# Symmetry in string theory

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A general method for determining the target-space symmetries, both broken and unbroken, of string theory is presented. Symmetries are shown to be consequences of a class of automorphisms of the world-sheet operator algebra. This formalism is used to prove general coordinate invariance, two-form gauge invariance, and non-Abelian gauge invariance in heterotic string theory.

## I. INTRODUCTION

First-quantized string theory[1] is formulated in a way that is not very transparent. Our only handle on dynamics is a set of rules for calculating on-shell scattering amplitudes in perturbation theory. In principle, this is all we need to discover the outcome of any experiment (indeed all we need is tree amplitudes, the rest of the theory being fixed by unitarity), but these rules do not make the physics particularly obvious, nor are they particularly easy to handle beyond the computation of a small number of loops in a few particularly tractable vacua.

One of the most powerful tools in the elucidation of important physical properties of a theory is its symmetry. Indeed most twentieth-century physical theories have been constructed by requiring that they exhibit certain types of symmetry, chosen for reasons that were variously aesthetic, phenomenological, or mathematical. Symmetries are usually quite striking in their physical consequences, yielding conservation laws or dramatic regularities in experimental data. Most importantly for the theoretical physicist, symmetries are usually much easier to analyze than detailed dynamics. Modulo anomalies (which are frequently exactly computable at one loop) symmetries are manifested in the classical theory (i.e., at the tree level) but yield results that are exact, even nonperturbatively. Historically, the study of symmetries (and especially the way in which they were realized) gave important information about both the strong interactions, which is the other complicated, nonperturbative problem that has concerned particle physicists, as well as the electroweak force.

String theory, on the other hand, has developed along different lines, emphasizing the perturbative calculation of scattering amplitudes over the study of invariance principles. We have previously outlined a formalism for discovering target-space symmetries in string theory.[2] In this and subsequent papers we will present a general theory of symmetry in string theory. It is our hope that a study of symmetries (and the way in which they are realized) will prove to be as illuminating in string theory as it has been in the standard model. For example, this approach may provide a useful guide for such problems as a phenomenological (or perhaps even first principles) choice of the string vacuum.

In field theory, finding symmetries is in principle straightforward: the theory is defined by a classical action, and we must seek transformations of the space-time fields that leave it invariant. In first-quantized string theory we cannot proceed in this way. We are furnished with an action for a conformally invariant *world-sheet* field theory in which the *space-time* fields (i.e., the particles with which experiments are performed) do not appear. What we do have is a set of rules for computing amplitudes as vacuum expectation values of certain primary fields (also called vertex operators) in this two-dimensional theory. Historically, the claim that string theory possesses general coordinate and gauge invariances is based on the existence of massless vector and tensor states of the string, and on the comparison of a few tree amplitudes (at low energy) with the corresponding amplitudes of general relativity and gauge field theories.[1] Even for these generally accepted symmetries this is a rather unsatisfactory state of affairs. Are *all* tree amplitudes gauge invariant at low energies? Does this invariance hold at higher energies? After all, one of the great virtues of string theory is that it is so *unlike* field theory at high energies.

The problem of symmetry in string theory is much deeper than this, though, because it is probable that string theory possesses a much richer symmetry structure than we yet know, of which gauge and general coordinate invariance are but remnants. First, the particle content and interactions of a string theory seem to be so tightly constrained that they are essentially unique. Normally this occurs because a symmetry is relating one coupling and particle to another, as gauge invariance relates the cubic and quartic couplings of gauge bosons. Second, Gross and Mende[3] have discovered that at high energies (where we might expect a spontaneously broken symmetry to be restored) scattering processes share certain universal features, suggesting that all states of the string are related by some enormous spontaneously broken symmetry. Elucidation of this symmetry will probably go a long way towards answering the oft-posed question "What is the underlying principle of string theory?"

