Quantum limited detectors for weak classical signals

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When a classical signal is very weak, the quantum features of its detector cannot be ignored. There appears then to be a conflict between the continuous nature of the classical signal and the discrete spectrum of a quantum device. Moreover, the final output cannot be read directly from a quantum system: The latter has to be "measured" by another device (the "meter") which then yields another classical signal—a real number. This paper examines the amount of distortion caused by the presence of a quantum interface between two classical signals. It is shown that the meter should have a moderate resolution, so as to lump together numerous levels of the detector. A finer resolution deteriorates the correspondence between the input and output signals. A perfect resolution, down to isolated eigenvalues, may completely lock the output signal (this is the quantum Zeno effect).

I. INTRODUCTION AND SUMMARY

The amplification of random signals is a common feat of engineering, as any owner of a radio receiver may know. In the language of theoretical physics, this is done by coupling the signal to a detector (a dynamical system of known structure) so that the Hamiltonian H of the detector contains a term depending on the unknown signal. Then, by observing the time dependence of a dynamical variable of the detector, we can reconstruct the entire H, including the unknown signal. However, when the signal is extremely weak, for example, if our aim is to detect gravitational radiation,[1] radically new problems arise: If we attempt to describe the detector dynamics by classical methods, namely, as the motion of a point in phase space, this motion may encompass a domain smaller than $\hbar$. This means, effectively, that classical mechanics breaks down and the detector must be treated as a quantum system. (It is assumed here that the noise temperature is low enough to allow neglecting all noise and other dissipative effects.)

We are then faced with a familiar problem: a quantum system driven by a time-dependent *classical* force[2] (even very weak signals can be treated classically, because they contain enormous numbers of photons or gravitons[3]). The next problem is to decode the information stored in the quantum system and to convert it into a new classical signal—the reading of our meter. This is the classic "quantum measurement" problem. In general, quantum measurements lead to an entropy increase[4] and therefore to a degradation of information. Thus, in summary, the problem is to examine the amount of distortion caused by the presence of a quantum interface between two classical signals. In this paper, it is shown that the resolving power of the meter must be matched to the spectral properties of the quantum system. While a poor resolution obviously gives inaccurate results, a resolution that is too sharp is also undesirable, because in that case the meter overwhelms the detector and yields results having little relationship to the original signal.

Section II of this paper discusses the *final* link of the amplification chain: the conversion of information encoded in a quantum state into the reading of a classical meter. The treatment is strictly quantum mechanical. It invokes no controversial notion such as the "collapse" of a wave function. The meaning of the wave function is that of a mere mathematical tool, allowing us to compute the probabilities for the occurrence of specified macroscopic events, following a given preparation.[5–7] It is shown that the rule saying "the observable values of a dynamical variable are the eigenvalues of the corresponding operator" is valid only in the limit of idealized meters.

Consecutive measurements are discussed in Sec. III. There must be as many independent meters as there are data to be taken (naturally, all these "meters" can be incorporated into a single instrument, but then they must belong to different degrees of freedom[8]). In the limiting case of very precise measurements performed repeatedly at very short time intervals, the meters lock at one of the eigenvalues. This is the so-called "quantum Zeno paradox" (which has nothing paradoxical: the meters literally overwhelm the quantum system).

The general results obtained in Sec. III are illustrated in Sec. IV for a model where the unknown signal is a variable torque causing the precession of a rotor (a particle of known spin). First, we consider the simple case of a spin-$\frac{1}{2}$ system, for which every detail of the calculation can be followed explicitly. As expected, a spin-$\frac{1}{2}$ particle cannot be a good detector: its Hilbert space is too small. We then consider the case of a particle having a large spin. It is shown that arbitrarily weak torques can be observed without appreciably disturbing the precession of the detector, provided that the meters have a resolution suitably matched to the spin spectrum.

Throughout this paper $\hbar=1$ and, moreover, the unit of length is chosen in such a way that the meter's scale directly gives the eigenvalues of the measured operator.

II. QUANTUM MEASUREMENTS

A "measurement" is a process which generates a *correlation* between a property of the measured object and a property of the meter. As a simple example, consider a particle of spin $\frac{1}{2}$ whose initial state is described by a spinor $\binom{\alpha}{\beta}$. Suppose for simplicity that there are no other degrees of freedom and no forces acting on that particle. The problem is to measure σ_z, a dynamical variable purported to have observable values ± 1.

The "meter" which performs this measurement is idealized as being another particle, with position q, momentum p, and mass M. Its free Hamiltonian is $p^2/2M$. Its interaction with the spin-$\frac{1}{2}$ particle is described by

$$H_{\rm int} = g(t)\sigma_z p \; , \tag{1}$$

where $g(t)$ is an externally controlled function of time[9] with narrow support near $t=0$, and such that $\int g(t)dt = 1$, in appropriate units.

