Path-integral bosonization for a nonrenormalizable axial four-dimensional fermion model

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We study the bosonization and exact solubility of a nonrenormalizable four-dimensional axial fermion model in the framework of anomalous chiral path integrals.

INTRODUCTION

The study of two-dimensional fermion models in the framework of chiral anomalous path integrals[1] has been shown to be a powerful nonperturbative technique to analyze the two-dimensional bosonization phenomenon.[2,3]

It is the purpose of this paper to implement this nonperturbative technique to solve exactly a nontrivial and nonrenormalizable four-dimensional axial fermion model which generalizes for four dimensions the two-dimensional model studied in Ref. 2.

THE MODEL

Let us start our analysis by considering the (Euclidean) Lagrangian of the proposed Abelian axial (mathematical) model

$$\mathcal{L}_1(\psi,\bar{\psi},\phi)=\bar{\psi}\gamma_\mu(i\partial_\mu-g\gamma_5\partial_\mu\phi)\psi+\tfrac{1}{2}g^2(\partial_\mu\phi)^2+V(\phi)\ , \tag{1}$$

where $\psi(x)$ denotes a massless four-dimensional fermion field, $\phi(x)$ a pseudoscalar field interacting with the fermion field through a pseudoscalar derivative interaction, and $V(\phi)$ is a ϕ self-interaction potential given by

$$V(\phi)=\frac{-g^4}{12\pi^4}\phi(\partial_\mu\phi)^2(-\partial^2\phi)+\frac{g^2}{4\pi^2}(-\partial^2\phi)(-\partial^2\phi)\ . \tag{2}$$

The presence of the above ϕ potential is necessary to afford the exact solubility of the model as we will show later [Eq. (18)].

The Hermitian γ matrices we are using satisfy the (Euclidean) relations

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu},\quad \gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3\ . \tag{3}$$

The Lagrangian $\mathcal{L}_1(\psi,\bar{\psi},\phi,)$ is invariant under the global Abelian and chiral Abelian groups

$$\psi\to e^{i\alpha}\psi,\quad \psi\to e^{i\gamma_5\beta}\psi,\quad (\alpha,\beta)\in\mathbb{R}\ , \tag{4}$$

with the Noether conserved currents at the classical level:

$$\partial_\mu(\bar{\psi}\gamma^5\gamma^\mu\psi)=0,\quad \partial_\mu(\bar{\psi}\gamma_\mu\psi)=0\ . \tag{5}$$

In the framework of path integrals, the generating functional of the correlation functions of the mathematical model associated with the Lagrangian $\mathcal{L}_1(\psi,\bar{\psi},\phi)$ is given by

$$Z[J,\eta,\bar{\eta}]=\frac{1}{Z[0,0,0]}\times\int D[\phi]D[\psi]D[\bar{\psi}]\times\exp\left[-\int d^4x[\mathcal{L}_1(\psi,\bar{\psi},\phi)+J\phi+\bar{\eta}\psi+\bar{\psi}\eta](x)\right]\ . \tag{6}$$

In order to generalize for four dimensions the chiral anomalous path integral bosonization technique as in Refs. 2 and 3, we first rewrite the full Dirac operator in the following suitable form:

$$\begin{aligned}\mathcal{D}[\phi]&=i\gamma_\mu(\partial_\mu+ig\gamma_5\partial_\mu\phi)\\&=\exp(ig\gamma_5\phi)(i\gamma_\mu\partial_\mu)\exp(ig\gamma_5\phi)\ .\end{aligned} \tag{7}$$

Now we proceed as in the two-dimensional case by decoupling the fermion field from the pseudoscalar field $\phi(x)$ in the Lagrangian $\mathcal{L}_1(\psi,\bar{\psi},\phi)$ by making the chiral change of variables:

$$\begin{aligned}\psi(x)&=\exp[ig\gamma_5\phi(x)]\chi(x)\ ,\\ \bar{\psi}(x)&=\bar{\chi}(x)\exp[ig\gamma_5\phi(x)]\ .\end{aligned} \tag{8}$$

On the other hand, the fermion measure $D[\psi]D[\bar{\psi}]$, defined by the eigenvectors of the Dirac operator $\mathcal{D}[\phi]$, is not invariant under the chiral change and yields a nontrivial Jacobian, as we can see from the relationship

$$\begin{aligned}\int D[\psi]D[\bar{\psi}]\exp\left[-\int d^4x(\bar{\psi}\mathcal{D}[\phi]\psi)(x)\right]&=\mathrm{Det}(\mathcal{D}[\phi])\\&=J[\phi]\int D[\chi]D[\bar{\chi}]\exp\left[-\int d^4x(\bar{\chi}i\gamma_\mu\partial_\mu\chi)(x)\right]\ .\end{aligned} \tag{9}$$

Here $J[\phi]=\mathrm{Det}(\mathcal{D}[\phi])/\mathrm{Det}(\mathcal{D}[\phi=0])$ is the explicit expression for this Jacobian.