In this paper we shall describe a general method for proving the existence of symmetries in first-quantized string theory, and illustrate the method by proving the existence of the various massless gauge invariances of the heterotic string:[4] general coordinate invariance, two-form gauge invariance, and non-Abelian gauge invariance (including the full $E_8 \times E_8$ case with spin operators). In the latter case, we shall show how world-sheet anomalies lead to the two-form transforming under the gauge transformation, a manifestation of the Green-Schwarz anomaly cancellation mechanism,[5] first shown by Hull and Witten for the SO(32) case.[6] A closely related example associated with an auxiliary gauge field of the four-dimensional supergravity multiplet has already been discussed elsewhere.[7] We postpone to a subsequent paper a general theory of the method, and a proof of the existence of the higher symmetries discussed in the preceding paragraph.[8]

## II. THE METHOD

The tool that we shall use to prove the existence of symmetries is the generating functional[9] for on-shell amplitudes $Z[\Phi^i]$. $Z$ is a functional of the space-time fields $\Phi^i$ defined by the requirement that

$$A_N(\{k_r\},\zeta_\mu)=\int \prod_r d^D Y_r e^{ik_r\cdot Y_r}\zeta^r_\mu \delta^N Z[\Phi^i]/\delta\Phi^1_\mu(Y_1)\cdots\delta\Phi^N_\mu(Y_n)|_{\Phi=0}\ , \tag{1}$$

where $A_N$ is the amplitude for the scattering of $N$ particles of momenta $k_r$ and polarizations $\zeta_\mu$. However, the amplitude is given by[1]

$$A_N(\{k_r\},\zeta_\mu)=\left\langle 0\left|\int \prod_r d^2 z_r V^\mu_r(z_r)\zeta^r_\mu e^{ik\cdot X(z_r)}\right|0\right\rangle\ . \tag{2}$$

The right-hand side of Eq. (2) is the vacuum expectation value of a product of primary fields of the two-dimensional (super)conformal field theory,[10] with action $S_0$, that defines the string-theory vacuum.[11] [We have written Eq. (2) assuming that this conformal field theory contains some free bosonic fields $X$ corresponding to some number of flat-space-time dimensions, although this assumption is not important.] Writing Eq. (2) in functional-integral form and comparing with Eq. (1) it is not hard to see that the generating functional is given by

$$Z[\Phi^i_\mu]=\int DX\exp(-S_\Phi)\ ,$$
$$S_\Phi=S_0-\int d^2z\, \Phi^i_\mu(X)V^\mu_i(z)\ . \tag{3}$$

Thus the generating functional is the partition function for a family of two-dimensional field theories, parametrized by the space-time fields. Since we use $Z$ only to generate on-shell amplitudes we need only differentiate $Z$ in on-shell directions. We therefore only need consider the partition functions for conformal two-dimensional theories,[11] which is just as well, since only in this case does Eq. (3) make literal sense. Nevertheless, it is very tempting to extend Eq. (3) to include renormalizable rather than simply conformal field theories (i.e., to include an ultraviolet regulator and counterterms), and to hope that in so doing we are getting some sort of off-shell string theory.[12]

As an example, consider the generating functional for gravitons in the bosonic string. In this case

$$S_0=\int d^2z\ \eta_{\mu\nu}\partial X^\mu\bar\partial X^\nu$$

and the truncated graviton vertex is given by

$$V^{\mu\nu}_g(z)=\partial X^\mu\bar\partial X^\nu(z)\ . \tag{4}$$

(The complete vertex operator would include an additional factor of $\xi_{\mu\nu}e^{ik\cdot X}$). Thus the generating functional becomes

$$Z[G_{\mu\nu}]=\int DX\exp\left[-\int d^2z\ G_{\mu\nu}(X)\partial X^\mu\bar\partial X^\nu\right]\ , \tag{5}$$

where

$$G_{\mu\nu}(X)=\eta_{\mu\nu}+h_{\mu\nu}(X)\ .$$

$h_{\mu\nu}(X)$ is the graviton field and we interpret $G_{\mu\nu}(X)$ as the space-time metric, so that the family of conformal field theories is the set of torsion free nonlinear $\sigma$ models.

A symmetry of the theory is a transformation on the space-time fields that leaves $Z$ invariant for all values of the space-time fields:

$$Z[\Phi^i]=Z[\Phi^i+\delta\Phi^i]\ . \tag{6}$$

Repeatedly differentiating Eq. (6) with respect to $\Phi^i$ and making use of Eq. (1) yields Ward identities for amplitudes.[13] This is completely analogous to working with an action, the only difference is that the action is the generating functional for one-particle-irreducible Green's functions. However working with a generating functional for amplitudes does have one advantage worth mentioning (in addition to tractability). If, as is frequently the case, we are interested in asking questions about amplitudes involving only a subset of the possible states of the string, we simply put other fields equal to their vacuum values in the generating functional. Precisely because we are using an explicitly on-shell formalism we do not have to worry about "integrating out" the other fields as we would if we were working with an action. In string theory, with its large number of possible states, this is a considerable advantage.