Both σ_z and p are constants of the motion. The Heisenberg equation of motion for q is

$$\dot{q} = i[H,q] = g(t)\sigma_z + (p/M) \; . \tag{2}$$

The last term of (2) can be neglected during the brief interaction. The solution of (2) thus is

$$q_f = q_i + \sigma_z \; . \tag{3}$$

This result is better visualized in the Schrödinger picture. Let $\phi(q)$ denote the initial wave function of the meter. Assume that this is a function with a sharp maximum at $q=0$ (this means that, before the measurement, it is most probable to find the meter close to $q=0$). Moreover assume that M, the mass of the meter, is so large that $\phi(q)$ will not appreciably spread during the experiment. In other words, it is legitimate to take $H \equiv H_{\rm int}$. The time evolution generated by this Hamiltonian is

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\phi(q) \to \begin{bmatrix}\alpha\\ 0\end{bmatrix}\phi(q-1) + \begin{bmatrix}0\\ \beta\end{bmatrix}\phi(q+1) \; . \tag{4}$$

This process is illustrated in Fig. 1. The meaning of the right-hand side of (4) is the following: There is a probability amplitude $\alpha\phi(q-1)$ to find the meter near $q=1$, and the particle with spin up; and a probability amplitude $\beta\phi(q+1)$ to find the meter near $q=-1$, and the particle with spin down. I emphasize that quantum mechanics allows one to compute only probabilities of events. It does not describe the events themselves.[10]

What we call "the observed value of σ_z" is given by the final position of the meter. The wave function ϕ gives the probabilities of observing the meter at its possible final positions. Ideally, $\phi(q \mp 1)$ should have zero width and the result of the measurement should be ± 1, i.e., one of the eigenvalues of σ_z. However, the actually observed value of σ_z may differ from the ideal result by a quantity of order Δq, the width of the meter wave packet. This discrepancy is not a trivial "technical difficulty," but a matter of principle. It will be seen in the following sections that in some cases it may be necessary to have Δq larger than the separation of consecutive eigenvalues.

Although there is no advantage to such a situation in the present problem (which involves a single measurement of σ_z) let us examine the consequences of having $\Delta q > 2$. The situation is represented in Fig. 2. The probability of observing the meter between q_0 and $q_0 + dq_0$ is

$$P\,dq_0 = [\,|\alpha|^2\,|\phi(q_0-1)|^2 + |\beta|^2\,|\phi(q_0+1)|^2]dq_0 \; , \tag{5}$$

where both terms may have contributions of the same order of magnitude.

For future reference, it is convenient to rewrite the preceding equations in terms of density matrices. Let λ_m be the eigenvalues of the operator A being measured (in the simple case considered above, A was σ_z and we had $\lambda_m = \pm 1$). Let ρ_{mn} be the density matrix of the quantum system, in a representation where A is diagonal. Let $\Phi(q',q'') = \phi(q')\phi^*(q'')$ be the initial density matrix of the meter. The combined density matrix thus is

$$\rho_{mn}(q',q'') = \rho_{mn}\Phi(q',q'') \; . \tag{6}$$

The interaction Hamiltonian (1) becomes $g(t)Ap$ and, instead of (4), we now have

$$\rho_{mn}(q',q'') \to \rho_{mn}(q'-\lambda_m, q''-\lambda_n) \; . \tag{7}$$

This expression contains all the information about the combined state of the quantum system and the meter used to observe it.

At this point, if we are no longer interested in the quantum system, we may trace out the indices referring to it. Then, the average value of any observable function $f(q)$—after completion of the interaction between the meter and the quantum system— is

$$\langle f(q) \rangle = \sum_j w_j \int \Phi(q-\lambda_j, q-\lambda_j) f(q)dq \; , \tag{8}$$

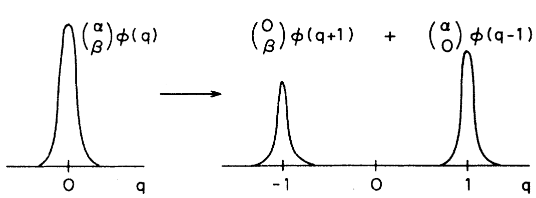

FIG. 1. The evolution of the wave function due to a measurement process. In this drawing and the following one, $\alpha=0.8$ and $\beta=0.6$ (the vertical scales are in arbitrary units).

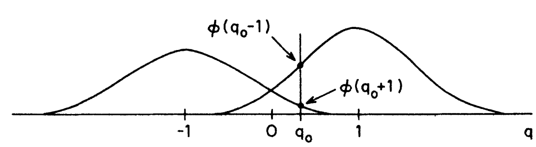

FIG. 2. This is the same as the right-hand side of Fig. 1, except that the initial location of the meter is uncertain by more than the separation of the eigenvalues ± 1, so that the measurement is "fuzzy."

where $w_j = \rho_{jj}$ is the probability of occurrence of the jth eigenstate of A, just before (or after) the interaction. For example, in the simple case where the meter is prepared in a symmetric state, $\phi(q) = \phi(-q)$, we have

$$\langle q \rangle = \sum_j w_j \lambda_j \tag{9}$$

and

$$\langle q^2 \rangle = \sum_j w_j \left[\int q^2 |\phi(q)|^2 dq + \lambda_j^2 \right] . \tag{10}$$

Therefore,

$$\langle q^2 \rangle - \langle q \rangle^2 = (\Delta A)^2 + (\Delta q)^2 , \tag{11}$$

where

$$(\Delta A)^2 = \sum_j w_j \lambda_j^2 - \left[\sum_j w_j \lambda_j \right]^2 , \tag{12}$$

and

$$(\Delta q)^2 = \int q^2 |\phi(q)|^2 dq , \tag{13}$$

are the variances associated with the quantum system and the meter, respectively.

