Decoherence in the density matrix describing quantum three-geometries and the emergence of classical spacetime

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We construct the quantum gravitational density matrix $\rho(g_{\alpha\beta},g'_{\alpha\beta})$ for compact three-geometries by integrating out a set of unobserved matter degrees of freedom from a solution to the Wheeler-DeWitt equation $\Psi[g_{\alpha\beta},q_{k(\text{matter})}]$. In the adiabatic approximation, ρ can be expressed as $\exp(-l^2)$ where $l^2(g_{\alpha\beta},g'_{\alpha\beta})$ is a specific "distance" measure in the space of three-geometries. This measure depends on the volumes of the three-geometries and the eigenvalues of the Laplacian constructed from the three-metrics. The three-geometries which are "close together" ($l^2 \ll 1$) interfere quantum mechanically; those which are "far apart" ($l^2 \gg 1$) are suppressed exponentially and hence contribute decoherently to ρ. Such a suppression of "off-diagonal" elements in the density matrix signals classical behavior of the system. In particular, three-geometries which have the same intrinsic metric but differ in size contribute decoherently to the density matrix. This analysis provides a possible interpretation for the semiclassical limit of the wave function of the Universe.

I. INTRODUCTION AND SUMMARY

If gravitational effects are ignored, then physical interactions can be studied in a fixed, flat, Lorentzian spacetime. When the gravitational field is present this situation changes drastically. Since we cannot distinguish between the effects of gravity and that of a curved spacetime we can no longer work with a flat spacetime. Moreover, changes in the energy density of matter fields, which are inevitable in any nontrivial dynamical situation, will lead to a time-dependent spacetime structure. We have, therefore, to treat spacetime as a dynamical entity.

Classically, this can be done by using Einstein's equations. A classical solution to Einstein's equations will describe a dynamical spacetime evolving in consonance with the energy density of the matter fields.

Such a picture, however, cannot be completely correct. We know that the matter fields are described by a quantum theory and not by a classical theory. In particular, the laws governing the matter fields are of a probabilistic nature. The gravitational field produced by such a source should necessarily display this probabilistic character at some level.

The only consistent way of introducing such a probabilistic character into the description of spacetime is to quantize gravity as well.[1] In a fully quantized version of the theory, we expect the Universe to be described by a grand wave function $\Psi(g_A,q_n)$ which depends on both the gravitational degrees of the freedom g_A and the matter (field) degrees of freedom q_n. Broadly speaking, we expect the quantity $|\Psi(g_A,q_n)|^2$ to be proportional to the probability of occurrence of the values $[g_A]$ and $[q_n]$ "simultaneously." (To define the notion of simultaneity we have to use some matter variable as a "clock"; it is assumed that this nontrivial task can be accomplished.) We also hope that the expectation values of physical observables can be computed from Ψ in the usual manner.

This wave function satisfies the Wheeler-DeWitt equation which can be written, in a concise notation, as

$$[-\tfrac{1}{2}l^2\nabla^2+l^{-2}V(g_A)+H_m(p_n,q_n,g_A)]\Psi(g_A,q_n)=0 \ , \tag{1}$$

where g_A stands for the metric $g_{\alpha\beta}(\alpha,\beta=1,2,3)$ on the three-space, ∇^2 is the Laplacian in the superspace of three-geometries constructed using the DeWitt metric,[2] V is the superspace potential, l is the Planck length, and $H_m(p_n,q_n,g_A)$ is the matter Hamiltonian. While Eq. (1) offers a formal solution to the problem of quantizing gravity, it is not of much practical value. In addition to the technical difficulties in solving (1), which are formidable, we are also faced with several nontrivial interpretational issues.

Consider, for example, the following situation. Let Ψ_1 and Ψ_2 be two solutions of Eq. (1). We will assume that these two solutions are characterized by the following feature. The expectation values of a commuting set of physical observables, O_i ($i=1,2,\ldots,N$), are *macroscopically* different in these two states. That is, we assume $\langle\Psi_1|O_i|\Psi_i\rangle$ and $\langle\Psi_2|O_i|\Psi_2\rangle$ to be measurably different for all $i=1,2,\ldots,N$. So, if the Universe is in state $|\Psi_1\rangle$ or $|\Psi_2\rangle$, that fact can be easily ascertained by measuring the observables O_i.