How do we demonstrate such an invariance? The first method we discuss involves finding an appropriate change of variables in the functional integral.[13] Consider, for example, general coordinate invariance in the bosonic string. By making the change of variables

$$X^\mu\to X^\mu+\xi^\mu(X) \tag{7}$$

the generating functional becomes

$$Z[G_{\mu\nu}]=\int DX\,J\exp\left[-\int d^2z\,G_{\mu\nu}(X+\xi)\partial(X^\mu+\xi^\mu)\bar\partial(X^\nu+\xi^\nu)\right]$$
$$=\int DX\,J\exp\left[-\int d^2z(G_{\mu\nu}+\xi^\lambda\partial_\lambda G_{\mu\nu}+\partial_\mu\xi^\lambda G_{\lambda\nu}+\partial_\nu\xi^\lambda G_{\mu\lambda})\partial X^\mu\bar\partial X^\nu\right]$$
$$=Z[G_{\mu\nu}+\xi^\lambda\partial_\lambda G_{\mu\nu}+\partial_\mu\xi^\lambda G_{\lambda\nu}+\partial_\nu\xi^\lambda G_{\mu\lambda}]\ , \tag{8}$$

where the third equality holds if the Jacobian of the transformation, $J$, is unity. Under these circumstances, Eq. (8) is a demonstration of general coordinate invariance (at least for graviton amplitudes).

There is frequently some obstruction to choosing the measure in such a way that the Jacobian is unity (though not in the case at hand)[14] and we shall return to discuss these anomalies later. However there is a more immediate problem with Eq. (8) insofar as the Jacobian is not unity even naively. In constructing the generating functional $Z$ we consider amplitudes in a flat background. Since we did nothing to the functional measure, it remains independent of $G_{\mu\nu}$, and the Jacobian invalidates the desired result. We could, however, restore the invariance by changing the measure by introducing a factor of the square root of the determinant of the space-time metric pulled back to each point on the world sheet. That is, let

$$DX\to DX[-G(X)]^{1/2}\ ,\quad G=\det[G_{\mu\nu}(X)]\ .$$

This would yield a new functional truly invariant under coordinate transformations. To complete the proof of coordinate invariance of string theory we must now argue that this new, invariant functional remains a generating functional for on-shell amplitudes. Differentiating with respect to $G_{\mu\nu}$ [as in Eq. (1)] yields the desired amplitudes plus unwanted terms from differentiating the measure. These unwanted terms do not contribute to amplitudes, however, by the tracelessness of the graviton polarization tensor. This form for the generating functional is by far the most convenient and, henceforth, will be used exclusively. Thus general coordinate invariance is proved.

The method described above works well for a number of symmetries, but has certain drawbacks.

(i) Handling Jacobians properly takes considerable care. This includes the anomaly problem alluded to above.

(ii) There does not appear to be a general rule for finding the appropriate changes of variables in the functional integral. The nontrivial requirement is that the generating functional be rewritten in the same form before and after the change of variables, with only the space-time fields changing.

(iii) As a particular case of the above, spin operators play a significant role in heterotic and type-II strings. These operators are normal-ordered exponentials of chiral scalar fields, which we need to transform to discover gauge invariance or supersymmetry. Unfortunately the desired transformations appear to be very complicated and nonlocal.

The origin of these difficulties is that we have not made use of all the information available to us, in particular the algebraic properties of the primary fields that are the vertex operators. To remedy this we rewrite the generating functional in the following way:

$$Z[\Phi]=\int DX\exp(-S_\Phi)$$
$$=\lim_{\beta\to\infty}\mathrm{tr}[\exp(-\beta H_\Phi)]\ , \tag{9}$$

where $H_\Phi$ is the two-dimensional Hamiltonian corresponding to the two-dimensional action $S_\Phi$, both of which are parametrized, of course, by the space-time fields $\Phi$.