Conversely, if we are interested in the quantum system (for later use) but not in the meter itself, the new density matrix ρ'_{mn} of the quantum system is obtained by tracing out q' and q'':

$$\rho'_{mn} = \int \rho_{mn}(q - \lambda_m, q - \lambda_n) dq = \rho_{mn} S_{mn} , \tag{14}$$

where S_{mn} is the *coherence matrix:*

$$S_{mn} = \int \phi(q - \lambda_m) \phi^*(q - \lambda_n) dq = \langle e^{ip(\lambda_m - \lambda_n)} \rangle . \tag{15}$$

For example, if

$$\phi(q) = (2\pi\sigma)^{-1/4} e^{-q^2/4\sigma^2} , \tag{16}$$

we have $\Delta q = \sigma$ and

$$S_{mn} = \exp[-(\lambda_m - \lambda_n)^2 / 8\sigma^2] . \tag{17}$$

Obviously, $S_{mm} = 1$ (for any ϕ) so that the diagonal matrix elements of ρ are not affected. On the other hand, the off-diagonal elements of ρ are depressed by a factor S_{mn}, and may even be reduced to zero if the displaced wave functions are mutually orthogonal, as in Fig. 1. However, if $\Delta q \gtrsim |\lambda_n - \lambda_{n-1}|$, as in Fig. 2, the off-diagonal elements of ρ are not completely suppressed. (In particular, if some eigenvalues are degenerate, the submatrix of ρ_{mn} corresponding to these eigenvalues is not affected at all.)

In the following sections, we shall consider a sequence of consecutive measurements. In particular, it will be seen that a low resolution such as the one illustrated in Fig. 2 may sometimes be advantageous when we want to monitor the time evolution of a dynamical variable. Some authors discuss this problem in the formalism of *effects* and *operations.*[11,12] The present paper uses the standard Hamiltonian formalism, which is, in my opinion, simpler and clearer.

III. CONSECUTIVE MEASUREMENTS

The detection and analysis of time-dependent signals necessitates numerous measurements, distributed in time,[13–15] so as to get a *sequence of numbers,* corresponding to times $t_1, t_2, \ldots$, and so on. There can be no continuous measurement, because a "measurement" was defined as a *brief and intense interaction* between the meter and the measured system, as shown, for example, in Eq. (1). In particular, the support of $g(t)$ must be smaller than the difference $t_k - t_{k-1}$, so that the measurements do not overlap.

It is possible for sure to consider measurements of finite duration, where the function $g(t)$ is spread over an appreciable time, but such a measurement yields only a *single number* corresponding, in the best case, to a time average of the observed variable.[16] (One can even consider a passive detector, such as a Geiger counter waiting for the decay of a nucleus, but this situation does not fit at all with our definition of a measurement. This setup is best described as a single metastable system with several decay channels.[17])

Quantum theory by itself does not impose any fundamental limitations to monitoring arbitrarily weak signals within arbitrarily short time intervals. Limitations arise solely because we want to use, or are forced to use, some particular detectors. For example, gravitational radiation couples very weakly to matter, and detectors must be very massive, and expensive, antennas.[1] This means, effectively, that it is impracticable to have a large number of identical detectors from which quantum averages are to be obtained. We must extract as much information as possible from *each* measurement.

Two strategies are possible. We may prepare the detector in a known state, which is an eigenstate of its free Hamiltonian (that is, the detector's Hamiltonian when no signal is present). We wait some preassigned time and then measure an operator which commutes with the free Hamiltonian. If the measurement is sharp, as in Fig. 1, the detector is left in a known eigenstate and the process can be repeated. This stategy is conceptually simple, but very inefficient: Each resetting of the detector destroys latent information, namely, the relative amplitudes and phases of the various components of the detector's wave function just prior to the measurement (one component is selected by the measurement, the other ones are lost forever). In particular, if the time t elapsed between propagation and observation is smaller than $(\pi/2\Delta H)$, where

$$\Delta H = [\langle H^2 \rangle - \langle H \rangle^2]^{1/2} \tag{18}$$

is the energy uncertainty of the detector (including its coupling to the signal), there is a probability larger than $\cos^2(t\Delta H)$ that the *initial* eigenstate will again be observed.[18,19] Very weak signals therefore necessitate waiting a long time between consecutive measurements. How long is unpredictable, in the absence of extraneous information. Finally, if and when enough nontrivial data have been obtained, we have to reconstruct from these data the time dependence of the signal. For example, this time dependence can be represented by a trial function including some unknown parameters, and the latter can then be fitted to the experimental data.