But notice that the Wheeler-DeWitt equation is linear in Ψ. If $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are solutions of (1) then so is $|\Psi\rangle=a|\Psi_1\rangle+b|\Psi_2\rangle$. This is a state which is obtained by superposing two different states which macroscopically different values for certain physical variables. There is no *a priori* reason why the Universe could not be in this state. But our experience shows that this is not the case; the Universe behaves almost classically as far as macroscopic observations are concerned. Any sensible model for quantum gravity should attempt to explain this peculiar situation. This paper attempts to discuss this issue in some detail.

(One should be careful not to read too much into the notation used in the last two paragraphs. This notation is, of course, borrowed from standard quantum mechanics and has the connotations of a Hilbert space. It is not known whether such a Hilbert-space structure exists in a quantum cosmology; see the work of Hartle in Ref. 10.)

One can offer "explanations" for the above phenomenon at several different levels. For example, one can argue that this question is really one of "initial" or "boundary" conditions. Several suggestions exist in the literature explaining how one can choose a particular solution of Wheeler-DeWitt equation among all possible solutions.[3] Such suggestions, however, do not provide a real "explanation" unless the specific choice for the boundary condition can be motivated by an *independent* physical reason of compelling nature. The boundary conditions suggested in the literature so far do not seem to satisfy this criterion.

Another "explanation" involves the "collapse of the wave function." It is motivated by the following kind of argument. "The issue which is being raised has nothing to do with quantum gravity. We *never* see any system in a state which is a superposition of states with macroscopically different parameters.[4] For example, cats are never seen in states of the kind $a|\text{alive}\rangle+b|\text{dead}\rangle$. Similar considerations should apply to gravity." This argument does contain a grain of truth; indeed, we do not observe cats in such peculiar states. Probably we can understand the wave function of the Universe if we first understand cats.

The reason for the classical nature of cats is still controversial and is hotly debated among those who work in this subject. There are, essentially, two schools of thought. The first one advocates the "collapse of the wave function."[4] In this interpretation, the wave function merely quantifies the amount of information we have about the system; it is only natural that the wave function should change if we acquire more information about the system. Thus, according to this interpretation, the wave function evolves in two different ways: continuously, via the Schrödinger equation and discontinuously, when the measurements are performed on the system.

The second school of thought does not recognize such dichotomy. According to this school, any system with a large number of degrees of freedom will "automatically" behave as a classical system. This result arises from the dynamics (suppression of the off-diagonal elements of the density matrix) and, indeed, has been explicitly demonstrated for several simple systems.[5]

(It is probably worth making a brief comment at this stage regarding the "many worlds interpretation" of the wave function.[6] This interpretation merely replaces the *ad hoc* nature of the collapse by the untestable branching of one world into many. It does not add anything to the predictive power of the formalism and is devoid of any dynamical machinery. Therefore, as far as this paper is concerned, it is not necessary to discuss this approach separately.)

The continuing controversy among experts in this field suggest that, as far as cats are concerned, it is difficult to decide which interpretation is correct. The situation, however, could be different as regards the Universe. There are some serious difficulties in using the approach based on the "collapse of the wave function" to interpret the quantum state of the Universe. It seems more natural to adopt the second interpretation, based on dynamics, in discussing the wave function of the Universe. There are, essentially, two reasons for this. (i) In the scenarios which use the "collapse interpretation," it is always tacitly assumed that the information about the system is being acquired by an *external* agency in an irreversible manner. This is clearly not the case with the Universe. (ii) The collapse of the wave function requires a description involving temporal evolution. It is not easy to introduce this feature in a solution to Wheeler-DeWitt equation, which is a "timeless" equation. Because of these reasons, it seems worthwhile to consider the second approach seriously and work out its consequences for quantum gravity. This is what we plan to do in this paper.

