3+1 formulation of general-relativistic perfect magnetohydrodynamics

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The laws of perfect general-relativistic magnetohydrodynamics (GRMHD) are rewritten in $3+1$ language in a general spacetime. The laws are expressed in terms of quantities (magnetic field, flow velocity, ...) that would be measured by the "fiducial observers" whose world lines are orthogonal to the hypersurfaces of constant time. This $3+1$ formalism of GRMHD should be of interest in numerical relativity, numerical astrophysics, and the membrane paradigm for black holes. The GRMHD equations are also specialized to a stationary spacetime and a stationary MHD flow with one arbitrary spatial symmetry (e.g., a stationary MHD magnetosphere for a rotating Kerr black hole); and the general features of stationary, symmetric GRMHD solutions are discussed.

I. INTRODUCTION

In astrophysics one often encounters magnetic fields. When interstellar clouds condense into stars, when stars collapse to neutron stars, and when accreting material falls into black holes, they all carry magnetic flux with them into a much smaller region thereby producing a relatively large-scale, ordered magnetic field. Meanwhile other processes, e.g., the dynamo effect, also extract energy from the fluid's motion and further intensify the already existing magnetic field. In order to understand many interesting phenomena in our Universe, we need a well-developed theory of magnetized plasmas. However, the full plasma theory is difficult to handle even in some of the most simple situations, and there are many situations where, as a first approximation to plasma theory, a theory of magnetohydrodynamics (MHD) can be rather accurate and reveal much interesting physics. For example, to study magnetic phenomena inside our Sun, Newtonian MHD is sufficient.[1] To study magnetospheres and interiors of a neutron star, special-relativistic MHD gives one a good understanding[2] but general-relativistic MHD is desirable.[3] However, to study the innermost regions of accretion disks and jets of magnetized, accreting black holes, the strength of gravity demands a general-relativistic MHD treatment.

There have been many efforts to develop a fully general-relativistic magnetohydrodynamic (GRMHD) theory and to apply it to interesting astrophysical situations (Refs. 3–8 and references cited therein). Thus GRMHD is already a rather mature subject. However, it is found in research that some versions of the theory are more helpful in intuitive thinking than others, or more convenient to use for some problems. A $3+1$ formulation is particularly useful for numerical calculations,[6,9] and it shows promise for intuitive understanding in black-hole situations (the "membrane paradigm,"[10] a $3+1$ version of black-hole theory based on a special family of fiducial observers). There has been one previous $3+1$ formulation of GRMHD: that of Sloan and Smarr.[6] However, that formalism expressed the theory in terms of a set of variables (energy density, energy flux, stress tensor) that are not optimal for intuitive understanding. The objective of this paper is to reexpress GRMHD in a more intuitively useful form: a form based on fluid and field quantities that are measured by a preferred family of fiducial observers (FIDO's) directly (the FIDO-measured magnetic field **B**, fluid velocity **V**, and the mass density ρ and pressure p as seen in fluid's rest frame). Expressed in this way the $3+1$ equations of GRMHD are Eqs. (2.5) or (2.13), (2.6), (2.12), (2.22), (2.24), and (2.26) below.

The work reported here has particularly been motivated by the "Blandford-Znajek effect"; i.e., the extraction of rotational energy from a black hole by the coupling of magnetic fields threading the hole to the hole's gravitomagnetic field (its "dragging of inertial frames"). In their seminal paper on this subject, Blandford and Znajek[11] idealized the magnetosphere as force-free, with its plasma consisting of electron-positron pairs created by magnetic-gravitomagnetic-induced electric fields. Macdonald and Thorne[12] analyzed this Blandford-Znajek process using the membrane paradigm and retaining the force-free idealization near the hole. More recently, Phinney[7] has developed and applied to the Kerr geometry a (non-$3+1$) formulation of GRMHD theory and has used it in an improved, MHD analysis of the Blandford-Znajek process. All of this past research has dealt with equilibrium states of the magnetosphere. A natural extension of these studies would be an investigation of the magnetosphere's dynamical properties, or as a first step, the behavior of MHD waves propagating in it. The author is carrying out an initial study of such waves in a black-hole magnetosphere. As a foundation for that study, a $3+1$ version of GRMHD is developed and presented in this paper.

Although the formulation presented in this paper was motivated by the black-hole problem and meshes nicely with the membrane paradigm, the formalism is not restricted to black holes or the membrane paradigm. It is presented initially (Sec. II) in a much more general form than that. However, in the Kerr geometry we have a set of preferred FIDO's [the zero angular momentum observers[13] (ZAMO's)]; and our general formalism can be easily specialized to the Kerr geometry with the ZAMO's playing the role of the FIDO's. The result is the membrane paradigm version of GRMHD.

In Sec. II of this paper the full and general set of GRMHD equations is given in terms of quantities measured by the FIDO's. In Sec. III we demonstrate that, without much extra effort, Phinney's results[7] on stationary, GRMHD, black-hole magnetospheres can actually be generalized to any stationary MHD system with one spatial symmetry; and for such a system we reduce the full set of GRMHD equations to a set of algebraic relations and an (algebraic) wind equation which, plus one nonlinear partial differential equation also contained in this set of GRMHD equations, fully determine the structure of MHD flows. In a subsequent paper we will use those equations to build equilibrium models, which we will then perturb in order to get insight into dynamical black-hole magnetospheres.

II. GENERAL-RELATIVISTIC MAGNETOHYDRODYNAMIC EQUATIONS

A. Notation

For the concept of a $3+1$ split of spacetime into space plus time and the concept of the FIDO's, associated with such a split, readers are referred to York,[14] to the membrane paradigm book,[10] and to references cited therein. Here only the basic points will be summarized. The foundation for the $3+1$ split is a particular choice of time coordinate t (i.e., a particular "foliation" of spacetime into "universal time" t and "absolute space," the hypersurfaces of constant t). With a specific choice of time t and spatial coordinate x^i, the spacetime line element takes the form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{2.1}$$

and the FIDO's (whose world lines are orthogonal to the hypersurfaces of constant t) have four-velocities

$$\boldsymbol{n} = \frac{1}{\alpha}\left[\frac{\partial}{\partial t} - \beta^i \frac{\partial}{\partial x^i}\right]. \tag{2.2}$$

The FIDO's proper time τ is related to the "universal time" t by $d\tau = \alpha\, dt$. The rate of change of any scalar physical quantity as seen by a FIDO is

$$\frac{df}{d\tau} \equiv \boldsymbol{n}\cdot{}^{(4)}\nabla f = \frac{1}{\alpha}\left[\frac{\partial}{\partial t} - \boldsymbol{\beta}\cdot\nabla\right] f, \tag{2.3a}$$

and the FIDO-measured rate of change of any three-dimensional vector $\mathbf{S}$ or tensor $\overleftrightarrow{\mathbf{D}}$ that lies in absolute space (i.e., orthogonal to $\boldsymbol{n}$) is defined by

$$\frac{d\mathbf{S}}{d\tau} \equiv \frac{1}{\alpha}\left[\mathcal{L}_t \mathbf{S} - (\boldsymbol{\beta}\cdot\nabla)\mathbf{S}\right], \quad \frac{d\overleftrightarrow{\mathbf{D}}}{d\tau} \equiv \frac{1}{\alpha}\left[\mathcal{L}_t \overleftrightarrow{\mathbf{D}} - (\boldsymbol{\beta}\cdot\nabla)\overleftrightarrow{\mathbf{D}}\right]. \tag{2.3b}$$

Here ${}^{(4)}\nabla$ denotes the gradient in four-dimensional spacetime, ∇ is the gradient in three-dimensional space, and $\mathcal{L}_t$ is the Lie derivative along $\partial/\partial t$, so $\mathcal{L}_t\mathbf{S}$ is the three-vector whose components in the coordinate system (2.1) are $\partial S^j/\partial t$.