A more efficient strategy is to let cumulative effects of the signal develop while the measurements are being performed. We shall see that this is possible if the measurements are "fuzzy," as illustrated in Fig. 2. When Δq (of the meter) is much larger than the separation of consecutive eigenvalues (of the detector), the relative amplitudes and phases of the corresponding wave-function components are not completely lost. The dynamical evolution of the detector can thus proceed, although it cannot be the same as in the absence of measurements. The reason is that if we want to measure a time-dependent variable $A(t)$ and if $[A(t), A(t')]\neq 0$, a measurement of $A(t)$ "disturbs the value of $A(t')$." Stated more precisely, given an ensemble of identically prepared and identically measured systems, the histogram of observed values of $A(t')$ depends on whether or not there is a prior measurement of $A(t)$.

Although the evolution of a quantum detector is inevitably modified by continually measuring it, we would like the meters' readings to remain reasonably reliable, even after numerous observations. Ideally, we would like the quantum evolution to mimic the classical one, and the output signal to be an amplified replica of the input signal. The signal distortion (and concomitant loss of information) should be minimized. The purpose of this paper is to investigate how closely these aims can be achieved.

Let us consider a sequence of measurements performed at times $t_1,\ldots,t_N$, by means of meters with coordinates $q_1,\ldots,q_N$, respectively. The initial density matrix of the detector and the meters is a generalization of (6):

$$\rho_{mn}(q_1',q_1'',\ldots,q_N',q_N'')=\rho_{mn}\prod_j \Phi_j(q_j',q_j'') \ . \tag{19}$$

The Hamiltonian of the combined system is

$$H=H_0(t)+A\sum_j g(t-t_j)p_j \ , \tag{20}$$

with the same notation as in Eq. (1). Here H_0 involves only the dynamical variables of the detector and in particular H_0 has a known functional dependence on the unknown signal. The masses of the meters are assumed so large that we can neglect their contributions $p_j^2/2M_j$ to H_0. In other words, we can ignore the spontaneous spreading of each meter's wave packet.

In the interval between measurements, H_0 generates the unitary evolution $\rho\rightarrow U\rho U^\dagger$. On the other hand, circa each $t=t_j$, there is an evolution similar to Eq. (7): namely,

$$\rho_{mn}(\ldots,q_j',q_j'',\ldots)\rightarrow\rho_{mn}(\ldots,q_j'-\lambda_m,q_j''-\lambda_n,\ldots) \ . \tag{21}$$

Note that the new density matrix entangles in a nontrivial way the discrete indices mn of the detector and the coordinates of the jth meter.

Equation (21) contains all the information about the state of the detector and the various meters. We can then ask a variety of questions, such as those at the end of the preceding section. For example, if we are interested only in the detector, not in the meters that have already interacted with it, the net result of a measurement is given by Eq. (14):

$$\rho_{mn}\rightarrow\rho_{mn}'=\rho_{mn}S_{mn} \ . \tag{22}$$

After that, the following meter, if observed, will give results similar to (9) and (10), with

$$w_j=\sum_{mn} U_{jm}\rho_{mn}' U_{jn}^* \ , \tag{23}$$

where U is the unitary matrix representing the free evolution of the detector since the preceding measurement, which left the detector in state ρ_{mn}'.

We may also be interested in comparing the readings of different meters. For example, if there are N consecutive measurements, let us predict the expected $\langle (q_1-q_N)^2\rangle$ (regardless of the results obtained at $t_2,\ldots,t_{N-1}$). As before, let ρ_{mn} be the density matrix of the detector just before the first measurement. The latter causes

$$\rho_{mn}\Phi_1(q_1',q_1'')\rightarrow\rho_{mn}\Phi_1(q_1'-\lambda_m,q_1''-\lambda_n) \equiv\rho_{mn}^{(1)}(q_1',q_1'') \tag{24}$$

(the q_j referring to subsequent measurements are omitted, for brevity). Between the first and second measurements, there is a unitary evolution $\rho^{(1)}\rightarrow\rho^{(2)}=U^{(1)}\rho^{(1)}U^{(1)\dagger}$ so that, just before t_2, we have

$$\rho_{rs}^{(2)}(q_1',q_1'')=\sum_{mn} U_{rm}^{(1)}U_{sn}^{(1)*}\rho_{mn}^{(1)}(q_1',q_1'') \ . \tag{25}$$

Then, there is a second measurement whose result is ignored[20] (q_2' and q_2'' are traced out). According to (14) this leads to a reduction of the off-diagonal elements $\rho_{rs}^{(2)}\rightarrow\rho_{rs}^{(2)}S_{rs}$. Consider in particular the case where the measurements are sharp (as in Fig. 1) so that $S_{rs}=\delta_{rs}$. We thus have, immediately after t_2, a density matrix containing only the diagonal elements $\rho_{rr}^{(2)}$ of (25).