It is instructive to point out that the model displays the appearance of the axial anomaly as a consequence of the nontriviality of $J[\phi]$, i.e., $\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)(x)=\{(\delta/\delta\phi)J[\phi]\}(\chi)$.

So, to arrive at a complete bosonization of the model Eq. (6) we face the problem of evaluating $J[\phi]$.

Let us, thus, compute the four-dimensional fermion determinant $\det(\not{D}[\phi])$ exactly. In order to evaluate it, we introduce a one-parameter family of Dirac operators[4,5] interpolating the free Dirac operator and the interacting one $\not{D}^{(\zeta)}[\phi]$: namely,

$$\not{D}^{(\zeta)}[\phi]=\exp(ig\gamma_5\zeta\phi)(i\gamma_\mu\partial_\mu)\exp(ig\gamma_5\zeta\phi) \tag{10}$$

with $\zeta\in[0,1]$.

At this point we introduce the Hermitian continuation of the operators $\not{D}^{(\zeta)}[\phi]$ by making the analytic extension in the coupling constant $\bar{g}=ig$. This procedure has to be done in order to define the functional determinant by the proper-time method since only in this way $(\not{D}^{\zeta}[\phi])^2$ can be considered as a (positive) Hamiltonian.

The justification for this analytic extension in the model coupling constant is due to the fact that typical interaction energy densities such as $\bar{\psi}\gamma^5\psi, \bar{\psi}\gamma_\mu\Delta^\mu\psi$, which are real in Minkowski space-time, become complex after continuation in Euclidean space-time.[6] As a consequence, the above analytic coupling extension must be done in the proper-time regularization for the Dirac functional determinant.

By using the property

$$\frac{d}{d_\zeta}\not{D}^{\zeta}[\phi]=\bar{g}\gamma_5\phi\not{D}^{\zeta}[\phi]+\not{D}^{\zeta}[\phi]\gamma_5\bar{g}\phi\ , \tag{11}$$

we can write the following differential equation for the functional determinant (see Ref. 4):

$$\frac{d}{d_\zeta}\ln \mathrm{Det}\not{D}^{\zeta}[\phi] =2\lim_{\epsilon\to 0^+}\int d^4x\ \mathrm{Tr}\langle x|(\bar{g}\gamma_5\phi)\exp\{-\epsilon(\not{D}^{\zeta}[\phi])^2\}|x\rangle\ , \tag{12}$$

where Tr denotes the trace over Dirac indices.

The diagonal part of $\exp\{-\epsilon(\not{D}^{\zeta}[\phi])^2\}$ has the asymptotic expansion[5,7]

$$\langle x|\exp\{-\epsilon(\not{D}^{\zeta}[\phi])^2\}|x\rangle \underset{\epsilon\to 0^+}{\sim} \frac{1}{16\pi^2\epsilon^2}\left[\mathbb{1}+\epsilon H_1(\phi)+\frac{\epsilon^2}{2!}H_2(\phi)\right] \tag{13}$$

with the Seeley coefficients given by (see the Appendix)

$$H_1(\phi)=\zeta\bar{g}\gamma_5\partial^2\phi-\bar{g}^2\zeta^2(\partial_\mu\phi)^2 \tag{14}$$

and

$$H_2(\phi)=2\partial^2H_1(\phi)+2\zeta\bar{g}\gamma_5(\partial_\mu\phi)\partial_\mu H_1(\phi)+2[H_1(\phi)]^2\ .$$

By substituting Eqs. (13)–(15) into Eq. (12), we obtain finally the result for the above-mentioned Jacobian:

$$J[\phi]=J_0[\phi,\epsilon]J_1[\phi]\ . \tag{16}$$

Here $J_0[\phi,\epsilon]$ is the ultraviolet cutoff-dependent Jacobian term

$$J_0[\phi,\epsilon]=\exp\left[\frac{g^2}{4\pi^2\epsilon}\int d^4x[\phi(-\partial^2)\phi](x)\right] \tag{17}$$

and $J_1(\phi)$ is the associated Jacobian finite part

$$J_1[\phi]=\exp\left[-\frac{g^2}{4\pi^2}\int d^4x(-\partial^2\phi)(-\partial^2\phi)(x)\right] \times\exp\left[\frac{g^4}{12\pi^2}\int d^4x[\phi(\partial_\mu\phi)^2(-\partial^2\phi)](x)\right]\ . \tag{18}$$

