitely large separations to a value proportional to G_4 when the separations become comparable to $1/m$.

The above argument will break down if $\det a$ is singular. We now investigate this possibility. We proceed through the following argument.

(i) Given $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, and $\mathbf{X}_4$ we find $\mathbf{Z}$. Then we change our coordinates so that $\mathbf{Z}=0$.

(ii) Then we compute the symmetric matrix a. Its eigenvalues are real and so are its eigenvectors. Further, the eigenvectors will also be orthonormal. Therefore a can be diagonalized by a real orthogonal matrix, i.e., by a rotation of the coordinate system. We will now perform this rotation which makes a diagonal.

(iii) In the new coordinate system we can write

$$a_{ij}=\sum_k\left[\frac{1}{|\mathbf{X}_k|}-\frac{(\mathbf{X}_k)_i(\mathbf{X}_k)_j}{|\mathbf{X}_k|^3}\right]\delta_{ij} . \tag{2.19}$$

For $\det a$ to be zero, one or more a_{ii} should be zero. But

$$\frac{1}{|\mathbf{X}_k|}-\frac{(\mathbf{X}_k)_i(\mathbf{X}_k)_i}{|\mathbf{X}_k|^3}\geq 0 \quad \text{for all } i \text{ and } k . \tag{2.20}$$

Therefore $a_{ii}=0$ if

$$1-\frac{(\mathbf{X}_k)_i(\mathbf{X}_k)_i}{|\mathbf{X}_k|^2}=0 \quad \text{for all } k . \tag{2.21}$$

In particular if $a_{11}=0$, we have

$$[(\mathbf{X}_k)_2]^2+[(\mathbf{X}_k)_3]^2=0 \quad \text{for all } k , \tag{2.22}$$

which implies that $(\mathbf{X}_k)_2=(\mathbf{X}_k)_3=0$ for all k; i.e., all four points lie on the $\mathbf{X}$ axis. Therefore, the matrix a is singular only if all points lie on a straight line. This is a very special case. This result can be easily understood by realizing that if all four points lie on a line, then $\mathbf{Z}$ is not fixed and can be anywhere on the segment between the two inner points.

We now remark on the other special case where $\mathbf{Z}$ coincides with one of the four points. With a slight modification, we can apply Laplace's method to this case also. The result of the calculation is the same as the generic case.

Now consider an interaction that is more general than the point interaction in Eq. (2.10). We assume that $\Gamma^{(4)}(\mathbf{Y}_1,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4)$ has a maximum when $\mathbf{Y}_1=\mathbf{Y}_2=\mathbf{Y}_3=\mathbf{Y}_4$ and falls off with a scale greater than m with any increasing Y separation. This assumption means that the glueball mass, which is one of the scales setting the fall of $\Gamma^{(4)}$, is bigger than the scale set by the string tension $[m=\sigma(T)/T]$ near $T=T_c$. Let us now look at Eq. (2.9). Near $\mathbf{Y}_1=\mathbf{Y}_2=\mathbf{Y}_3=\mathbf{Y}_4$ we can write

$$|\mathbf{X}_2-\mathbf{Y}_2|\simeq|\mathbf{X}_2-\mathbf{Y}_1|+\frac{X_2^i-Y_1^i}{|\mathbf{X}_2-\mathbf{Y}_1|}(Y_1^i-Y_2^i)$$
$$=|\mathbf{X}_2-\mathbf{Y}_1|+\hat{\mathbf{n}}\cdot(\mathbf{Y}_1-\mathbf{Y}_2)\ , \qquad (2.23)$$

where

$$n^i=\frac{X_2^i-Y_1^i}{|\mathbf{X}_2-\mathbf{Y}_1|} \qquad (2.24)$$

is a unit vector. Similar equations can be written for $|\mathbf{X}_3-\mathbf{Y}_3|$ and $|\mathbf{X}_4-\mathbf{Y}_4|$. This means that $e^{-m|\mathbf{X}_2-\mathbf{Y}_2|}$, $e^{-m|\mathbf{X}_3-\mathbf{Y}_3|}$, and $e^{-m|\mathbf{X}_4-\mathbf{Y}_4|}$ fall or rise at most as fast as m, near $\mathbf{Y}_2=\mathbf{Y}_1$, $\mathbf{Y}_3=\mathbf{Y}_1$, and $\mathbf{Y}_4=\mathbf{Y}_1$. Therefore, we can use Laplace's method to integrate over $\mathbf{Y}_2$, $\mathbf{Y}_3$, and $\mathbf{Y}_4$. This result is