Between the second and the third measurement, there is another unitary evolution $\rho^{(2)}\rightarrow\rho^{(3)}=U^{(2)}\rho^{(2)}U^{(2)\dagger}$, with result

$$\rho_{st}^{(3)}(q_1',q_1'')=\sum_r U_{sr}^{(2)}U_{tr}^{(2)*}\rho_{rr}^{(2)}(q_1',q_1'') \ . \tag{26}$$

Again, the third measurement is sharp and only the diagonal elements survive. These are

$$\rho_{ss}^{(3)}(q_1',q_1'')=\sum_r V_{sr}^{(2)}\rho_{rr}^{(2)}(q_1',q_1'') \ , \tag{27}$$

where

$$V_{sr}^{(2)}=|U_{sr}^{(2)}|^2 \ . \tag{28}$$

The rule to continue is obvious. After the last unregistered measurement[20] at t_{N-1}, we have a diagonal density matrix with elements

$$\rho_{xx}^{(N-1)}(q_1',q_1'') =\sum_{wv\cdots}\sum_{\cdots sr} V_{xw}^{(N-2)}V_{wv}^{(N-3)}\cdots V_{ts}^{(3)}V_{sr}^{(2)}\rho_{rr}^{(2)}(q_1',q_1'') \ . \tag{29}$$

Then, finally, the Nth measurement gives

$$\rho_{zy}^{(N)}(q_1',q_1'';q_N',q_N'')=\sum_x U_{zx}^{(N-1)}U_{yx}^{(N-1)*}\rho_{xx}^{(N-1)}(q_1',q_1'')\times\Phi_N(q_N'-\lambda_z,q_N''-\lambda_y) \ . \tag{30}$$

Recall that we are interested in $\langle(q_1-q_N)^2\rangle$, for which we need only the reduced density matrix involving q_1 and q_N, with the detector's indices traced out. This is

$$\rho(q_1',q_1'';q_N',q_N'')=\sum_{zr} W_{zr}\rho_{rr}^{(2)}(q_1',q_1'')\times\Phi_N(q_N'-\lambda_z,q_N''-\lambda_z) \ , \tag{31}$$

where

$$W_{zr}=\sum_{x\cdots s} V_{zx}^{(N-1)}V_{xw}^{(N-2)}\cdots V_{ts}^{(3)}V_{sr}^{(2)} \ , \tag{32}$$

and

$$\rho_{rr}^{(2)}(q_1',q_1'')=\sum_{mn} U_{rm}^{(1)}U_{rn}^{(1)*}\rho_{mn}\Phi_1(q_1'-\lambda_m,q_1''-\lambda_n). \tag{33}$$

Here, we may be tempted to increase N and to make the time intervals very short, so as to have a quasicontinuous monitoring. However, as we shall presently see, this would lead to a complete *loss* of information. Consider in particular the case where the measurements are equally spaced, at intervals $\tau=(t_N-t_1)/(N-1)$, and let $N\to\infty$ while $T=t_N-t_1$ is kept fixed. Let

$$h=\int_{t_n}^{t_n+\tau} H(t)dt \ . \tag{34}$$

We then have, in the brief time interval τ,

$$U=e^{-ih}=I-ih-\tfrac{1}{2}h^2+\cdots \ , \tag{35}$$

whence

$$V_{sr}\equiv|U_{sr}|^2=\delta_{sr}-\delta_{sr}\sum_j|h_{sj}|^2+|h_{sr}|^2+O(h^4) \ . \tag{36}$$

It follows that each V in (32) differs from the unit matrix by terms of order $(TH/N)^2$. Since there are $N-2$ such terms in (32), $W_{zr}=\delta_{zr}+O(T^2H^2/N)$. Thus, in the limit $N\to\infty$, we can replace all the U and V by unit matrices and we obtain

$$\rho(q_1',q_1'';q_N',q_N'')=\sum_j w_j\Phi_1(q_1'-\lambda_j,q_1''-\lambda_j)\times\Phi_N(q_N'-\lambda_j,q_N''-\lambda_j) \ , \tag{37}$$

where $w_j=\rho_{jj}$, as usual. We then have, by virtue of (9) and (10),

$$\langle(q_1-q_N)^2\rangle=(\Delta q_1)^2+(\Delta q_N)^2 \ . \tag{38}$$

The dynamical evolution of the detector has been "frozen" by its continual interaction with the meters. This is the well-known quantum Zeno effect.[21–25]

This effect was proved here without invoking the controversial "collapse" postulate. It has nothing paradoxical, notwithstanding its name "Zeno paradox." What happens simply is that the quantum system is overwhelmed by the meters which continually interact with it. Note that the derivation essentially depends on the assumption $S_{mn}=\delta_{mn}$ or, in other words, $\Delta q\ll|\lambda_j-\lambda_{j-1}|$. Meters with a coarser resolution do not completely block the detector's motion. Indeed, it was shown by Caves and Milburn[26] that τ can be made arbitrarily small, provided that σ increases as τ^{-1}.