From Eqs. (17) and (1), we can see that the (bare) coupling constant g^2 gets an additive (ultraviolet) renormalization. Besides the Jacobian term cancels with the chosen potential $V(\phi)$ in Eq. (2).

As a consequence of all these results, we have the following expression for $Z[J,\eta,\bar{\eta}]$ with the fermions decoupled:

$$Z[J,\eta,\bar{\eta}]=\frac{1}{Z[0,0,0]}\int D[\phi]D[\chi]D[\bar{\chi}]\exp\left[-\int d^4x[\tfrac{1}{2}\bar{\chi}(i\gamma_\mu\partial_\mu)\chi+\tfrac{1}{2}g_R^2(\partial_\mu\phi)^2 +\bar{\eta}\exp(ig_R\gamma_5\phi)\chi+\bar{\chi}\exp(ig_R\gamma_5\phi)\eta+J\phi](x)\right]\ . \tag{19}$$

This expression is the main result of our paper and should be compared with the two-dimensional analogous generating functional analyzed in Ref. 2. Now we can see that the quantum model given by Eq. (6) although being nonrenormalizable by usual power counting and Feynman-diagrammatic analysis it still has nontrivial and exactly soluble Green's functions. For instance, the two-point fermion correlation function is easily evaluated and produced the result

$$\langle\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle=S^F_{\alpha\beta}(x_1-x_2;m=0) \times\exp[-\Delta_F(x_1-x_2;m=0)]\ . \tag{20}$$

Here $\Delta(x_1-x_2;m=0)$ is the Euclidean Green's function of the massless free scalar propagator. We notice that correlation functions involving fermions $\psi(x),\bar{\psi}(x)$ and the pseudoscalar field $\phi(x)$ are easily computed too (see Ref. 2).

As a conclusion of our paper let us comment to what extent our proposed mathematical Euclidean axial model describes an operator quantum field theory in Minkowski space-time.

In the operator framework the fields $\psi(x)$ and $\phi(x)$ satisfy the following wave equations:

$$\begin{aligned} &i\gamma_\mu\partial_\mu\psi=ig\gamma_\mu\gamma_5(\partial_\mu\phi)\psi\ ,\\ &\Box\phi=\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)+\frac{\delta\Delta}{\delta\phi}\ . \end{aligned} \tag{21}$$

It is very difficult to solve exactly Eq. (21) in a pure operator framework because the model is axial anomalous $[\partial_\mu(\bar{\psi}\gamma_\mu\gamma_s\psi)\neq 0]$. However the path-integral study Eq. (19) shows that the operator solution of Eq. (21) [in terms of free (normal-ordered) fields] is given by

$$\psi(x) = :\exp[i\gamma_5\phi(x)]:\chi(x) \tag{22}$$

since it is possible to evaluate exactly the anomalous divergence of the above-mentioned axial-vector current and, thus, choose a suitable model potential $V(\phi)$ which leads to the above simple solution.

It is instructive to point out that the operator solution Eq. (22) coincides with the operator solution of the U(1) vectorial model analyzed by Schroer in Ref. 8 [the only difference between the model's solutions being the γ_5 factor in the phase of Eq. (22)].

Consequently we can follow Schroer's analysis to conclude that the Euclidean correlation functions Eq. (20) define Wightman functions which are distributions over certain class of analytic test functions.[8-10] But the model suffers the problem of the nonexistence of time-ordered Green's functions which means that the proposed axial model in Minkowski space-time does not satisfy the Einstein causality principle.[8]

Finally we remark that the proposed model is to a certain extent less trivial than the vectorial model since for nondynamical $\phi(x)$ field the associated S matrix is nontrivial and is given in a regularized form by the result

$$\begin{aligned} S &= T\left[\exp\left[i\int_{-\infty}^{+\infty}(\bar{\psi}\gamma_\mu\gamma_5\partial_\mu\phi\psi)(x)d^4x\right]\right] \\ &= \exp\left[-i\int_{-\infty}^{+\infty}\phi(x)\left[\frac{\delta}{\delta\phi(x)}J_\epsilon[\phi]\right](x)d^4x\right] \end{aligned} \tag{23}$$

with $J_\epsilon[\phi]$ expressed by Eq. (16).