With the generating functional in this new form we need to understand how to prove the existence of symmetries: i.e., what is the analogue of a change of variables in the functional integral? The answer is an automorphism of the operator algebra. In particular we shall consider automorphisms of the type

$$\delta O=i[h,O]\ , \tag{10}$$

where $O$ is an arbitrary operator of the algebra, and $h$ is any fixed operator. It is not hard to see that the Jacobi identity guarantees that Eq. (10) is indeed an automorphism. Automorphisms of this type are termed "inner," and it is inner automorphisms that will concern us here, but it should be emphasized that outer (i.e., noninner) automorphisms also yield symmetries, and it seems likely that discrete symmetries and the relationship of string theory to the sporadic finite simple groups can be understood in this way.

In more concrete terms an inner automorphism reduces to the cyclic property of the trace:

$$Z[\Phi]=\mathrm{tr}[\exp(-\beta H_\Phi)]=\mathrm{tr}[\exp(-\beta UH_\Phi U^{-1})] \tag{11}$$

for any operator $U$. In particular consider an operator close to the identity ($h$ is infinitesimal)

$$U=1+ih$$

so that Eq. (11) becomes

$$Z[\Phi]=\mathrm{tr}\{\exp[-\beta(H_\Phi+i[h,H_\Phi])]\}\ .$$

The argument of the exponential is a new Hamiltonian. It might be possible to choose the operator $h$ in such a way that this new Hamiltonian is also in the family that enters the generating functional; i.e., it might be possible to make this deformation of the Hamiltonian correspond to a deformation of the space-time fields. In this case

$$Z[\Phi]=\mathrm{tr}\{\exp[-\beta(H_\Phi+i[h,H_\Phi])]\}$$
$$=\mathrm{tr}[\exp(-\beta H_{\Phi+\delta\Phi})]=Z[\Phi+\delta\Phi]$$

and we have proved a symmetry. The problem of finding a symmetry is thus reduced to finding an operator such that the second equality above holds. Although it is not yet obvious, this is progress.

## III. BOSONIC GAUGE INVARIANCES

In this section we shall use the Hamiltonian method described in Sec. II to prove the standard gauge invariances for amplitudes involving massless external bosons, specifically, general coordinate invariance, Abelian two-form gauge invariance and regular non-Abelian gauge invariance. For definiteness we shall consider the heterotic string in ten dimensions.

The superconformal field theory corresponding to this vacuum is, of course, just the free (1,0) supersymmetric field theory with ten bosonic superfields $X^\mu$ and a right-moving sector that can be written[4] in terms of 32 free Majorana-Weyl fermionic superfields $\Psi^A$. The action is[15]

$$S=-i\int d^2z\, d\theta\, \eta_{\mu\nu}DX^\mu\bar{\partial}X^\nu+i\Psi^A D\Psi^A+\text{ghosts}\ , \quad (12)$$

where $\theta$ is the single odd superspace coordinate and $D$ is the superderivative

$$D=\partial_\theta+i\theta\partial\ . \quad (13)$$

The vertex operators may similarly be written[1,4]

$$\begin{aligned} V_G(z,\bar{z})&=-i\int d\theta\, \zeta_{(\mu\nu)}DX^\mu\bar{\partial}X^\nu e^{ik\cdot X}\ ,\\ V_B(z,\bar{z})&=-i\int d\theta\, \zeta_{[\mu\nu]}DX^\mu\bar{\partial}X^\nu e^{ik\cdot X}\ ,\\ V_A(z,\bar{z})&=\int d\theta\, \epsilon_\mu^A DX^\mu J^A e^{ik\cdot X}\ . \end{aligned} \quad (14)$$