The work described here is motivated by some recent papers by Zeh, Kiefer, and Joos.[7–9] Zeh and Kiefer consider the emergence of a semiclassical *time coordinate* due to the "continuous measurement" of a scale factor of a Friedmann universe by other degrees of freedom. This work is done in the context of minisuperspace models and supplements the work of several others[10,11] on the emergence of semiclassical time coordinate. Zeh and Kiefer however do not consider the emergence of classical *spacetime* from a general wave function. The paper by Joos does address this question but only in a qualitative manner using nonrelativistic concepts. The work described here may be considered as a generalization of the ideas which are implicit in the above papers. (For another recent application of these ideas in quantum cosmology, see Ref. 12.)

The basic idea and the result of the present work is as follows. Consider a solution to the Wheeler-DeWitt equation which can be adequately approximated in the form

$$\Psi(\gamma_{\alpha\beta},f_k)\approx\Psi_0(\gamma_{\alpha\beta})\psi(\gamma_{\alpha\beta},f_k)\ ,$$

where $\gamma_{\alpha\beta}$ is the three-metric on a compact three-space and f_k's describe the degrees of freedom of all the matter variables populating the Universe. Among these matter variables there will be a large number of modes which are unobserved. It is therefore necessary to use a density matrix while describing the quantum nature of the three-geometries. Such a density matrix for the three-geometries, $\rho(\gamma_{\alpha\beta},\gamma'_{\alpha\beta})$, can be obtained by "tracing out' the unobserved matter degrees of freedom. We compute this density matrix in the adiabatic approximation and show that its off-diagonal elements, corresponding to "widely different" three-geometries, are strongly suppressed. It is well known that such a suppression of off diagonal elements reduces the density matrix to a decoherent sum and signals classical behavior. This is how we obtain classical spacetime from the wave function of the Universe.

It is, of course, necessary to make precise what is meant by three-geometries which are "widely different." This requires a notion of some metric in the space of all three-geometries. An interesting feature of this work is the following. *The analysis automatically provides such a metric.* This metric is constructed from the three-geometries in the following manner.

Let us use natural, dimensionless, angular coordinates on the compact three-geometries. Then the metric $\gamma_{\alpha\beta}$ will have the dimensions of (length)^2. We scale this out by writing $\gamma_{\alpha\beta}$ as $R^2 g_{\alpha\beta}$ where R^3 is the volume of the compact three-space. From the dimensionless metric $g_{\alpha\beta}$ we construct the three-dimensional Laplacian operator. Let the eigenvalues of this operator be $[\nu_k^2]$ where k is a labeling index. We have thus associated with every three-geometry the following set of numbers: (R,ν_k). We now define a positive-semidefinite, symmetric "distance" between any two three-geometries by the expression

$$l^2=\frac{1}{2}\sum_k \ln\left[1+\frac{(R\sqrt{\nu_k}-R'\sqrt{\nu_k'})^2}{2RR'\sqrt{\nu_k}\sqrt{\nu_k'}}\right] .$$

Two three-geometries will be considered 'widely different" if "distance" l^2 between them calculated using the above expression is large compared to unity. Such three-geometries contribute decoherently to the density matrix and hence are macroscopically distinguishable. In contrast, the three-geometries between which the distance l^2 is small compared to unity retain their phase relation and interfere quantum mechanically. We have thus a precise characterization of three-geometries which are classically distinguishable.

Note that the distance l^2 defined above depends both on the "size" (R) and on the "shape" (ν_k's). Previous work by Zeh, Kiefer, etc., was based on specific minisuperspace models in which the shape is fixed. For example, suppose we consider only the three-geometries which are three-spheres. Two such three-geometries will differ in R but will have the same intrinsic eigenvalue spectrum $[\nu_k]$. The distance between two such three-geometries will depend only on the sizes. In this particular case we will recover the results[8] of Kiefer.

The most commonly used metric in the space of three-geometries is the DeWitt metric. It is not clear whether l^2 is related in any simple manner to the DeWitt metric. This issue is still under investigation.