Note added

After completion of our research we became aware of a paper by J. A. Mignaco and M. A. Rego Monteiro [Phys. Lett. B **175**, 77 (1986)] where it is claimed that quantum chromodynamics with axial-vector coupling is exactly bosonized by evaluating exactly the fermion functional determinant. Unfortunately the analysis of these authors is wrong for the following reasons.

First, their arbitrary external gauge field satisfies the zero-field-strength constraint $F_{\mu\nu}(B)(x)\equiv 0$ as a direct consequence of its chosen form for the gauge $B_\mu(x)$ as a Wu-Yang phase factor [see, for instance, Itzhak Bars, Phys. Rev. Lett. **40**, 688 (1977), for a correct treatment of the theory of gauge prepotentials in four dimensions and Botelho, Centro Brasileiro de Pesquisas Físicas Report No. CBFF-NF-084 (unpublished), for the case of two-dimensional QCD].

Second, even in the case of pure gauge fields their analysis remains incomplete in the sense that only the well-known *chiral change of the phase of the Dirac functional determinant* was evaluated, and, thus, missed important (cutoff-dependent) terms associated with the chiral change of its modulus [see our Eq. (13) and D. Ebert and H. Reinhardt, Phys. Lett. B **173**, 453 (1986)].

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APPENDIX

We now briefly calculate the asymptotic term of the second-order positive differential elliptic operator:

$$(\not{D}^\zeta[\phi])^2 = (-\partial^2) - (\bar{g}\gamma_5\zeta\partial_\mu\phi)\partial_\mu - \bar{g}\zeta\gamma_5\partial^2\phi + (\partial_\mu\phi)^2 \,. \tag{A1}$$

For this study, let us consider the more general second-order elliptic four-dimensional differential operator (non-necessarily) Hermitian in relation to the usual normal in $L^2(\mathbb{R}^D)$ (Ref. 7):

$$\mathcal{L}_x = -(\partial^2)_x + a_\mu(x)(\partial_\mu)_x + V(x) \,. \tag{A2}$$

Its evolution kernel $K(x,y,\zeta) = \langle x|e^{-\zeta\mathcal{L}}|y\rangle$ satisfies the heat-kernel equation

$$\begin{aligned} &\frac{\partial}{\partial\zeta}K(x,y,;\zeta) = -\mathcal{L}_x K(x,y;\zeta) \,, \\ &\lim_{\zeta\to 0^+} K(x,y;\zeta) = \delta^{(D)}(x-y) \,. \end{aligned} \tag{A3}$$

The Green's function $K(x,y,\zeta)$ has the asymptotic expansion

$$\lim_{\zeta\to 0^+} K(x,y;\zeta) \sim K_0(x,y;\zeta)\left[\sum_{m=0}^{\infty}\zeta^m H_m(x,y)\right] , \tag{A4}$$

where $K_0(x,y,\zeta)$ is the evolution kernel for the n-dimensional Laplacian $(-\partial^2)$. By substituting Eq. (A4) into Eq. (A3) and taking the coinciding limit $x\to y$, we obtain the following recurrence relation for the coefficients $H_m(x,x)$:

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n H_{n+1}(x,x) = -\Bigg[&\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}(-\partial_x^2)H_n(x,x) \\ &+ a_n(x)\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\partial_\mu^x H_n(x,x) \\ &+ V(x)\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}H_n(x,x)\Bigg] \,. \end{aligned} \tag{A5}$$

For $D=4$, Eq. (A5) yields the Seeley coefficients

$$\begin{aligned} &H_0(x,x) = \mathbb{1}_{4\times 4} \,, \\ &H_2(x,x) = -V(x) \,, \\ &H_2(x,x) = 2[-\partial_x^2 V(x) + a_\mu(x)\partial_\mu V(x) + V^2(x)] \,. \end{aligned} \tag{A6}$$

Now substituting the value $a_\mu(x) = [-\zeta\bar{g}\partial_\mu\phi(x)]\mathbb{1}_{4\times 4}$ and $V(x) = [(-\bar{g}\zeta\gamma_5\partial_x^2\phi + \partial_\mu\phi)^2]\mathbb{1}_{4\times 4}$ into Eq. (A6) we obtain the result Eqs. (14) and (15) quoted in the main text.

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