In this paper geometrized units, with $G = c = 1$, will be used. Vectors and tensors living in four-dimensional spacetime will be denoted by boldface italic letters, such as the FIDO's four-velocity $\boldsymbol{n}$; vectors living in three-dimensional absolute space will be denoted by boldface roman or greek letters, such as the shift function $\boldsymbol{\beta}$; three-dimensional tensors are distinguished from vectors by a dyad over the letter, such as the three-dimensional metric $\overleftrightarrow{\gamma}$. All vector-analysis notations such as the gradient, curl, and vector cross product will be those of the three-dimensional absolute space whose three-metric is $\overleftrightarrow{\gamma}$, unless specified otherwise. The determinant of the three-metric is denoted as g:

$$g \equiv \det|\gamma_{ij}|. \tag{2.4}$$

Latin letters $i, j, k, \ldots$ represent indices in absolute space and thus run from 1 to 3; greek letters $\alpha, \beta, \gamma, \ldots$ represent indices in spacetime and thus run from 0 to 3. Summation on repeated indices is assumed.

B. Evolution of the magnetic field

In an MHD fluid, the motion of the fluid will change the magnetic field; and the magnetic field, in turn, will change the state of the fluid's flow through its Lorentz and Coulomb forces.

The FIDO-measured magnetic field $\mathbf{B}$ is governed by half of Maxwell's equations [Eqs. (3.4), (2.16), (2.17) of Ref. 15, together with (2.3b) above; see also Ref. 6]:

$$\frac{d\mathbf{B}}{d\tau} + \frac{1}{\alpha}\mathbf{B}\cdot\nabla\boldsymbol{\beta} + \theta\mathbf{B} = -\frac{1}{\alpha}\nabla\times(\alpha\mathbf{E}), \tag{2.5}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{2.6}$$

Here, θ is the expansion rate of the FIDO's four-velocity, i.e., three times the direction-averaged "Hubble expansion rate" of absolute space as seen by them,

$$\theta \equiv {}^{(4)}\nabla\cdot\boldsymbol{n}, \tag{2.7}$$

and is expressible in terms of $g = \det|\gamma_{ij}|$, the "lapse function" α, and "shift function" (or "gravitomagnetic potential") $\boldsymbol{\beta}$ by

$$\theta = \frac{1}{\alpha}\left[\frac{g_{,t}}{2g} - \nabla\cdot\boldsymbol{\beta}\right]. \tag{2.7'}$$

The FIDO-measured electric field $\mathbf{E}$, electric current $\mathbf{j}$, and electric charge density ρ_e are treated as auxiliary quantities in the GRMHD formalism. For imperfect MHD (MHD with finite electrical conductivity) they can be found from the other half of Maxwell's equations [Eqs. (3.4), (2.16), (2.17) of Ref. 15 together with (2.3b) above; see also Ref. 6],

$$\frac{d\mathbf{E}}{d\tau} + \frac{1}{\alpha}\mathbf{E}\cdot\nabla\boldsymbol{\beta} + \theta\mathbf{E} = \frac{1}{\alpha}\nabla\times(\alpha\mathbf{B}) - 4\pi\mathbf{j}, \tag{2.8}$$

$$\nabla\cdot\mathbf{E} = 4\pi\rho_e, \tag{2.9}$$

and from Ohm's law (the spatial part of the 3+1 version of $J^\mu+u^\mu u^\nu J_\nu=\sigma F^{\mu\nu}u_\nu$, where u_ν is the fluid four-velocity):

$$\mathbf{j}+\gamma^2(\mathbf{V}\cdot\mathbf{j})\mathbf{V}-\rho_e\gamma^2\mathbf{V}=\sigma\gamma(\mathbf{E}+\mathbf{V}\times\mathbf{B}). \tag{2.10}$$

Here $\mathbf{V}$ is the FIDO-measured fluid velocity, γ is the fluid's Lorentz factor as seen by the FIDO's,

$$\gamma\equiv(1-\mathbf{V}^2)^{-1/2}, \tag{2.11}$$

and σ is the electric conductivity as measured in the fluid rest frame, not in the FIDO's frame.

In this paper we will restrict attention to perfect MHD, i.e., to MHD with perfectly conducting ($\sigma\to\infty$) fluids; this is an excellent idealization for most astrophysical situations. For a detailed discussion of its validity in the context of active galactic nuclei (AGN's), see Sec.V3 of Phinney.[7] Under the perfect MHD assumption there can be no electric field in the fluid's rest frame, i.e.,

$$\mathbf{E}+\mathbf{V}\times\mathbf{B}=0. \tag{2.12}$$

This equation can be formally derived from $u^\mu F_{\mu\nu}=0$ or can be inferred from (2.10) with $\sigma\to\infty$. Note that for perfect MHD, $\mathbf{E}$ can be computed from $\mathbf{V}$ and $\mathbf{B}$ using Eq. (2.12); then $\mathbf{j}$ and ρ_e can be computed from (2.8) and (2.9). In the following Eqs. (2.8) and (2.9) will not be used again except to calculate the auxiliary quantities $\mathbf{j}$ and ρ_e when needed.

For perfect MHD the magnetic-field evolution equation (2.5) can be simplified by substituting $-\mathbf{V}\times\mathbf{B}$ for $\mathbf{E}$ and making use of Eq. (2.6). The result is

$$\frac{D\mathbf{B}}{D\tau}+\frac{1}{\alpha}\mathbf{B}\cdot\nabla(\boldsymbol{\beta}-\alpha\mathbf{V})+\left[\theta+\frac{\nabla\cdot(\alpha\mathbf{V})}{\alpha}\right]\mathbf{B}=0, \tag{2.13}$$

where

$$\frac{D}{D\tau}\equiv\frac{d}{d\tau}+\mathbf{V}\cdot\nabla=\frac{1}{\alpha}\left[\frac{\partial}{\partial t}+(\alpha\mathbf{V}-\boldsymbol{\beta})\cdot\nabla\right] \tag{2.14}$$

is the time derivative moving with the fluid. As Evans and Hawley[9] have pointed out, with a little bit of manipulation this evolution law can be reduced to a form more suitable for numerical calculations:

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\,\mathbf{B}}{\partial t}-\nabla\times[(\alpha\mathbf{V}-\boldsymbol{\beta})\times\mathbf{B}]=0. \tag{2.13'}$$

Because the evolution law (2.13) represents only half of the dynamic Maxwell equations [Eq. (2.5) but not (2.8)], a natural question arising at this stage is: Should one impose the constraint (2.6) ($\nabla\cdot\mathbf{B}=0$) at all times or just on the initial data? The answer is what we would guess intuitively: as in everyday physics we only need to impose it on the initial data. The proof is very straightforward; we will sketch it here to conclude this section. First we move $d/d\tau$ inside ∇ in $(d/d\tau)(\nabla\cdot\mathbf{B})$, taking care to include curvature terms when we change the orders of differentiation; then we use (2.5) to eliminate $d\mathbf{B}/d\tau$. The end result is

$$\frac{d}{d\tau}\nabla\cdot\mathbf{B}=-\theta\,\nabla\cdot\mathbf{B}, \tag{2.15}$$

which says explicitly that once $\nabla\cdot\mathbf{B}=0$ is imposed on the initial data, it will continue to hold at later times as the magnetic field is evolved using (2.5) [or its consequence, (2.13) or (2.13')].