The plan of the paper is as follows. The key idea behind the principle of dynamical decoherence is reviewed in the next section. It is applied to the wave function of the Universe in Sec. III and the results are discussed in Sec. IV.

II. DYNAMICAL ORIGIN OF CLASSICAL BEHAVIOR IN MACROSYSTEMS

To understand the basic idea behind this approach, let us first consider a simple example from quantum mechanics. Consider a system S described by a Hamiltonian $H(Q,P)$. Let us assume that this system is interacting with a bunch of N other degrees of freedom each described by a Hamiltonian $h_i(q_i,p_i,Q)$. (The presence of Q in h_i signals the coupling between the systems.) We can, for example, think of S interacting with a measuring apparatus which is classical; most classical systems will have large number of degrees of freedom. The total Hamiltonian for our discussion will be, therefore,

$$H_{\text{total}}=H(Q,P)+\sum_{i=1}^{N} h_i(p_i,q_i,Q) . \tag{2}$$

We can describe this total system consisting of S and the measuring apparatus by a single Schrödinger equation corresponding to H_{total}. The state of the system will be described by a wave function $\psi(Q,q_i)$ at any given time, which can be expressed as a superposition

$$\psi(Q,q_i)=\sum_n c_n f_n(Q,q_i) \tag{3}$$

in terms of some suitably chosen basis. We can associate with this *pure state* the density matrix

$$p(Q,Q';q_i,q_i')=\sum_n\sum_m c_n c_n^* f_n(Q,q_i) f_m^*(Q',q_i') . \tag{4}$$

Let us suppose that we are primarily interested in the observables of the system S. Then, we can construct from (4) a "reduced" density matrix in the Q-space by "tracing out" the q_i's. We will then obtain

$$\begin{aligned}\rho_{\text{red}}(Q,Q')&=\sum_{n,m} c_n c_m^* \int \prod_{i=1}^{N} dq_i f_n(Q,q_i) f_m^*(Q',q_i)\\ &=\sum_{n,m} c_n c_m^* F_{nm}(Q,Q') .\end{aligned} \tag{5}$$

This reduced density matrix preserves all the phase correlations which were originally present in the quantum state $\psi(Q,q_i)$. Expectation values of observables belonging to the system S, computed using ρ_{red}, will be identical to those obtained using ψ.

Consider now an entirely different density matrix, ρ_{class} which is *defined* as

$$\rho_{\text{class}}(Q,Q')=\delta(Q-Q')\sum_n |c_n|^2 F_{nn}(Q,Q) . \tag{6}$$

This density matrix, clearly, represents a *mixed* state. The quantity $|c_n|^2$ gives *probability* that our system can be found in the state represented by f_n; the quantity F_{nn} gives the probability that our variable has the same Q, *when* the system is in the nth state. Thus ρ_{class} is a decoherent sum of *probabilities* and represents a classical situation.

The central issue in the quantum theory of measurement is, of course, the following. Since the Schrödinger equation evolves pure states into pure states, we can never obtain ρ_{class} as a result of the Schrödinger evolution from a pure state. However, macroscopic observations invariably suggest a description in terms of ρ_{class}. Faced with this dilemma, Von Neumann and Heisenberg suggested that the density matrix changes to ρ_{class} due to "uncontrollable *external* influences" which accompany a measurement.[4] (This is equivalent to assuming the "collapse" of the wave function.) Since we plan to dispense with this idea, we have to provide an alternative resolution of this dilemma.

This can be done as follows. Let us suppose that the true density matrix of the system is indeed $\rho_{\rm red}$ of Eq. (5). But suppose our quantum state is such that

$$\int \prod_i dq_i f_n(Q,q_i) f_m^*(Q',q_i) \approx \delta_{nm} F_{nn}(Q,Q') \ , \tag{7}$$

where $F_{nn}(Q,Q')$ is a *sharply peaked function* of $(Q-Q')^2$. Then it will be impossible to distinguish the true density matrix $\rho_{\rm red}$ from the $\rho_{\rm class}$. Equation (7) directly suppresses all the off-diagonal correlations in $\rho_{\rm red}$.