$$\langle L(\mathbf{X}_1)L(\mathbf{X}_2)L(\mathbf{X}_3)L(\mathbf{X}_4)\rangle_c\simeq -Z^2\int d^3Y_1\frac{e^{-m(|\mathbf{X}_1-\mathbf{Y}_1|+|\mathbf{X}_2-\mathbf{Y}_1|+|\mathbf{X}_3-\mathbf{Y}_1|+|\mathbf{X}_4-\mathbf{Y}_1|)}}{|\mathbf{X}_1-\mathbf{Y}_1||\mathbf{X}_2-\mathbf{Y}_1||\mathbf{X}_3-\mathbf{Y}_1||\mathbf{X}_4-\mathbf{Y}_1|}$$
$$\times\int d^3Y_2d^3Y_3d^3Y_4\Gamma^{(4)}(\mathbf{Y}_1,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4)\ . \qquad (2.25)$$

Now

$$\int d^3Y_2d^3Y_3d^3Y_4\Gamma^{(4)}(\mathbf{Y}_1,\mathbf{Y}_2,\mathbf{Y}_3,\mathbf{Y}_4)$$

will be independent of $\mathbf{Y}_1$ due to translational invariance, and it is precisely the renormalized four-point function at zero momentum $\tilde{\Gamma}^{(4)}(0,0,0,0)=\lambda$. We can conclude again that if there is no scale smaller than m, then for distances of order $1/m$, f is proportional to G_4.

III. $\lambda(\phi^4)_3$ THEORY

We begin by considering scalar theory on a lattice in three dimensions. The two parameters are the bare mass m_0 and the bare coupling λ_0. The critical surface intersects the λ_0,m_0^2 plane giving the critical line indicated in Fig. 1. The many irrelevant directions are out of the page. In three dimensions, there are two fixed points,[5] one trivial and one nontrivial. The phase transition is second order, and the order parameter is $\langle\phi\rangle$. We will be interested in the approach to the critical surface from the symmetric phase.

We define a quantity g as[6]

$$g=\frac{\tilde{\Gamma}^{(4)}(0,0,0,0)}{m}\ , \qquad (3.1)$$

where $\tilde{\Gamma}^{(4)}(0,0,0,0)$ is the renormalized, 1PI, four-point function at zero momentum and m is the renormalized mass. We notice that g is well defined away from the critical surface where m does not vanish.

We first show that g is a constant along a renormalization flow line. We proceed along the lines of Wilson and Kogut.[5] The lattice field theory is formulated in momentum space. The momentum scale is normalized, so that it ranges from 0 to 1. A renormalization-group iteration is done as follows. The momentum integration is carried out from $\frac{1}{2}$ to 1. This results in a redefinition of the coupling constants and fields. Then the momentum is rescaled to range from 0 to 1. This means that $\mathbf{P}_i\rightarrow\mathbf{P}_i'=2\mathbf{P}_i$ under an iteration. Under such an iteration $\phi_{\mathbf{P}}\rightarrow\phi_{\mathbf{P}}'$, and is defined by

$$\phi_{\mathbf{P}}=\zeta\phi'_{\mathbf{P}'}=\zeta\phi'_{2\mathbf{P}}\ . \qquad (3.2)$$

The n-point Green's function in terms of $\mathbf{P}_i'$ is given by

$$\delta^3(\mathbf{P}_1'+\mathbf{P}_2'+\cdots+\mathbf{P}_n')G_n'(\mathbf{P}_1',\ldots,\mathbf{P}_n')$$
$$=\int[d\phi']\phi'_{\mathbf{P}_1'}\phi'_{\mathbf{P}_2'}\cdots\phi'_{\mathbf{P}_n'}\frac{e^{\mathcal{H}'}}{\mathcal{Z}'}\ , \qquad (3.3)$$

where $\mathcal{H}'$ is the Hamiltonian and $\mathcal{Z}'$ the normalization factor after one iteration. Using Eq. (3.2) in Eq. (3.3), it follows that

$$\zeta^nG_n'(2\mathbf{P}_1,\ldots,2\mathbf{P}_n)\delta^3(2\mathbf{P}_1+2\mathbf{P}_2+\cdots+2\mathbf{P}_n)$$
$$=G_n(\mathbf{P}_1,\ldots,\mathbf{P}_n)\delta^3(\mathbf{P}_1+\mathbf{P}_2+\cdots+\mathbf{P}_n)$$
$$\Longrightarrow G_n(\mathbf{P}_1,\ldots,\mathbf{P}_n)=\zeta^n2^{-3}G_n'(2\mathbf{P}_1,\ldots,2\mathbf{P}_n)\ . \qquad (3.4)$$