C. Motion of the fluid

The total energy-momentum tensor of an MHD system must obey the conservation law

$$^{(4)}\nabla\cdot(\boldsymbol{T}_{\text{fluid}}+\boldsymbol{T}_{\text{EM}})=0. \tag{2.16}$$

Here $\boldsymbol{T}_{\text{fluid}}$ is the four-dimensional energy-momentum tensor of the fluid and $\boldsymbol{T}_{\text{EM}}$ is that of the electromagnetic field. Each of these $\boldsymbol{T}$'s is broken into the FIDO-measured energy density ϵ, energy flux or momentum density $\mathbf{S}$, and stress tensor $\overleftrightarrow{\mathbf{W}}$.[6,15] For the electromagnetic field alone we have [Eq. (3.10) of Ref. 15]

$$\epsilon_{\text{EM}}=\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{B}^2), \tag{2.17a}$$

$$\mathbf{S}_{\text{EM}}=\frac{1}{4\pi}(\mathbf{E}\times\mathbf{B}), \tag{2.17b}$$

$$\overleftrightarrow{\mathbf{W}}_{\text{EM}}=\frac{1}{4\pi}[-(\mathbf{E}\otimes\mathbf{E}+\mathbf{B}\otimes\mathbf{B})+\tfrac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)\overleftrightarrow{\gamma}]. \tag{2.17c}$$

Here $\overleftrightarrow{\gamma}$ is the three-metric of absolute space and $\otimes$ denotes the tensor product. For the perfect fluid we have [Eq. (3.11) of Ref. 15]

$$\epsilon=(\rho+p\mathbf{V}^2)\gamma^2, \tag{2.18a}$$

$$\mathbf{S}=(\rho+p)\gamma^2\mathbf{V}, \tag{2.18b}$$

$$\overleftrightarrow{\mathbf{W}}=(\rho+p)\gamma^2\mathbf{V}\otimes\mathbf{V}+p\overleftrightarrow{\gamma}, \tag{2.18c}$$

where ρ is the mass density and p is the presure as seen in the fluid's rest frame, $\gamma=(1-\mathbf{V}^2)^{-1/2}$ is the fluid's Lorentz factor [Eq. (2.11)].

The conservation law (2.16) can be viewed in two equivalent ways. One is to treat its two parts separately, and regard the electromagnetic part as an external force acting on the fluid, an approach used by Sloan and Smarr.[6] We shall also adopt this approach in deriving our dynamic GRMHD equations. The other approach is to treat the total energy-momentum tensor as a whole.[3,7] This is found to be more useful in deriving conservation laws when symmetry exists and will be used in Sec. III below to deduce properties of equilibrium solutions.

When we project (2.16) along a FIDO's world line we get the local energy-conservation law as seen by the FIDO; when we project (2.16) into absolute space, i.e., orthogonal to the FIDO's world line, what we get is a force balance equation as seen by the FIDO's. The two resulting equations are[6,10,15]

$$\frac{d\epsilon}{d\tau}+\theta\epsilon+\frac{1}{2\alpha}W^{ij}\mathcal{L}_t\gamma_{ij}=-\frac{1}{\alpha^2}\nabla\cdot(\alpha^2\mathbf{S})+\frac{1}{\alpha}(\nabla\boldsymbol{\beta}):\overleftrightarrow{\mathbf{W}}+\mathbf{E}\cdot\mathbf{j}, \tag{2.19}$$

$$\frac{dS_i}{d\tau}+\theta S_i+(\mathcal{L}_t\gamma_{ij})S^j=-\epsilon a_i-W_{ij}a^j+\frac{1}{\alpha}\beta_{j|i}S^j-W_i^j{}_{|j}+\rho_e E_i+(\mathbf{j}\times\mathbf{B})_i . \quad (2.20)$$

Here ϵ, S_i, and W_{ij} are the FIDO-measured energy density, energy flux, and stress in the fluid alone [Eq. (2.18)];

$$\mathbf{a}\equiv(\nabla\alpha)/\alpha \quad (2.21)$$

is the negative of the FIDO-measured gravitational acceleration; and the covariant derivative in three-dimensional absolute space is denoted by a slash $|i$. The auxiliary quantities ρ_e, $\mathbf{j}$, $\mathbf{E}$ are to be found from Eqs. (2.8), (2.9), and (2.12). If we replace ϵ, $\mathbf{S}$, $\overleftrightarrow{\mathbf{W}}$ in (2.19) and (2.20) by the appropriate expressions for a fluid with dissipation and retain a finite conductivity σ, what we get are the "imperfect" general GRMHD equations.[8] However since we will concentrate on perfect MHD in this paper we shall replace ϵ, $\mathbf{S}$, $\overleftrightarrow{\mathbf{W}}$ by their perfect-fluid expressions in (2.18). When this is done, the FIDO-measured law of force balance (2.20) becomes

$$\left[\left[\rho_0\gamma^2\mu+\frac{\mathbf{B}^2}{4\pi}\right]\gamma_{ij}+\rho_0\gamma^4\mu V_iV_j-\frac{1}{4\pi}B_iB_j\right]\frac{DV^j}{D\tau}+\rho_0\gamma^2\mu V_i\frac{D\mu}{D\tau}-\left[\frac{\mathbf{B}^2}{4\pi}\gamma_{ij}-\frac{1}{4\pi}B_iB_j\right](V^j{}_{|k}V^k)$$
$$=-\rho_0\gamma^2\mu\left[a_i-\frac{1}{\alpha}\beta_{j|i}V^j-(\mathcal{L}_t\gamma_{ij})V^j\right]-p_{|i}+\frac{1}{4\pi}(\mathbf{V}\times\mathbf{B})_i\nabla\cdot(\mathbf{V}\times\mathbf{B})-\frac{1}{8\pi\alpha^2}(\alpha\mathbf{B})^2{}_{|i}+\frac{1}{4\pi\alpha}(\alpha B_i)_{|j}B^j$$
$$-\frac{1}{4\pi\alpha}(\mathbf{B}\times\{\mathbf{V}\times[\nabla\times(\alpha\mathbf{V}\times\mathbf{B})+(\mathbf{B}\cdot\nabla)\boldsymbol{\beta}]+(\mathbf{V}\times\mathbf{B})\cdot\nabla\boldsymbol{\beta}\})_i . \quad (2.22)$$

Here a subscript i on a vector quantity means the i component of that vector;

$$\mu\equiv\frac{\rho+p}{\rho_0} \quad (2.23)$$

is the specific enthalpy of the fluid [and also the inertial mass per unit rest mass, cf. Exercise 5.4 of Misner, Thorne, and Wheeler[16] (MTW)]; and ρ_0 is the fluid's rest mass density. In deriving the above equation, the local law of conservation of rest mass [3+1 version of $(\rho_0u^\mu)_{;\mu}=0$]

$$\frac{D\rho_0}{D\tau}+\rho_0\gamma^2\mathbf{V}\cdot\frac{D\mathbf{V}}{D\tau}+\frac{\rho}{\alpha}\left[\frac{1}{2g}g_{,t}+\nabla\cdot(\alpha\mathbf{V}-\boldsymbol{\beta})\right]=0 \quad (2.24)$$

was used.