IV. EXAMPLE: DETECTION OF A WEAK TORQUE

As a concrete example, assume that a random signal $\omega(t)$ can be used as a torque acting on a rotor with angular momentum $\mathbf{J}$. The Hamiltonian of the rotor is

$$H=H_0+\omega(t)J_y \ . \tag{39}$$

If the rotor is spherically symmetric, $H_0=\mathbf{J}^2/2I$ is a constant of the motion, which can be ignored. The equations of motion (in classical or quantum theory) are

$$\dot{J}_x=J_z, \quad \dot{J}_y=0, \quad \dot{J}_z=-J_x \ . \tag{40}$$

Their solution is

$$J_x=J_{x0}\cos\tau+J_{z0}\sin\tau \ , \tag{41a}$$

$$J_y=J_{y0} \ , \tag{41b}$$

$$J_z=-J_{x0}\sin\tau+J_{z0}\cos\tau \ , \tag{41c}$$

where

$$\tau=\int_0^t\omega(t)dt \ . \tag{42}$$

In classical physics, the initial values J_{k0} are known and continual measurements of J_x and J_z give $\tau=\tau(t)$:

$$\tau=\arctan(J_x/J_z)-\arctan(J_{x0}/J_{z0}) \ , \tag{43}$$

whence we can obtain $\omega(t)$. Unfortunately, this method is not readily applicable to quantum systems, because J_x and J_z do not commute and therefore (43) is not valid. We shall now see in detail what can be done when the "rotor" is a particle of spin $\frac{1}{2}$. Thereafter, we shall consider a particle of large spin j.

The state of a spin-$\frac{1}{2}$ particle can be described by a density matrix

$$\rho=\tfrac{1}{2}(I+\mathbf{m}\cdot\boldsymbol{\sigma}) \ , \tag{44}$$

where $\mathbf{m}=2\langle\mathbf{J}\rangle$. Since the equations of motion (41) are linear, they are satisfied by $\mathbf{m}$ as well as by $\mathbf{J}$. In this example, we shall assume that initially $\mathbf{m}=(0,0,1)$. Therefore $\mathbf{m}$ will remain in the xz plane. The coherence matrix S_{mn} has diagonal elements 1, and off-diagonal elements $S_{12}=S_{21}=S<1$ (assumed real, for simplicity). Thus, in the present notation, the density matrix reduction (14) simply is

$$m_x'=Sm_x \text{ and } m_z'=m_z \ . \tag{45}$$

The combined effect of a rotation and a reduction is the mapping

$$\begin{bmatrix} m_x \\ m_z \end{bmatrix}\to\begin{bmatrix} m_x' \\ m_z' \end{bmatrix}=\Omega\begin{bmatrix} m_x \\ m_z \end{bmatrix} \ , \tag{46}$$

where

$$\Omega = \begin{bmatrix} S\cos\theta & S\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} . \tag{47}$$

Here, $\theta = \int \omega(t)dt$ during the time interval since the preceding measurement. Recall that, for a Gaussian shaped $\phi(q)$, we have $S = \exp[-1/8(\Delta q)^2]$. The extreme cases are $\Delta q \gg 1$, giving $S \simeq 1$ and therefore an unperturbed rotation of the vector **m** (but no true measurement, of course); and $\Delta q \ll 1$ (whence $S \simeq 0$) corresponding to a sharp measurement. Suppose that we repeatedly measure J_z. Initially, we have $\langle J_z \rangle = \frac{1}{2}$. Thereafter, all we can obtain is a sequence of $+\frac{1}{2}$ and $-\frac{1}{2}$ from which we have to reconstruct the function $\tau(t)$: Obviously, a spin-$\frac{1}{2}$ particle is not a good torque detector—its Hilbert space is too small. On the other hand, these particles come cheap (that is, for gedanken experiments) so that we can afford to use a large number N of identical detectors. It therefore makes sense to compute the expected average $m_z = \langle J_z \rangle$, under various scenarios.

The following figures illustrate the behavior of $m_z(t)$, in the simple case $\omega = 1$ (so that $\tau = t$) from $t = 0$ to $t = 5\pi/2$. The dotted line is the undisturbed $m_z = \cos t$ evolution, corresponding to $\Delta q \gg 1$ [we would of course need $N \gg (\Delta q)^2$ to actually observe it, as an average over N data]. This ideal result is compared, in Fig. 3, to the case $S = 0$ (sharp measurements) for 10, 30, and 90 equally spaced samplings. Obviously, the more frequent the measurements, the less m_z moves (this is the Zeno effect). In every case, m_z decays exponentially—it does not oscillate as the unperturbed m_z. This can be verified by computing the eigenvalues of Ω in (47). The latter are given by the secular equation $\lambda^2 - \lambda\cos\theta(1+S) + S = 0$, whence

$$\lambda_\pm = \tfrac{1}{2}\{(1+S)\cos\theta \pm [(1+S)^2\cos^2\theta - 4S]^{1/2}\} . \tag{48}$$

If $S = 0$, we obtain $\lambda_+ = \cos\theta$, with eigenvector $u_+ = \binom{0}{1}$; and $\lambda_- = 0$, with eigenvector $u_- = \binom{1}{0}$. As we started from u_+, the vector **m** is shortened by a factor $\cos\tau$ at each measurement.