Here $G$, $B$, and $A$ refer to the metric (i.e., graviton), antisymmetric tensor and gauge fields, respectively. The polarizations are denoted by $\zeta$ and $\epsilon$, and the momenta by $k$. $J^A$ are currents whose lowest components are most naturally constructed out of $\phi^i$, the bosonization of the fermions that are the lowest components of the superfields $\Psi$. The currents take the normal-ordered form

$$J^\alpha=:\exp(i\alpha_i\phi^i):,\quad J^i=i\bar{\partial}\phi^i\ ,$$

where $\alpha$ is a root of the gauge algebra [in this case either $E_8\times E_8$ or SO(32)] and the fields $\phi^i$ parametrize the group's maximal torus (the second form listed above applies to currents belonging to the Cartan subalgebra). They satisfy the equal-time commutation relations

$$\begin{aligned} [J^L(\sigma),J^M(\sigma')]=&\sqrt{2}f^{LMN}J^N(\sigma)\delta(\sigma-\sigma')\\ &+(i\kappa/\pi)\delta^{MN}\delta'(\sigma-\sigma')\ , \end{aligned} \quad (15)$$

where $f^{LMN}$ are the relevant structure constants and the current algebra has a central extension proportional to $\kappa$, which in this case has a value of 1. Having passed to the Hamiltonian formalism, all operators are evaluated at fixed world-sheet time, and so are functions only of the spatial coordinate $\sigma$. A prime denotes differentiation with respect to $\sigma$.

With these vertices we may write the generating functional as

$$Z[G_{\mu\nu},B_{\mu\nu},A_\mu]=\int DX(-G)^{1/2}\exp(-S_\Phi)\ , \quad (16)$$

$$\begin{aligned} S_\Phi=&-i\int d^2z\, d\theta[G_{\mu\nu}(X)+B_{\mu\nu}(X)]DX^\mu\bar{\partial}X^\nu+i\Psi^A D\Psi^A\\ &+iA_\mu^M(X)J^M DX^\mu+\text{ghosts}\ . \end{aligned}$$

The ghosts will play no further role in the discussion in this section, and so will be omitted from future formulas.

From Eq. (16) we may construct the associated classical energy-momentum tensors. The results of a straightforward calculation are

$$\begin{aligned} T_F=&\lambda^\mu\partial X_\mu+(i/2)\partial_\kappa B_{\mu\nu}\lambda^\kappa\lambda^\mu\lambda^\nu\ ,\\ T=&G^{\mu\nu}\partial X_\mu\partial X_\nu-(i\sqrt{2})\lambda^a\lambda^{a'}-(i\sqrt{2})\omega_\mu{}^{ab}\lambda^a\lambda^b X^{\mu'}\\ &-\tfrac{1}{2}F_{ab}{}^M J^M\lambda^a\lambda^b\ ,\\ \bar{T}=&G^{\mu\nu}\bar{\partial}X_\mu\bar{\partial}X_\nu+(i\sqrt{2})\Psi^A\Psi^{A'}+(i\sqrt{2})A_\mu^M J^M X^{\mu'}\\ &-\tfrac{1}{2}F_{ab}^M J^M\lambda^a\lambda^b\ , \end{aligned} \quad (17)$$

where

$$\begin{aligned} \partial X_\mu=&(1/\sqrt{2})[\pi_\mu+(G_{\mu\nu}+B_{\mu\nu})X^{\nu'}+(i/\sqrt{2})\omega_\mu^{ab}\lambda^a\lambda^b\\ &+(i/\sqrt{2})A_\mu^M J^M]\ ,\\ \bar{\partial}X_\mu=&\partial X_\mu-\sqrt{2}G_{\mu\nu}X^{\nu'}\ . \end{aligned} \quad (18)$$

Equations (17) and (18) are written in component fields, so that previously introduced symbols are now to be interpreted as lowest components instead of complete superfields. $\lambda^\mu$ is the higher component of the superfield $X^\mu$. $F$ is the field strength constructed from the gauge potential $A$ and $\omega$ is the vielbein compatible spin connection with torsion

$$H_{\mu\nu\lambda}=\tfrac{3}{2}\partial_{[\mu}B_{\nu\lambda]}\ . \quad (19)$$

These equations deserve some commentary. It is necessary to work with world-sheet fields with equal-time commutation relations independent of the space-time fields, so that all dependence of the generating functional on the space-time fields is explicit. Thus the generators of the algebra from the bosonic sector are $X^\mu$ and their conjugate momenta $\pi_\mu$. To achieve the same goal for the Majorana-Weyl fermions $\lambda$, it is convenient to choose an orthonormal basis to the tangent space at each point of space-time [a vielbein $E(X)_\mu^a$] and refer the fermions to that basis:

$$\lambda^a=E_\mu^a\lambda^\mu\ . \quad (20)$$

The algebra of $\lambda^a$ is then field independent:

$$\begin{aligned} &\{\lambda^a(\sigma),\lambda^b(\sigma')\}=-\eta^{ab}\delta(\sigma-\sigma')/\sqrt{2}\ ,\\ &[\pi_\mu(\sigma),X^\nu(\sigma')]=-i\delta_\mu^\nu\delta(\sigma-\sigma')\ ,\\ &[\lambda^a(\sigma),X^\nu(\sigma')]=[\pi_\mu(\sigma),\lambda^b(\sigma')]=0\ . \end{aligned} \quad (21)$$