The renormalized mass is defined by

$$\frac{1}{m^2}=-\frac{1}{G_2(\mathbf{P})}\frac{d^2G_2(\mathbf{P})}{d\mathbf{P}^2}\bigg|_{\mathbf{P}=0}\ . \qquad (3.5)$$

This implies that under $\mathbf{P}\rightarrow\mathbf{P}'=2\mathbf{P}$, $m\rightarrow m'=2m$. The renormalized mass is also related to G_2 by

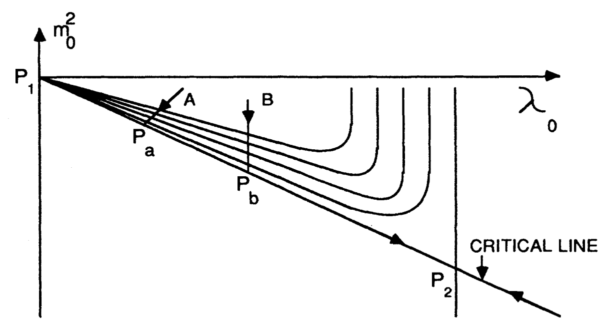

FIG. 1. The λ_0,m_0^2 plane for $(\phi^4)_3$ theory. The critical line passes through fixed points at P_1 and P_2.

$$G_2(0)=Zm^{-2} . \tag{3.6}$$

Combining Eqs. (3.4) and (3.6) we get the transformation for Z, the wave-function renormalization, as

$$Z=2^{-5}\zeta^2 Z' . \tag{3.7}$$

The renormalized, 1PI, four-point function at zero momentum is given by

$$\begin{aligned}\tilde{\Gamma}^{(4)}(0,0,0,0)&=Z^2\tilde{\Gamma}_0^{(4)}(0,0,0,0)\\&=Z^2\frac{G_4(0,0,0,0)}{[G_2(0)]^4} .\end{aligned} \tag{3.8}$$

Combining Eqs. (3.4), (3.7), and (3.8) we get

$$\tilde{\Gamma}^{(4)'}(0,0,0,0)=2\tilde{\Gamma}^{(4)}(0,0,0,0) . \tag{3.9}$$

This, along with the result that $m'=2m$, implies that the g defined in Eq. (3.1) is constant along a renormalization flow line.

Next, we define g at a point on the critical surface as the limit obtained by approaching that point along any direction in the parameter space. Since g is expected to behave smoothly as a function of the parameters, this limit is valid. Now we show that g defined this way is a constant along a flow that is on the critical surface. Consider approaching P_a along A and P_b along B as shown in Fig. 1. We first note that both A and B cut the same renormalization flow lines in the same order. Since g is a constant on each of the flow lines, the sequence of values of g that both paths A and B define are the same and hence they must approach the same limit. Hence g at P_a is the same as g at P_b. Therefore g is a constant along a flow on the critical surface. As a side comment, we note that to obtain different values of g for the continuum field theory one has to approach the Gaussian fixed point along different flow lines.

Next, we focus on the value of g on the critical surface. It can be shown in the context of field theory defined on the critical surface that g is finite.[8] It is shown in Ref. 6 that $g=0$ if hyperscaling is violated and that $g\neq 0$ if hyperscaling is valid. In the latter case, $g\simeq 24^6$.

Since g is defined by zero-momentum quantities, it should be universal. If it is continuous at the nontrivial fixed point, then it must have the same value on different flows on the critical surface. This means that it is constant in the region of attraction of the nontrivial fixed point, i.e., it is universal.

IV. CONNECTION

This section will serve to review the relation between (3+1)-dimensional, finite-temperature, SU(2), gauge theory and three-dimensional, Euclidean, Z_2-symmetric, scalar field theory.[9]

The Wilson line is the order parameter for the finite-temperature confinement-deconfinement phase transition. There is a line operator $L(\mathbf{I})$ on each site of the three-dimensional spatial lattice. The gauge theory has a global Z_2 symmetry that acts as $L\rightarrow -L$ on the lines.

An effective theory for the lines is obtained by holding the values of all of the lines fixed while all of the other variables of the gauge theory are integrated out. The field theory of lines that results will be very complicated.

A few general remarks can be made. Evidently it is a zero-temperature, three-dimensional, Euclidean, scalar field theory. The form of the theory will depend on the gauge coupling and the number of lattice spacings in the inverse temperature direction N_T. The line theory will have a global, Z_2 field-reflection symmetry.

The line theory will include complicated interactions among lines at different sites. There are theoretical arguments and numerical results which suggest that these interactions are short ranged.[2] There is also a large body of evidence which supports the view that the gauge theory phase transition is second order. The line theory has an associated second-order, disorder-order phase transition.