Here we deliberately will not make the law of energy conservation (2.19) explicit because in perfect MHD, a combination of (2.19) and (2.20) is easier to use. We shall turn to this in some detail in the next subsection.

Because of the underlying plasma processes, where the fluid particles are locked onto magnetic field lines, it is easier for fluid to move along magnetic field lines than across them. If we think of the coefficient of $DV^j/d\tau$ on the left-hand side (LHS) of (2.22) as an "effective inertia," we can clearly see this anisotropy in the fluid's inertia caused by the magnetic field. The quartic term in γ on the LHS is a relativistic correction: fluid is harder to accelerate at higher speed. The second term on the LHS of (2.22) represents the force needed for a moving fluid when its specific enthalpy is changing. The last term on the LHS is a correction to the first inertial term. On the right-hand side (RHS) of (2.22), the first term in the large parentheses is the standard gravitational acceleration (due to failure of the FIDO's to fall freely); the second term in that set of parentheses is the gravitomagnetic acceleration; and the third term comes from the coupling of the motion of the fluid to nonstatic spatial curvature. The second term on the RHS is the familiar pressure gradient. The third and fourth terms on the RHS are just the Coulomb and Lorentz forces. The curl of the $\alpha\mathbf{V}\times\mathbf{B}$ term is the coupling of the induced electric field to the fluid velocity and the magnetic field. The rest of the term comes from the coupling of the magnetic field to the gravitomagnetic field, a force underlying the Blandford-Znajek effect.

D. Thermodynamic variables

In perfect MHD there is no Ohmic dissipation nor viscous loss, so entropy is strictly conserved locally. Therefore we can write the first law of thermodynamics as

$$d\rho=\frac{\rho+p}{\rho_0}d\rho_0 \quad (2.25)$$

as seen in the fluid's rest frame, or

$$\frac{D\rho}{D\tau}=\mu\frac{D\rho_0}{D\tau} . \quad (2.25')$$

We can also derive Eq. (2.25) from the law of energy conservation as seen by the fluid $T^{\alpha\beta}{}_{;\alpha}u_\beta=0$ plus the frozen-in condition and the conservation of rest mass, or equivalently from a linear combination of (2.19) and

(2.20) plus (2.6) and (2.24). Notice that in Eq. (2.25) only the fluid's variables appear; and energy conservation has the same form as for an ordinary fluid with no magnetic field. This comes from the perfect MHD assumption that in the fluid's rest frame there is no electric field and therefore no exchange of energy between the fluid and magnetic field. In our MHD equations we choose to use (2.25) instead of (2.19). Of course our MHD equations are not complete without an equation of state

$$F(\rho_0,p,s)=0, \tag{2.26}$$

where s is the specific entropy. In this paper we assume, for simplicity, that the system of equations for perfect MHD is closed by a barotropic equation of state (i.e., s is constant throughout the fluid and, of course, constant in time)

$$p=p(\rho_0) \tag{2.26'}$$

and correspondingly ρ can be computed once and for all from [cf. Eq. (2.25)]

$$\rho=\rho(\rho_0)=\rho_0\int\frac{p}{\rho_0^2}d\rho_0. \tag{2.27}$$

To summarize, in our 3+1 equations for perfect MHD the basic variables are the FIDO-measured magnetic field $\mathbf{B}$ and fluid velocity $\mathbf{V}$, and the rest-mass density ρ_0 as measured in the fluid's rest frame. The total density of mass-energy ρ and pressure p (in the fluid rest frame) are computed from ρ_0 via Eqs. (2.26′) and (2.27); the FIDO-measured electric field $\mathbf{E}$ is computed from $\mathbf{V}$ and $\mathbf{B}$ via Eq. (2.12); the FIDO-measured current density $\mathbf{j}$ and charge density ρ_e are computed from Eqs. (2.8) and (2.9); the magnetic field $\mathbf{B}$ is evolved via Eq. (2.13); the fluid velocity $\mathbf{V}$ is evolved via Eq. (2.22); and the rest-mass density ρ_0 is evolved via Eq. (2.24). These are the perfect GRMHD equations in their most general form. Using these equations, we can study stationary configurations, dynamic evolution of conducting fluid with appropriate boundary conditions, or a small perturbation to an equilibrium state.

III. GRMHD IN A STATIONARY, SYMMETRIC BACKGROUND

In this section we restrict attention to a stationary spacetime with one spatial symmetry and demand that the MHD flow have the same symmetries. More specifically, we assume that spacetime has a timelike Killing vector field (KVF) $\boldsymbol{k}=\partial/\partial t$ and a spacelike KVF $\mathbf{m}=\partial/\partial\xi$ which commute with each other, and we insist that all fluid and electromagnetic quantities have vanishing Lie derivatives along $\boldsymbol{k}$ and $\mathbf{m}$. Moreover, we also insist (as is the case for a rotating, Kerr black hole) that the gravitomagnetic potential point along the symmetry direction $\mathbf{m}$,

$$\boldsymbol{\beta}\equiv\beta\mathbf{m}. \tag{3.1}$$

These restrictions guarantee that (i) our metric (2.1) will be independent of t and ξ; (ii) $\boldsymbol{\beta}$ and $\mathbf{m}$ will both lie in the hypersurfaces of constant time; i.e., they are three-vectors in absolute space; (iii) the congruence of FIDO world lines will not expand,

$$\theta=0; \tag{3.2}$$

and (iv) the gravitational acceleration will have vanishing projection along $\boldsymbol{\beta}$:

$$\mathbf{a}\cdot\boldsymbol{\beta}=0. \tag{3.3}$$

We shall see how 3+1 electrodynamics can be simplified under these conditions (Sec. III A) and how the conservation laws associated with these KVF's can be used to simplify the analysis of equilibrium configurations (Secs. III B and III C). This discussion is a 3+1 treatment of Phinney,[7] and an extension to MHD of Macdonald and Thorne[12] (but with the spacetime slightly more general than in those cases).

A. Electrodynamics

To study electrodynamics in a stationary, symmetric spacetime, we first introduce, as auxiliary quantities used in intermediate steps, some special components of the four-vector potential. They are[15]

$$\mathbf{A}\equiv\overleftrightarrow{\gamma}\cdot(\text{four-vector potential})$$
$$=(\text{three-vector potential living in three-dimensional absolute space}), \tag{3.4}$$

$$A_0\equiv\boldsymbol{k}\cdot(\text{four-vector potential})$$
$$=(\text{time component of four-vector potential}), \tag{3.5}$$

$$A_\xi\equiv\mathbf{m}\cdot(\text{four-vector potential})$$
$$=\mathbf{m}\cdot(\text{three-vector potential})$$
$$=(\xi\text{ component of three-vector or four-vector potential}). \tag{3.6}$$

Using these potentials, we can write the electric and magnetic fields as [Eqs. (5.9) and (5.10) of Ref. 15]

$$\mathbf{E}=\frac{1}{\alpha}(\nabla A_0-\beta\nabla A_\xi), \tag{3.7}$$

$$\mathbf{B}=\nabla\times\mathbf{A}. \tag{3.8}$$

Because A_ξ will play an important role in determining the structure of stationary MHD flows, let us examine the physical content of A_ξ first. Consider a curve C in absolute space with tangent vector $\mathbf{m}$ (i.e., an "integral curve" of $\mathbf{m}$) and a magnetic flux tube bounded by C (see Fig. 1). The magnetic flux Ψ inside such a flux tube is related to A_ξ in the following way:

$$\Psi(\mathbf{x})=\int\int_S\mathbf{B}\cdot d\mathbf{S}=\oint_C A_\xi d\xi. \tag{3.9}$$