In the case of simple quantum-mechanical models, it has been shown that such a suppression *does* take place.[5] In these models, the q_i's are essentially the degrees of freedom of the measuring apparatus. There are several people who believe that this is the proper approach to classical limit to quantum systems. Systems which we consider to be classical are merely those which are in constant interaction with other systems containing large number of degrees of freedom. Thus, the environment constantly "measures" a given system with which it is coupled and thus forces it to be classical. Rigorously speaking, the density matrix does not become exactly diagonal; but it becomes a very sharply peaked function of $(Q-Q')^2$.

The problem is thus reduced to understanding the conditions under which we will be led to the behavior suggested in (7). We have to first understand which quantum states lead to (7). After identifying them we have to ask ourselves whether these states are sufficiently generic to be of some value.

Let us illustrate, in the present, familiar setting the kind of states we will be working with in quantum gravity. To do this we go back to the Schrödinger equation describing our full system:

$$\left[H(P,Q) + \sum_{i=1}^{N} h_i(p_i,q_i,Q) \right] \psi(Q,q_i) = E\psi(Q,q_i) \ . \tag{8}$$

We look for *approximate* solutions to this equation which can be written in the form

$$\psi(Q,q_i) \approx \psi_0(Q) \prod_{i=1}^{N} \eta_i(q_i Q) \ , \tag{9}$$

where ψ_0 and each of the η_s's satisfy the equations

$$h_i(p_i,q_i)\eta_i(q_i,Q) \approx \epsilon_i(Q)\eta_i(q_i,Q) \ , \tag{10}$$

$$H(P,Q)\psi_0(Q) \approx \left[E - \sum_{i=1}^{N} \epsilon_i \right] \psi_0(Q) \approx E\psi_0(Q) \ . \tag{11}$$

The nature of the approximation is clear from (10) and (11). Equation (10) treats $h_i(p_i,q_i,Q)$ is a time-dependent Hamiltonian in which Q enters merely as a parameter. This allows us to define approximate energy eigenvalues $\eta_i(q_i,Q)$ which also depend on Q only as a parameter. This approximation is valid as long as the variable Q evolves adiabatically. To arrive at the second equality in (11) we have ignored the $\sum \epsilon_i$ term in comparison with E. This is equivalent to ignoring the "back reaction" of q_i's on Q and can be done if the two parts of the Hamiltonian differ widely in scale. We may assume that this is the case.

Let us now work out the reduced density matrix $\rho(Q,Q')$ corresponding to the state ψ in Eq. (9). We have

$$\rho(Q,Q') = \psi_0(Q)\psi_0^*(Q') \prod_{i=1}^{N} \int dq_i \eta_i(q_i,Q)\eta_i^*(q_i,Q') \tag{12}$$

$$= \psi_0(Q)\psi_0^*(Q') \prod_{i=1}^{N} A_i(Q,Q') \ . \tag{13}$$

Everything depends on the behavior of the product in (13). Classical behavior will arise if this term is sharply peaked around $Q=Q'$. It is clear that

$$A_i(Q,Q) = \int dq_i \eta_i(q_i,Q)\eta_i^*(q_i,Q) = 1 \tag{14}$$

since η_i's are normalized wave functions depending on Q merely as a parameter. Thus the product in (13) will be unity at $Q=Q'$ for all N. If $|A_i|^2$ decreases sufficiently rapidly as Q moves away from Q', then the classical behavior will be assured for *large N*. In this case, the product in (13) will consist of N terms each less than unity. Thus the product will decrease rapidly, as we move away from $Q=Q'$, if N is large. This is precisely what happens in the situations which we shall consider.

We shall now apply the above considerations to the quantized gravitational field.