Assuming short-range interactions and a second-order transition, the lore of the renormalization group and universality can be evoked. This asserts that all three-dimensional, Z_2-symmetric, scalar theories are in the same universality class. Each such theory can be thought of as a point in the infinite-dimensional space of such theories. Included in this space are the line theory, lattice $(\phi^4)_3$ theory, and the three-dimensional Ising model.

The infinite-dimensional space has one relevant direction and a critical surface of codimension one. There is an infrared unstable Gaussian fixed point at the origin and an infrared stable nontrivial fixed point. Except for the fixed point at the origin, the critical behavior is the same all over the critical surface and is controlled by the nontrivial fixed point.

The 1PI, n-point functions have a parametric dependence on position in the theory space. One can think of the correlation functions of the line theory and of $(\phi^4)_3$ theory as being the same function evaluated at different points in the theory space. Since G_4 and g are defined in the same way in terms of zero-momentum correlation functions, they are the same quantity evaluated at different points in the theory space.

When the parameters m_0^2 and λ_0 of $(\phi^4)_3$ theory and N_T and the gauge coupling of the line theory are adjusted to criticality, the theories hit the critical surface at two different points. The crucial assumption is that these two points are within the region of attraction of the nontrivial fixed point. The Gaussian fixed point at the origin is the only point with critical behavior different from that of the rest of the surface. The line theory is unlikely to hit the critical surface at that point unless there is some physics reason. We are not aware of any.

The conclusions are that the line theory and $(\phi^4)_3$ theory have the same critical behavior, and that G_4 and g are essentially the same quantity.

V. SUMMARY

The ideas from the last three sections are combined and applied to the work of I. A complete understanding of the results in I is obtained.

Section II has explained the relationship between G_4 and the physical interaction of heavy quarks at finite temperature. G_4 gives a measure of the size of the contributions to the four-quark free energy that arise from the interactions of the strings that connect the quarks.

Section IV has shown that G_4 of the line theory and g of $(\phi^4)_3$ theory are essentially the same quantity close to criticality. Thus, each property of g that was noted in Sec. III can be translated into a physical statement about G_4 and the four-quark free energy.

The statement that g and therefore G_4 are less than infinity is sensible in the context of the gauge theory. It indicates that the picture of static quarks connected by interacting strings remain valid as the temperature increases toward the critical temperature.

With hyperscaling, g and G_4 have finite, nonzero limits as the critical surface is approached. This means that the strings of color-electric flux interact significantly near the critical temperature. One certainly expects this to be the case in a non-Abelian gauge theory.

It was noted that g is a renormalization-group invariant with a constant, universal value on the critical surface. Thus G_4 should have the same numerical value as g. High-temperature series expansions[6] and simulations[10] of $(\phi^4)_3$ theory give $g \simeq 24$. (The validity of hyperscaling is assumed.) The result for I is $G_4 \simeq 24$. The determination of G_4 in I was quite rough: 30 ± 18. $G_4 \simeq 24$ is easily accommodated.

The sign of G_4 indicates that the interactions of the strings make a repulsive contribution to the four-quark free energy. At scales that are comparable to or larger than the scale $T/\sigma(T)$, this four-body repulsion is a correction to the two-body attractions.

---

[1] J. Kiskis, Phys. Rev. D **37**, 3679 (1988) (called I in the text).

[2] Theoretical arguments are given in Ref. 1 of I. Numerical work can be found in J. Kiskis, Phys. Rev. D **37**, 1597 (1988); M. Okawa, Phys. Rev. Lett. **60**, 1805 (1988).

[3] Research that establishes the picture described in this paragraph can be found in Refs. 1–3 of I.

[4] F. Takagi, Phys. Rev. D **34**, 1646 (1986).

[5] K. G. Wilson and J. Kogut, Phys. Rep. **12C**, 77 (1974).

[6] G. A. Baker, Jr. and J. M. Kincaid, J. Stat. Phys. **24**, 469 (1981); G. A. Baker, Jr., in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1984), Vol. 9, p. 233.

[7] The notation in this paper differs somewhat from that in I. For example, $\tilde{\Gamma}^{(4)}(0)$ was called Γ_4^R in I.

[8] R. Schrader, Phys. Rev. B **14**, 172 (1976); G. A. Baker, Jr. and S. Krinsky, J. Math. Phys. **18**, 590 (1977); J. Glimm and A. Jaffé, *Quantum Physics, A Functional Integral Point of View* (Springer, New York, 1981).

[9] See Ref. 1 in I.

[10] B. Freedman, P. Smolensky, and D. H. Weingarten, Phys. Lett. **113B**, 481 (1982).