Here Ψ, regarded as a scalar field in absolute space, has the above value at any point $\mathbf{x}$ that lies on curve C. Because $\mathbf{m}\equiv\partial/\partial\xi$ is assumed to be a KVF, A_ξ is indepen-

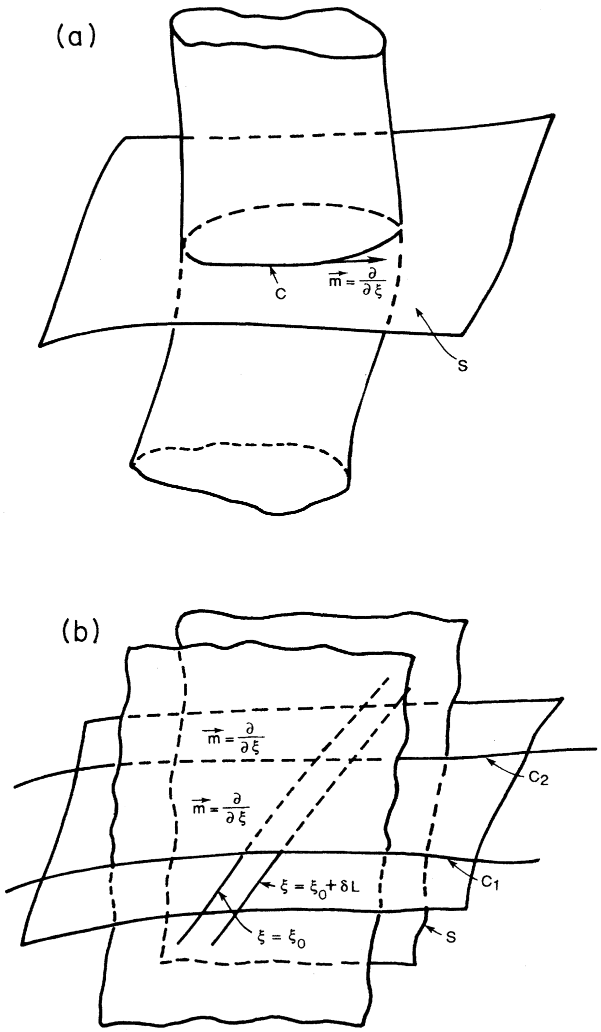

FIG. 1. Magnetic flux inside flux tube bounded by C is $\Psi(\mathbf{x})=\oint_C A_\xi d\xi = A_\xi \oint_C d\xi$. (a) When Ψ and $\oint_C d\xi$ are finite, A_ξ is clearly the flux per unit length of C within the tube; (b) when $\oint_C d\xi$ is not finite, e.g., if $\mathbf{m}$ is a translational symmetry and C extends to infinity, Ψ is infinite unless A_ξ vanishes. For any finite A_ξ, we use the ξ symmetry and consider only a small portion δL of curve C. Then $\delta\Psi = A_\xi \delta L$ and we can still think of A_ξ as the flux per unit length of C within the flux tube.

dent of ξ; so we can consider A_ξ as the flux per unit ξ length of C within the flux tube. In an axisymmetric spacetime, ξ is equal to the angle ϕ around the axis of symmetry, the line integral (3.9) runs from 0 to 2π, and $\Psi(\mathbf{x})$ is the flux inside the circle that passes through $\mathbf{x}$ and is generated by $\partial/\partial\xi$. In a translation-symmetric spacetime ξ runs from $-\infty$ to $+\infty$ and for finiteness we make use of the ξ symmetry and restrict the line integral (3.9) to a small, fixed interval [say $\xi_0(\mathbf{x})<\xi<\xi_0(\mathbf{x})+\delta L$ for some small δL], and regard $\Psi(\mathbf{x})$ as the flux in a tube bounded by (i) the integral curve of $\mathbf{m}$ through $\mathbf{x}$, (ii) a curve at $\xi=\xi_0(\mathbf{x})$, (iii) a curve at $\xi=\xi_0(\mathbf{x})+\delta L$, and (iv) some fixed fiducial integral curve of $\mathbf{m}$.

Before we try to relate A_ξ to $\mathbf{B}$ and $\mathbf{E}$, let us first decompose, for later convenience, any vector $\mathbf{U}$ in absolute space into its ξ component U^ξ and the part $\mathbf{U}^P$ perpendicular to $\mathbf{m}=\partial/\partial\xi$:

$$\mathbf{U}=\mathbf{U}^P+U^\xi \mathbf{m}. \tag{3.10}$$

(The superscript P stands for "poloidal"—a terminology adapted from the axisymmetric case where ξ is the angle around the symmetry axis.) Then it turns out that (see below)

$$\mathbf{V}^P \| \mathbf{B}^P. \tag{3.11}$$

Here $\mathbf{V}^P$ and $\mathbf{B}^P$ are the "poloidal" parts of the fluid velocity and magnetic field, respectively. Thus the poloidal magnetic field lines coincide with the fluid's poloidal stream lines on surfaces orthogonal to $\mathbf{m}$. This can be derived by taking a scalar product of (3.7) with $\mathbf{m}$ and then using the ξ symmetry and the frozen-in condition (2.6) to conclude that

$$\mathbf{m}\cdot(\mathbf{V}\times\mathbf{B})=0, \tag{3.12}$$

or

$$\mathbf{m}\cdot(\mathbf{V}^P\times\mathbf{B}^P)=0, \tag{3.13}$$

which says that $\mathbf{m}$, $\mathbf{V}^P$, and $\mathbf{B}^P$ are not linearly independent; hence the statement in (3.11). Because $\mathbf{V}^P$ and $\mathbf{B}^P$ are parallel, we can define a proportionality coefficient k to relate them. By a careful choice, it is defined as[3,7]

$$\mathbf{V}^P \equiv \frac{k}{4\pi\alpha\rho_0\gamma}\mathbf{B}^P. \tag{3.14}$$

With this choice we can show, using (2.6) and (2.24), that

$$\mathbf{B}^P\cdot\nabla k=0. \tag{3.15}$$

The proof involves combining the stationary, symmetric versions of the law of mass conservation (2.24)

$$\nabla\cdot(\alpha\rho_0\gamma\mathbf{V}^P)=0 \tag{3.16}$$

and the law of flux conservation $\nabla\cdot\mathbf{B}^P=0$. Therefore k will be constant on magnetic surfaces, or flux tubes, though it typically will vary from one magnetic surface to another.

Now we wish to find a relation between $\mathbf{B}$ and A_ξ. The argument here is that of Thorne and Macdonald.[15] Consider an integral curve C' of $\mathbf{m}$ which differs slightly from C. The flux per unit length of C, between C and C', is

$$dA_\xi=(\nabla A_\xi)\cdot d\mathbf{x}=\mathbf{B}\cdot(\mathbf{m}\times d\mathbf{x})=(\mathbf{m}\times\mathbf{B})\cdot d\mathbf{x}, \tag{3.17}$$

where $d\mathbf{x}$ is any vector reaching from C to C'. Because C' is arbitrary, $d\mathbf{x}$ is also arbitrary, and we thus have

$$\nabla A_\xi=\mathbf{m}\times\mathbf{B}=\mathbf{m}\times\mathbf{B}^P. \tag{3.18}$$

By taking a cross product with $\mathbf{m}$, we can invert (3.18) to obtain

$$\mathbf{B}^P=-\frac{\mathbf{m}\times\nabla A_\xi}{\gamma_{\xi\xi}}. \tag{3.19}$$

Here, $\gamma_{\xi\xi}=\mathbf{m}\cdot\mathbf{m}$ is the "$\xi\xi$ component" of the three-metric. From (3.19) we can deduce that

$$\mathbf{B}^P\cdot\nabla A_\xi=0. \tag{3.20}$$

This guarantees that A_ξ, as is k, is a constant on magnetic surfaces and that k can be regarded as a function of A_ξ:

$$k=k(A_\xi). \tag{3.21}$$

The coefficient k is called the stream function in Phinney[7] because it is constant along a "stream line," i.e., along an integral curve of $\mathbf{V}=\mathbf{V}^P+V^\xi\mathbf{m}$. As we shall see in Sec. III C, stream functions play an important role in determining the structure of stationary flows.