Unfortunately, Eqs. (17) and (18) do not describe a superconformal field theory. We are by now familiar with the idea that only space-time fields that satisfy the string equations of motion correspond to a conformally invariant world-sheet theory.[11] We shall simply assume that this condition is satisfied, and shall need explicit forms for neither the equations of motion nor their solutions.

However there is another source of conformal noninvariance that we must confront. In order to be a superconformal field theory it is necessary that the moments of the normal-ordered stress-energy tensor satisfy the Neveu-Schwarz or Ramond algebras. However, the algebra of the currents $J$ [Eq. (15)] contains a central extension proportional to $\kappa$ that does not appear in the classical Poisson brackets. The same is true for bilinears in $\lambda$:

$$[\lambda^a\lambda^b(\sigma),\lambda^c\lambda^d(\sigma')]=(1/\sqrt{2})[\eta^{ac}\lambda^b\lambda^d(\sigma)+\eta^{bd}\lambda^a\lambda^c(\sigma)-\eta^{bc}\lambda^a\lambda^d(\sigma)-\eta^{ad}\lambda^b\lambda^c(\sigma)]\delta(\sigma-\sigma') + (i\rho/4\pi)(\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc})\delta'(\sigma-\sigma') \ . \tag{22}$$

In this case the central extension is proportional to $\rho$, which again has a value of 1 (it is convenient to leave $\rho$ and $\kappa$ arbitrary to keep track of the terms which are due to the central extension, and to make our formulas easier to apply in other situations).

Since currents $J$ and $\lambda$ bilinears appear in the energy-momentum tensors of Eq. (17), these central extensions will contaminate their equal-time commutators, which will cease to be those of the superconformal algebra. To correct this sad state of affairs (recall that we argued earlier that the generating functional for on-shell amplitudes must be the partition function for a superconformal field theory) we must amend Eqs. (17)–(19) as follows:

$$\begin{aligned}
T_F&=\lambda^\mu\partial X_\mu+(i/3)H_{\kappa\mu\nu}\lambda^\kappa\lambda^\mu\lambda^\nu \ ,\\
T&=G^{\mu\nu}\partial X_\mu\partial X_\nu-(i\sqrt{2})\lambda^a\lambda^{a'}-(i\sqrt{2})[\omega_\mu^{ab}-(\kappa/4\pi)A_\mu^M F_{ab}^M]\lambda^a\lambda^b X^{\mu'}\\
&\quad-\tfrac{1}{2}F_{ab}^M J^M\lambda^a\lambda^b-(i\rho/4\sqrt{2}\pi)\omega_\mu^{ab}F_{ab}^M J^M X^{\mu'}+[(\rho/4\pi)\omega_\mu^{ab}\omega_\nu^{ab}-(\kappa\rho/8\pi^2)A_\nu^M\omega_\mu^{ab}F_{ab}^M]X^{\mu'}X^{\nu'} \ ,\\
\bar{T}&=G^{\mu\nu}\bar\partial X_\mu\bar\partial X_\nu+(i\sqrt{2})\Psi^A\Psi^{A'}+(i\sqrt{2})[A_\mu^M-(\rho/8\pi)\omega_\mu^{ab}F_{ab}^M]J^M X^{\mu'}\\
&\quad-\tfrac{1}{2}F_{ab}^M J^M\lambda^a\lambda^b+(i\kappa/2\sqrt{2}\pi)A_\mu^M F_{ab}^M\lambda^a\lambda^b X^{\mu'}+[(\kappa/2\pi)A_\mu^M A_\nu^M-(\kappa\rho/8\pi^2)A_\nu^M\omega_\mu^{ab}F_{ab}^M]X^{\mu'}X^{\nu'} \ ,
\end{aligned} \tag{23}$$