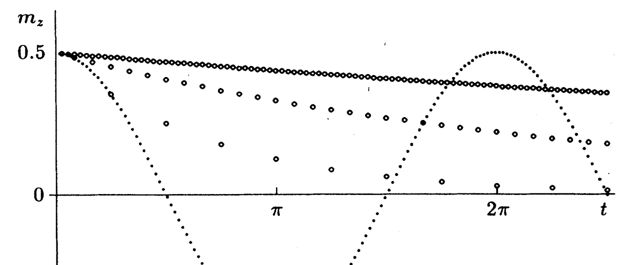

FIG. 3. Average $\langle J_z \rangle$ of a spin-$\frac{1}{2}$ particle precessing around the y axis with constant $\omega = 1$. Initially, $\langle J_z \rangle \equiv m_z = 0.5$. The dotted line corresponds to the undisturbed evolution (that is, each point represents the value of m_z which would be obtained if there were no measurement before that time). The circles represent consecutive values of m_z that are observed if *sharp* measurements are performed at intervals (from top to bottom) $5\pi/180$, $5\pi/60$, and $5\pi/20$.

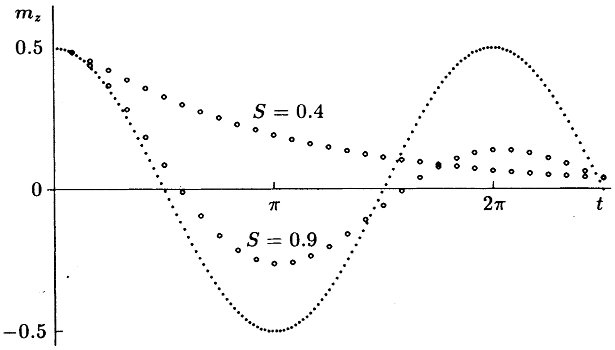

FIG. 4. Under the same conditions as in Fig. 3, each set of circles represents consecutive values of m_z that are observed if *fuzzy* measurements are performed at intervals $5\pi/60$. The coherence factors $S = 0.4$ and 0.9 correspond to $\Delta q = 0.37$ and 1.09, respectively. The critical value of S determining the onset of oscillations of m_z is $S_{\rm cr} = 0.589$.

When $0 < S < 1$, the eigenvalues $\lambda_\pm$ may be real or complex conjugate, according to the sign of $(1+S)^2\cos^2\theta - 4S$ (Ref. 27). Figure 4 illustrates the case $\theta = 5\pi/60$, with $S = 0.4$ and 0.9, corresponding to $\Delta q = 0.37$ and

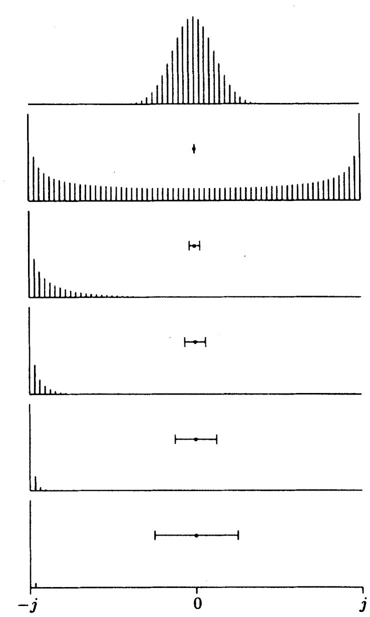

FIG. 5. The top diagram is the expected probability distribution for the results of measurements of J_z, following a preparation of the state with $j_z = +j$, and a rotation by $\pi/2$. The calculations were done for $j = 32$ and the standard deviation for this diagram is $\Delta J_z = 4$. The five other diagrams are the expected probability distributions for a subsequent measurement of J_z, after another rotation by $\pi/2$. The results depend on the resolution of the meter which performed the first measurement. From top to bottom, Δq (shown as a horizontal error bar) is 0, 1, 2, 4, and 8. (Different vertical scales were chosen in the various diagrams, for better visibility. The sum of lengths of the vertical bars is always 1, by definition.)