We can also express $\mathbf{E}$ in terms of the gradient of A_ξ. The frozen-in condition (2.12) implies that $\mathbf{E}$ is orthogonal to $\mathbf{B}$; and thus (2.12) plus Eqs. (3.7) and (3.20) and ξ symmetry implies

$$\mathbf{B}^P\cdot\nabla A_0=0, \tag{3.22}$$

which in turn implies that

$$A_0=A_0(A_\xi). \tag{3.23}$$

Since $\nabla^P A_0$ and $\nabla^P A_\xi$ are both poloidal (by ξ symmetry) and are both orthogonal to $\mathbf{B}^P$, they must be parallel to each other:

$$\nabla A_0=-V^F\nabla A_\xi, \tag{3.24}$$

for some scalar field V^F. Correspondingly, Eq. (3.7) implies

$$\mathbf{E}=-\frac{1}{\alpha}(\boldsymbol{\beta}+V^F)\nabla A_\xi=-\frac{1}{\alpha}(V^F\mathbf{m}+\boldsymbol{\beta})\times\mathbf{B}. \tag{3.25}$$

By taking derivatives along $\mathbf{B}^P$ of both sides of (3.24) and using (3.20) and (3.22) we conclude that V^F must be a function of A_ξ, i.e., $\mathbf{B}^P\cdot\nabla V^F=0$, and V^F is also a stream function. We can think of V^F as the coordinate speed of the magnetic field, because observers who move with $d\xi/dt=V^F$ [i.e., at velocity $(V^F\mathbf{m}+\boldsymbol{\beta})/\alpha$ as measured by FIDO's] see an electric field

$$\mathbf{E}'\propto\left[\mathbf{E}+\left[\frac{V^F\mathbf{m}+\boldsymbol{\beta}}{\alpha}\right]\times\mathbf{B}\right]=0 \tag{3.26}$$

that vanishes; i.e., they regard the magnetic field as at rest with respect to themselves. A comparison of Eqs. (2.12) and (3.26) gives us an algebraic relation

$$V^\xi=\frac{kB^\xi}{4\pi\alpha\rho_0\gamma}+\frac{V^F+\beta}{\alpha}, \tag{3.27}$$

which relates V^ξ to B^ξ.

B. MHD flow

For the stationary and symmetric MHD flow, since the FIDO's move along symmetry directions, $d(\text{everything})/d\tau=0$. FIDO's do not see any changes in the MHD flow around themselves. Moreover, associated with the two KVF's, we have two conserved fluxes [Eqs. (3.65b) and (3.69b) of Ref. 10]

$$\mathbf{S}_{p_\xi}\equiv\overleftrightarrow{\mathbf{W}}\cdot\mathbf{m},\quad \mathbf{S}_{E_\infty}\equiv\alpha\mathbf{S}-\boldsymbol{\beta}\cdot\overleftrightarrow{\mathbf{W}}, \tag{3.28}$$

whose products with the lapse function α are divergence free under stationary, symmetric assumptions:

$$\nabla\cdot(\alpha\mathbf{S}_{p_\xi})=0,\quad \nabla\cdot(\alpha\mathbf{S}_{E_\infty})=0 \tag{3.29}$$

[Eqs. (3.67) and (3.71) of Ref. 10]. Here $\mathbf{S}_{p_\xi}$ is the flux of the ξ component of momentum, and $\mathbf{S}_{E_\infty}$ is the flux of energy at infinity, or "red-shifted energy." Because of the symmetry, the ξ components of these fluxes give identically zero contribution to the conservation laws (3.29). Thus, the poloidal parts of the fluxes also satisfy the conservation laws (3.29) and we shall concentrate attention on them. Using expressions (3.14) and (3.25) for $\mathbf{E}$ and $\mathbf{B}$, we find

$$\mathbf{S}^P_{p_\xi}=(\mu\gamma V_\xi-\alpha B_\xi/k)\rho_0\gamma\mathbf{V}^P, \tag{3.30}$$

$$\mathbf{S}^P_{E_\infty}=[\gamma\mu(\alpha-\beta V_\xi)-V^F\alpha B_\xi/k]\rho_0\gamma\mathbf{V}^P, \tag{3.31}$$

which allows us to introduce two more stream functions:[7]

$$l\equiv\gamma\mu V_\xi-\alpha B_\xi/k,\quad l=l(A_\xi), \tag{3.32}$$

$$e\equiv\gamma\mu(\alpha+\beta V_\xi)-V^F\alpha B_\xi/k,\quad e=e(A_\xi). \tag{3.33}$$

That l and e are indeed stream functions (i.e., are constant along $\mathbf{B}^P$ and thus are expressible as functions of A_ξ) can be verified directly from (3.16) and (3.29)–(3.33). We can rewrite Eqs. (3.30) and (3.31), using l and e, as

$$\mathbf{S}^P_{p_\xi}=l(A_\xi)\rho_0\gamma\mathbf{V}^P, \tag{3.30$'$}$$

$$\mathbf{S}^P_{E_\infty}=e(A_\xi)\rho_0\gamma\mathbf{V}^P. \tag{3.31$'$}$$

As pointed out by Phinney[7] and also quite obvious here, l and e can be interpreted as the covariant ξ component of momentum (henceforth the "generalized momentum") and the energy at infinity carried by unit rest mass of fluid. Sometimes, especially when seeking solutions to the "wind equation" [Eq. (3.46) below], a combination of e and l, the field-rest-frame specific energy f, is more useful in determining the flow structure. It is defined as

$$f\equiv e-V^F l=\gamma\mu(\alpha-CV_\xi), \tag{3.34}$$

where

$$C\equiv\beta+V^F. \tag{3.35}$$

Using Eqs. (3.29)–(3.31) we can also write the extraction rate of ξ component of momentum, $\dot{L}$, and that of energy at infinity, $\dot{M}$, in terms of stream functions l, e, and k. Consider two flux tubes S_1 and S_2, which are bounded separately by integral curves C_1 and C_2 of $\mathbf{m}$ on a two-dimensional surface S_{up} with its normal orthogonal to $\mathbf{m}$ (see Fig. 2). Let us assume that S_{up} is located in a nearly flat region, so the total fluxes of generalized momentum and energy at infinity across it between S_1 and S_2 can be regarded as the rates of extraction of these quantities from the strong gravity region, e.g., leaving a surface S_{down} whose normal is also orthogonal to $\mathbf{m}$ inside that region. Using Eq. (3.29) we see that