where

$$\partial X_\mu=(1/\sqrt{2})\{\pi_\mu+[G_{\mu\nu}+B_{\mu\nu}+(\kappa/2\pi)A_\mu^M A_\nu^M]X^{\nu'} +(i/\sqrt{2})\omega_\mu^{ab}\lambda^a\lambda^b+(i/\sqrt{2})A_\mu^M J^M\} \ ,$$

$$\bar\partial X_\mu=\partial X_\mu-\sqrt{2}G_{\mu\nu}X^{\nu'} \ , \tag{24}$$

$$H=\tfrac{3}{2}dB+(\kappa/4\pi)\Omega_{\rm YM}-(\rho/4\pi)\Omega_L \ , \tag{25}$$

and

$$\begin{aligned}\Omega_{\rm YM}&=-2(A^M dA^M-\tfrac{1}{3}f^{LMN}A^L A^M A^N) \ ,\\ \Omega_L&=\omega^{ab}d\omega^{ba}+\tfrac{2}{3}\omega^{ab}\omega^{bc}\omega^{ca} \ .\end{aligned} \tag{26}$$

The most noteworthy amendment to the classical equations is that of Eq. (25) where the torsion is augmented by the difference between the Chern-Simons three-forms associated with Yang-Mills and spin connections. It is precisely such an amendment to the torsion that leads to the cancellation of anomalies in the effective field theories associated with the consistent superstrings.[5] The appearance of these three forms is thus seen to be a consequence of the central extension in the current algebra and the requirement of superconformal invariance.

With these formulas we may prove the usual bosonic gauge invariances. Recall from the discussion of Sec. II [Eq. (11)] that we need to find operators $h$ that satisfy

$$H_\Phi+i[h,H_\Phi]=H_{\Phi+\delta\Phi} \tag{27}$$

for some $\delta\Phi$, so that

$$Z[\Phi]=Z[\Phi+\delta\Phi] \ .$$

Of course the Hamiltonian is

$$H_\Phi=\int d\sigma\, T(\sigma)+\bar{T}(\sigma) \ . \tag{28}$$

The operators $h$ are not hard to find. The most interesting case is that of a non-Abelian gauge invariance. In this case $h$ is

$$h=-i\int d\sigma\,\Lambda^M(X(\sigma))J^M(\sigma) \ . \tag{29}$$

We may compute the commutator

$$\begin{aligned}
i[h,T(\sigma)]&=\sqrt{2}G^{\mu\nu}\partial X_\mu[(\kappa/\sqrt{2}\pi)\partial_\lambda\Lambda^M A_\nu^M X^{\lambda'}+i\partial_\nu\Lambda^M J^M+if^{LNM}\Lambda^L A_\nu^M]+(i\kappa/2\pi)\partial_\mu\Lambda^M F_{ab}^M\lambda^a\lambda^b X^{\mu'}\\
&\quad-(1/\sqrt{2})f^{LNM}\Lambda^L F_{ab}^N J^M\lambda^a\lambda^b-(i\rho/4\pi)\omega_\mu^{ab}f^{LNM}\Lambda^L F_{ab}^N J^M X^{\mu'}-(\kappa\rho/4\sqrt{2}\pi^2)\partial_\nu\Lambda^M\omega_\mu^{ab}F_{ab}^M X^{\mu'}X^{\nu'}\\
&=T_{\Phi+\delta\Phi}(\sigma)-T_\Phi(\sigma) \ ,
\end{aligned} \tag{30}$$

where the changes in the space-time fields (written in form notation) are

$$\delta A^M=\sqrt{2}(d\Lambda^M+f^{LNM}\Lambda^L A^N)\ ,$$
$$\delta B=-(\kappa/\sqrt{2}\pi)d\Lambda^M A^M\ . \tag{31}$$

A similar result may be obtained for $\bar{T}$, i.e.,

$$i[h,\bar{T}(\sigma)]=\bar{T}_{\Phi+\delta\Phi}(\sigma)-\bar{T}_{\Phi}(\sigma)$$

with the same changes in the space-time fields, given by Eq. (31).