1.09, respectively. We see that the larger Δq gives complex eigenvalues to Ω, and yields damped oscillations of m_z. Similar results were recently obtained by Milburn.[28]

The worst-case scenario is a pair of consecutive measurements performed at $\tau=\pi/2$ and $\tau=\pi$, respectively. The result can easily be obtained in closed form and will later be compared with the one for a rotor of spin j. From (47) we have, with $\sin\theta=1$,

$$\Omega^2=\begin{bmatrix} -S & 0 \\ 0 & -S \end{bmatrix} , \tag{49}$$

so that the initial $\mathbf{m}$ is simply multiplied by $-S$, after two measurements. For very fuzzy measurements ($\Delta q \gg 1$), $-S \simeq -1$ and the spin has been flipped, as expected. For sharp measurements, $\mathbf{m}$ is reduced to zero and, thereafter, no further information is available.

Obviously, a spin-$\frac{1}{2}$ particle cannot mimic the classical rotor described by Eq. (39). We therefore turn our attention to spin j. Let us prepare the rotor in an eigenstate of J_z, with $j_z=j$. Its rotation through an angle $\pi/2$ is generated by the unitary matrix U. We have, with Wigner's notations,[29,30] $U_{mn}=\mathcal{D}^{(j)}(\{0\theta0\})_{nm}$. In the "worst scenario" mentioned above, $\theta=\pi/2$ and the first measurement of J_z is performed on an eigenstate of J_x with eigenvalue j. The probability of getting $j_z=m$ is a binomial distribution

$$w_m=(2j)!/[2^j(j+m)!(j-m)!] , \tag{50}$$

with variance[31] $j/2$. This is shown in the upper diagram of Fig. 5, for the case $j=32$ [the standard deviation then is $\Delta J_z=(j/2)^{1/2}=4$].

Thereafter, the situation depends on whether the first measurement was sharp or fuzzy. If it was sharp ($S_{mn}=\delta_{mn}$) the histogram of expected results for the second measurement (at $\theta=\pi$) is given by the second diagram of Fig. 5: the distribution is almost uniform, and very little information is available. Better results are obtained if the first measurement is fuzzy.[32] The following diagrams of Fig. 5 show the distribution of results of the second measurement, depending on the Δq of the meter which was used for the first measurement. Obviously, a broad Δq allows the quantum state to reassemble near $j_z=-j$ (which would be its expected value at $\theta=\pi$, if the first measurement were not performed).[33]

On the other hand, if Δq is too broad, the "measurement" becomes useless. Actually, the expression which should be optimized is given by Eq. (11). From the example discussed above, it appears that the fuzziness of the meter should be about the same as the natural width of the detector's wave packet. A poorer resolution obviously gives inaccurate results, but a finer resolution destroys the information in which we are interested.

This problem is peculiar to quantum systems. It disappears in the semiclassical limit, where eigenvalues become extremely dense. From the quantum point of view, *classical measurements are always fuzzy.* This is why a watched pot may boil, after all: the observer watching it is unable to resolve the energy levels of the pot. Any hypothetical device which could resolve these energy levels would also radically alter the behavior of the pot. Likewise, the mere presence of a Geiger counter does not prevent a radioactive nucleus from decaying.[17] The Geiger counter does not probe the energy levels of the nucleus (it interacts with decay products whose Hamiltonian has a continuous spectrum). As the preceding calculations show, peculiar quantum effects, such as the Zeno "paradox" occur only when individual levels are resolved (or almost resolved).

APPENDIX: QNDD VARIABLES

This paper would not be complete without a mention of QNDD (quantum nondemolition detection) variables[34] satisfying $[A(t),A(t')]=0$. If such a dynamical variable were realizable in the laboratory, it could be monitored continuously with arbitrary precision, without disturbing in any way its evolution. For example, if $v(t)$ is a random classical signal and if $H=v(t)p$, then q is a QNDD variable because $\dot{q}=v(t)$, so that

$$q(t)=q(0)+\int_0^t v(t')dt' , \tag{A1}$$

whence $[q(t),(q(t')]=0$. Measuring q will of course disturb p, but this has no effect on the evolution of q itself. The difficulty, of course, is that the Hamiltonian $H=v(t)p$ is only a mathematical construct, with no experimental counterpart.

It is unlikely that any nontrivial QNDD can actually be realized, because of the conflicting requirements which it must satisfy: Its time evolution should be sensitive to weak external signals, but on the other hand it should not be affected by the intense interaction of the measurement process.

In particular, it should be noted that QNDD variables must have a continuous spectrum. Indeed, from $[A(t),(t')]=0$ it follows, by taking $t'=t+dt$, that $[A,\dot{A}]=0$, or

$$[A,[A,H]]=0 . \tag{A2}$$

Now, if A has discrete eigenvalues λ_m and λ_n, the matrix element of (A2) between the corresponding eigenstates is

$$(\lambda_m-\lambda_n)^2 H_{mn}=0 , \tag{A3}$$

so that $H_{mn}=0$ if $\lambda_m \neq \lambda_n$. Then, if (A3) holds, we also have $[A,H]=0$ and A is a constant of the motion. Therefore, nontrivial QNDD variables cannot have a discrete spectrum.

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