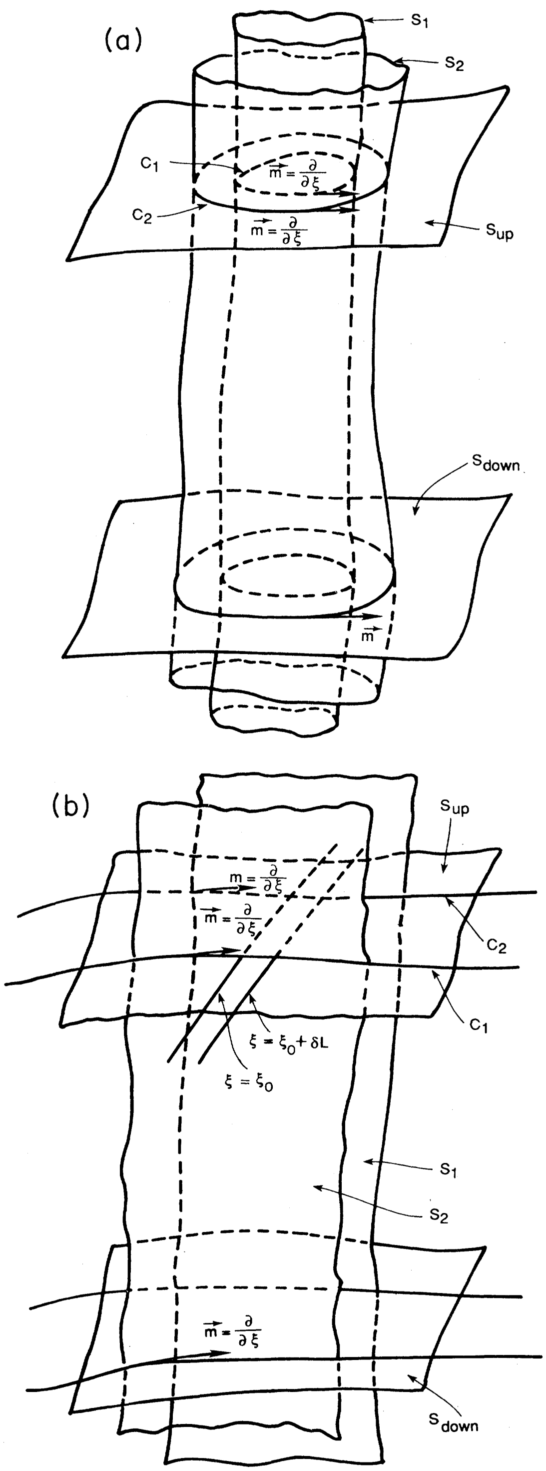

FIG. 2. (a) The generalized momentum and energy are transported from one region to another between flux tubes S_1 and S_2. On a two-dimensional surface $S_{\rm up}$, whose normal is orthogonal to $\mathbf{m}$, these flux tubes are bounded by integral curves C_1 and C_2 of $\mathbf{m}$. $S_{\rm down}$ is another two-dimensional surface whose normal is also orthogonal to $\mathbf{m}$. (b) When the length of integral curves of $\mathbf{m}$ is unbounded, e.g., when $\mathbf{m}$ corresponds to a translation, we consider a portion of flux sheets S_1 and S_2 of width δL and the generalized momentum and energy at infinity transported within δL between integral curves C_1 and C_2.

$$\dot{L}\equiv\int_{S_{\rm up}}\alpha\mathbf{S}_{P_\xi}\cdot d\mathbf{S}=\int_{S_{\rm up}}\alpha\mathbf{S}^P_{P_\xi}\cdot d\mathbf{S}$$
$$=-\int_{S_{\rm down}}\alpha\mathbf{S}^P_{P_\xi}\cdot d\mathbf{S}, \tag{3.36}$$

$$\dot{M}\equiv\int_{S_{\rm up}}\alpha\mathbf{S}_{E_\infty}\cdot d\mathbf{S}=\int_{S_{\rm up}}\alpha\mathbf{S}^P_{E_\infty}\cdot d\mathbf{S}$$
$$=-\int_{S_{\rm down}}\alpha\mathbf{S}^P_{E_\infty}\cdot d\mathbf{S}. \tag{3.37}$$

Thus, in stationary perfect GRMHD, the flux tubes act as wires in a dc circuit, guiding energy and momentum from one place to another without dissipation.[12] To study how the energy is converted into photons that observers can see requires an analysis that goes beyond perfect MHD.[14] Of course, this does not render perfect GRMHD uninteresting: perfect GRMHD is still a good approximation in which we can understand how the energy is transported from the strong gravity region to the dissipation region, where gravity is usually weak and non-general-relativistic analysis would be adequate. To express $\dot{L}$ and $\dot{M}$ in terms of stream functions, we rewrite $\mathbf{S}^P_{P_\xi}$ and $\mathbf{S}^P_{E_\infty}$ using (3.30′) and (3.31′), and we also make use of Eqs. (3.14), (3.18), and (3.19) to obtain

$$\dot{L}=\int_{S_{\rm up}}\alpha l\rho_0\gamma\mathbf{V}^P\cdot d\mathbf{S}=\int_{S_{\rm up}}\frac{lk}{4\pi}\mathbf{B}^P\cdot d\mathbf{S}$$
$$=-\frac{1}{4\pi}\int_{S_{\rm up}}\frac{lk(\mathbf{m}\times d\mathbf{S})\cdot\nabla A_\xi}{\gamma_{\xi\xi}}.$$

On $S_{\rm up}$,

$$d\mathbf{S}=d\mathbf{x}^P\times(\mathbf{m}d\xi);$$

therefore,

$$\dot{L}=-\frac{1}{4\pi}\int_{A_\xi^{(1)}}^{A_\xi^{(2)}}lk\,dA_\xi\int_C d\xi, \tag{3.38}$$

where $A_\xi^{(1)}$ is the value of A_ξ on flux tube S_1, $A_\xi^{(2)}$ is the value of A_ξ on flux tube S_2. To also include the case where $\int_C d\xi\to\infty$ we use $\dot{\mathcal{L}}$, the extraction rate per unit ξ length of momentum between flux tubes S_1 and S_2. Then

$$\dot{\mathcal{L}}=-\frac{1}{4\pi}\int_{A_\xi^{(1)}}^{A_\xi^{(2)}}lk\,dA_\xi. \tag{3.39}$$

Similarly, for the extraction rate per unit length of energy at infinity, we have

$$\dot{\mathcal{M}}=-\frac{1}{4\pi}\int_{A_\xi^{(1)}}^{A_\xi^{(2)}}ek\,dA_\xi. \tag{3.40}$$

For a Kerr black hole we shrink S_1 to the hole's symmetry axis and let S_2 approach the equatorial plane. Then Eqs. (3.39) and (3.40) become (7.5) of Ref. 7 modulo a factor of 4π (2π from the length $\int d\phi$, 2 from the reflection symmetry about the equatorial plane assumed for the MHD flow).

C. The wind equation and its solutions

For our stationary, symmetric system, we have just seen how most of the GRMHD equations can be integrated once, giving us one algebraic relation and five stream functions A_0, V^F, k, l, and e or f, whose values are determined by boundary conditions. From these and one other algebraic constraint

$$\gamma^2(1-\mathbf{V}^2)=1 \tag{3.41}$$

we can find out all the components of $\mathbf{V}$ and $\mathbf{B}$ once we have solved for A_ξ. To solve for A_ξ, we can express $\mathbf{V}$, $\mathbf{B}$, $\rho_0, \ldots,$ in terms of stream functions and metric coefficients, and then substitute them into the remaining unintegrated force-balance equation. The result will be a second-order nonlinear partial differential equation. In general we do not have analytic solutions to this equation. In the following, as did Phinney[7] (on whose work our 3+1 analysis is modeled), we will assume that this equation has already been solved for A_ξ and will leave the actual solution to numerical work; and following Phinney we shall concentrate on (3.41) and examine what constraint it puts on the stream functions in addition to those demanded by boundary conditions.