Thus gauge invariance is modified, in that the two-form field $B$ must transform also, as a consequence of the central extension of the current algebra. This result was previously obtained by Hull and Witten[6] using a functional integral approach for the SO(32) string. In that case (but not for $E_8\times E_8$) the gauge currents can all be written as fermion bilinears, and the variation of $B$ is due to a nontrivial Jacobian in the functional integral measure for these fermions (a "$\sigma$ model anomaly").[14]

Other gauge invariances may be demonstrated in the same way. Calculation of the commutators is straightforward but rather tedious, so we shall simply list the operators $h$ which generate the symmetries and describe the actual field transformations. The appropriate operators $h$ are

general coordinate invariance

$$\int d\sigma\, \xi^{\mu}(X(\sigma))\pi_{\mu}(\sigma)\ ,$$

two-form gauge invariance

$$\int d\sigma\, V_{\mu}(X(\sigma))X^{\mu\prime}(\sigma)\ , \tag{32}$$

local Lorentz invariance

$$-i\int d\sigma\, \Sigma^{ab}(X(\sigma))\lambda^a\lambda^b(\sigma)\ .$$

The general coordinate and two-form gauge transformations are canonical, but, because of the central extension of the current algebra in Eq. (22), local Lorentz transformations exhibit the same Green-Schwarz mechanism as occurred in the gauge case:

$$\delta\omega=\sqrt{2}(d\Sigma+[\omega,\Sigma])\ ,$$
$$\delta E^a=-\sqrt{2}\Sigma^{ab}E^b\ , \tag{33}$$
$$\delta B=(\rho/2\sqrt{2}\pi)d\Sigma^{ab}\omega^{ab}\ .$$

## IV. CONCLUSION

We have described a method for providing the existence of symmetries in string theory and used it to demonstrate the standard bosonic gauge invariances (including Green-Schwarz terms), at least for amplitudes involving only massless bosons.

The results are exact, at the string tree level, and hold to arbitrarily high energy. Since the results depend only on the operator algebra which is local on the string world sheet, it seems unlikely that these results will change for surfaces of higher genus. Stated more succinctly, closed-string theories appear to have no anomalies, a result suggested on other grounds by Schellekens and Warner.[16] A fuller investigation of this point would involve a proper understanding of conformal field theories on an arbitrary Riemann surface[17] and the behavior of its partition function at the boundary of moduli space.

It is natural to ask why the operators of Eq. (32) do indeed generate symmetries; that is, we would like to understand what properties of $h$ lead to its satisfying the nontrivial condition

$$i[h,H_{\Phi}]=H_{\Phi+\delta\Phi}-H_{\Phi}\ .$$

A general discussion of this problem will be presented elsewhere,[8] but for the particular cases discussed in this paper there is a straightforward answer. All possible terms of (naive) dimension two that can be written with the available world-sheet fields appear in the Hamiltonian of Eqs. (23), (26), and (28). Thus commuting it with the integral of an operator $h$ that is of dimension one will necessarily yield terms of the form that appeared in the original Hamiltonian. However, it is not yet obvious that this guarantees a symmetry, because there are relations between the coefficients of these terms. For example, in the classical expression for $T$ [Eq. (17)] the coefficient of $J^M\lambda^a\lambda^b$ is the field strength derived from the gauge potential that is the coefficient of $J^M X^{\mu\prime}$. We must explain why these relationships are preserved by commuting with $h$. The reason is that these relationships are a consequence of superconformal invariance. They will therefore be preserved if

$$T\rightarrow T+i[h,T]$$

preserves the superconformal invariance. This is certainly the case, simply because we are dealing with an (inner) automorphism of the algebra; the superconformal algebra is preserved by virtue of the Jacobi identity.

An instructive example to consider is non-Abelian gauge invariance. If we were to commute the operator $h$ of Eq. (29) with the *classical* form of the energy-momentum tensor given in Eqs. (17) and (18) (i.e., the energy-momentum tensor without the terms proportional to $\kappa$ or $\rho$) we would, by the dimensional arguments given above or by direct computation, discover dimension two terms of the same form as already appear in $T$. However, the relationships between the coefficients would not be preserved, precisely because of the central extensions in the current algebra. This is the easiest way to show that the energy-momentum tensor of Eq. (17) is *not* superconformally invariant. It is also the easiest way to find the corrections that restore superconformal invariance, Eqs. (23)–(26).

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