III. CLASSICAL SPACETIME FROM QUANTUM GRAVITY

In the quantum-mechanical example considered above we used two separate systems, a "quantum" system S and a "classical" system with N degrees of freedom, where N is a large number. In this particular context one could have thought of these two as separate systems external to each other, but coupled. Such a qualification, however, is completely unnecessary. *We can very well think of the Hamiltonian in (2) as describing a single system with (N+1) degrees of freedom.* The crucial properties of the system which went into our results of the last section were the following. (1) There were large numbers of degrees of freedom in the system which were unobserved, and hence, had to be traced out in computing the expectation values and (2) there was an intrinsic disparity between the Q and the q_i's in the following sense: Q influenced the q_i's adiabatically while the back reaction of q_i's on Q was an ignorable, higher-order, effect.

These two features exist in the case quantum gravity as well. The wave function of the Universe, in principle, depends on *all* the degrees of the freedom of the matter in the universe. It is clearly impossible to observe all of them. Therefore our system does contain several degrees of freedom which are unobserved, satisfying the first criterion.

The second feature depends on the nature of the Hamiltonian describing the system. The Hamiltonian in the Wheeler-DeWitt equation (1) separates nicely into two sets of terms: $H_{\rm WD} = H_{\rm gravity}(p_A,g_A) + H_m(p_n,q_n,g_A)$ where

$$H_{\rm gravity} = [-\tfrac{1}{2} l^2 \nabla^2 + l^{-2} V(g_A)] \tag{15}$$

depends only on the gravitational variables and H_m depends on both the matter variables and the gravitational variables. These two parts scale differently because of the appearance of an explicit length scale l in H_g. It has been shown by several people[10,11] that the existence of such a relative scale allows one to obtain approximate solutions to Wheeler-DeWitt equation of the following type:

$$\Psi(g_A,q_n)\approx\Psi_0(g_A)\psi(g_A,q_n)\ , \tag{16}$$

where $\Psi_0(g_A)$ and $\psi(g_A,q_n)$ satisfy the equations

$$H_g\Psi_0(g_A)\approx-\epsilon\Psi_0(g_A)\approx 0\ , \tag{17}$$

$$H_m\psi(g_A,q_n)\approx\epsilon\psi(g_A,q_n)\ . \tag{18}$$

Equation (18) deserves comment. Using an expansion in powers of l, we will be able to obtain the Schrödinger equation for the matter fields in a given background geometry ("quantum field theory in curved spacetime"). Such an equation will be the same as (18) with the right-hand side replaced by the "time derivative":

$$i\frac{\partial\psi}{\partial\tau}=G_{AB}\frac{\partial\ln\Psi_0}{\partial g_A}\frac{\partial\psi}{\partial g_B}\ . \tag{19}$$

If the background geometry is not violently varying in τ, then it is admissible to introduce approximate stationary states as in (18). This is the assumption of adiabaticity which we invoke. [In addition, we have ignored the "back reaction" of the matter fields on gravity by setting the right-hand side of (18) to zero. This approximation, however, is not very crucial to what follows.] The similarity between (18), (19), and (10), (11) is apparent.

The assumption of adiabaticity introduces the necessary disparity between the gravitational and matter degrees of freedom. To investigate the emergence of classical spacetime, we have to compute the reduced density matrix for the three-geometries. This, in turn, involves identifying a set of N degrees of freedom among the matter variables and evaluating the product

$$\prod_{i=1}^{N}\int dq_i\,\psi^*(g'_A,q_i)\psi(g_A,q_i)\ .$$

Classical behavior is assured if the off-diagonal elements in this product are strongly suppressed. We shall attempt this task next.

Among the various matter fields which populate the Universe, let there be a massless scalar field $\phi(t,\mathbf{x})$ described by the action

$$A=\tfrac{1}{2}\int dt\,d^3\mathbf{x}\sqrt{-g}\,g^{ik}\partial_i\phi\partial_k\phi\ . \tag{20}$$

We shall assume that the three-space is compact and has some metric $\gamma_{\alpha\beta}$. We shall work in a gauge in which lapse function is unity and the shift functions are zero. For the sake of convenience we shall treat the spatial coordinates x^α on the compact three-space to be some suitably chosen, dimensionless, angular coordinates. This will give $\gamma_{\alpha\beta}$ the dimensions of (length)^2, which we will scale out by writing $\gamma_{\alpha\beta}=R^2g_{\alpha\beta}$ where R^3 is the volume of the compact three-space.