More specifically, what we shall do is express $\mathbf{V}$ and γ completely in terms of stream functions and metric coefficients, and then insert them into (3.41) to get a so-called "wind equation" for $V^P \equiv |\mathbf{V}^P|$. Actually, it turns out that, using the components of four-velocity,

$$\mathcal{V}_\xi \equiv \gamma V_\xi, \tag{3.42a}$$

$$\mathcal{V}^P \equiv \gamma \mathbf{V}^P, \tag{3.42b}$$

makes the calculation simpler than working with V_ξ and $\mathbf{V}^P$ themselves. To derive the wind equation, we first use the definitions for l and f [Eqs. (3.32) and (3.34)] to eliminate B^ξ and γ from Eq. (3.27) and get an expression for $\mathcal{V}_\xi$:

$$\mathcal{V}_\xi = \frac{\alpha k l \mathcal{V}^P - C f \gamma_{\xi\xi} B^P/\mu}{\alpha k \mu \mathcal{V}^P + \gamma_{\xi\xi} B^P C^2 - \alpha^2 B^P}. \tag{3.43}$$

Here and below, we are to regard $\mathbf{B}^P$ as a function of the "known" quantity A_ξ, given by Eq. (3.19). By substituting (3.43) into (3.34), we get an expression for γ also in terms of known quantities and $\mathcal{V}^P$:

$$\gamma = \frac{(f+Cl)k\mu\mathcal{V}^P - \alpha f B^P}{\mu(\alpha k \mu \mathcal{V}^P + \gamma_{\xi\xi} B^P C^2 - \alpha^2 B^P)}. \tag{3.44}$$

In terms of $\mathcal{V}_\xi$ and $\mathcal{V}^P$, (3.41) is

$$\gamma^2 - \mathcal{V}_\xi^2/\gamma_{\xi\xi} - (\mathcal{V}^P)^2 = 1, \tag{3.45}$$

which can be manipulated into the form

$$D \equiv K \frac{(\mathcal{V}^P - F_1)(\mathcal{V}^P - F_2)}{(\mathcal{V}^P - F_3)^2} - (\mathcal{V}^P)^2 - 1 = 0. \tag{3.46}$$

Here,

$$K = \frac{(f+Cl)^2 k^2 \gamma_{\xi\xi} - \alpha^2 k^2 l^2}{\alpha^2 k^2 \mu^2 \gamma_{\xi\xi}}, \tag{3.47a}$$

$$F_1 = \frac{\sqrt{\gamma_{\xi\xi}}(\alpha + \sqrt{\gamma_{\xi\xi}} C f)}{\mu k[(f+Cl)\sqrt{\gamma_{\xi\xi}} + \alpha l]} B^P, \tag{3.47b}$$

$$F_2 = \frac{\sqrt{\gamma_{\xi\xi}}(\alpha - \sqrt{\gamma_{\xi\xi}} C f)}{\mu k[(f+Cl)\sqrt{\gamma_{\xi\xi}} - \alpha l]} B^P, \tag{3.47c}$$

$$F_3 = \frac{\alpha^2 - C^2 \gamma_{\xi\xi}}{\alpha k \mu} B^P. \tag{3.47d}$$

Equation (3.46), with (3.47) substituted in, is the "wind equation" from which we can determine the structure of MHD flows.[7,17]

For the special case of an "isothermal equation of state,"

$$\mu = \frac{\rho + p}{\rho_0} = \text{const}, \tag{3.48}$$

all quantities in the wind equation except $\mathcal{V}^P$ can be regarded as known from the solution for A_ξ and from boundary conditions. Thus, the wind equation (3.46) can be solved for $\mathcal{V}^P$, and the remaining flow structure can be computed algebraically from Eqs. (3.14), (3.27), (3.43), and (3.44). Thus, the flow structure is determined from (3.46) completely. But for more realistic equations of state

$$\mu = \mu(\rho_0), \tag{3.49}$$

we must use (3.49) to eliminate μ from Eqs. (3.43)–(3.45); and it then may be easiest to use (3.45) directly as our wind equation. For ease of discussion below, we will assume an isothermal equation of state. This type of equation of state is of interest also because it includes the cold-flow limit, $p = 0$ (Ref. 7).

For continuous flows (no shocks) the solutions to the wind equation (3.45) should extend smoothly from the region of interest to spatial infinity. However, for an arbitrary set of stream functions, D will generally become singular at critical surfaces where the flow speed equals one of the perturbation propagation speeds inside the stationary flow. We either have no solutions beyond these critical surfaces, or if energy and momentum conservation permit, we have shocks. To avoid such situations we have to constrain the stream functions in such a way that either these critical surfaces are pushed to or beyond spatial infinity or the solutions pass through the critical surfaces smoothly.[7,17] To make our discussion more concrete, let us assume that we have parametrized the stream lines by a parameter y^P, which can be regarded as the coordinate length along the stream lines. Then the constraint equations for smooth passage through a critical surface as found by Kennel, Fujimura, and Okamoto[17] for special-relativistic flow, which are also true general relativistically, are (see also Chap. V of Ref. 7)

$$\left.\frac{\partial D}{\partial \mathcal{V}^P}\right|_{\mathcal{V}^P = \mathcal{V}_c^P,\, y^P = y_c^P} = 0, \tag{3.50a}$$

$$\left.\frac{\partial D}{\partial y^P}\right|_{\mathcal{V}^P = \mathcal{V}_c^P,\, y^P = y_c^P} = 0, \tag{3.50b}$$

where $\mathcal{V}_c^P$ is the flow speed on one of the critical surfaces, and y_c^P denotes one of the locations of the critical surfaces. Solutions with shocks will introduce many interesting processes into our problem. However to handle shocks well, more careful analysis is required. In particular, to determine the structures of shocks, a detailed analysis of full plasma theory is required (see, for example, Ref. 18). Such an analysis is far beyond the scope of this paper.

IV. CONCLUSION

In this paper the GRMHD equations were rewritten in $3+1$ language in a general spacetime. They were expressed as (i) evolution equations for the FIDO-measured magnetic field $\mathbf{B}$ and flow velocity $\mathbf{V}$ [Eqs. (2.13) and (2.22)], and the fluid's rest mass density ρ_0 [Eq. (2.24)]; (ii) the frozen-in condition of perfect MHD [Eq.(2.12)]; and (iii) algebraic constraining equations on the magnetic field [Eq. (2.6)] and thermodynamic variables [the equation of state (2.26′) or (2.27)]. Then for a stationary, symmetric flow in a stationary, symmetric spacetime these equations were reduced to a wind equation (3.46) from which one determines $\mathcal{V}^P$ (given A_ξ as known), and the algebraic relations (3.14), (3.27), (3.43), and (3.44) from which one computes ρ_0, $\mathbf{V}^P$, V^ξ, and B^ξ for an isothermal equation of state. With A_ξ given or calculated from a nonlinear partial differential equation derived from (2.22), $\mathbf{B}^P$ is calculated using (3.19).

In a future paper the author will use this formalism to build stationary, symmetric MHD model magnetospheres, and will linearize the evolution equations to study dynamic perturbations of those magnetospheres so as to gain insight into the dynamical effects of the coupling of the magnetic field to the gravitomagnetic field.

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