Under these circumstances, the line element is

$$ds^2=dt^2-R^2g_{\alpha\beta}dx^\alpha dx^\beta \tag{21}$$

and the action in (20) becomes

$$A=\tfrac{1}{2}\int dt\,d^3x^\alpha\sqrt{g}\,R^3(\dot\phi^2-R^{-2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)\ . \tag{22}$$

We shall expand the scalar field using the normal modes of the three-dimensional Laplacian. We write

$$\phi(t,\mathbf{x})=\sum_n f_n(t)Q_n(\mathbf{x})\ , \tag{23}$$

where Q_n's are the normalized eigenfunctions of the Laplacian operator corresponding to the three-metric $g_{\alpha\beta}$ with some eigenvalue ν_n^2:

$$\nabla^2Q_n+\nu_n^2Q_n=0\ . \tag{24}$$

The normalization condition implies that

$$\int d^3x^\alpha\sqrt{g}\,Q_nQ_m^*=\delta_{nm}\ . \tag{25}$$

Substituting (23) into (22) and using (24) and (25) we can easily express the scalar field action as that of a bunch of harmonic oscillators:

$$A=\sum_n\tfrac{1}{2}R^3\int dt(\dot f_n^2-R^{-2}\nu_n^2f_n^2)\ . \tag{26}$$

Note that the geometry of the spacetime enters into this matter Hamiltonian essentially through the eigenvalues ν_n. *Because of our assumption of adiabaticity we can treat these eigenvalues as constant.* In such a situation the quantum state of the matter field can be written as a direct product of the harmonic-oscillator eigenstates for each of the modes. Let the kth harmonic oscillator be in a state labeled by the integer n_k. Then the full wave function of the system will be given by

$$\Psi(g_{\alpha\beta},[f_k])\approx\Psi_0(g_{\alpha\beta})\prod_k\psi_{n_k}(f_k,w_k)\ , \tag{27}$$

where w_k stands for the combination $R^2\nu_k$ and ψ_{n_k} are the harmonic oscillator wave functions:

$$\psi_{n_k}(f_k,w_k)=(2^{n_k}n_k!)^{1/2}(w/\pi)^{1/4}H_{n_k}(\sqrt{w_k}\,f_k)\exp(-\tfrac{1}{2}wf_k^2)\ . \tag{28}$$

The spacetime geometry enters these wave functions only through the combination $w_k=R^2\nu_k$. Let us now divide the matter modes $[f_k]$ labeled by k into two sets: those which are observed, $[a_k]$ and those which are not observed and hence traced out $[b_k]$. Suppose there are N unobserved degrees of freedom. The reduced density matrix for the observed degrees of freedom and the spacetime geometry can be easily found from (27) by integrating over the unobserved degrees of freedom:

$$\rho(g_{\alpha\beta},g'_{\alpha\beta};a_k,a'_k)=\Psi_0(g_{\alpha\beta})\Psi_0^*(g'_{\alpha\beta})\prod_k\psi_{n_k}(a_k,w_k)\psi_{n_k}^*(a'_k,w'_k)\prod_{k=1}^{N}\int db_k\,\psi_{n_k}(b_k,w_k)\psi_{n_k}^*(b_k,w'_k)\ . \tag{29}$$

The Nfold product in the above expression determines the decoherence between two geometries $g_{\alpha\beta}$ and $g'_{\alpha\beta}$. Such a decoherence will be indicated if the Nfold product is a sharply peaked function around $(w_k = w'_k)$.

Let us first consider the case in which all the unobserved modes are in the ground state. (We will discuss the general case in the end; the same results are obtained in the general case as well.) Then the integral over each b_k contributes the quantity

$$A_k = \int db_k \psi_0(b_k, w_k)\psi_0^*(b_k, w'_k)$$
$$= \int db_k \left[\frac{w_k w'_k}{\pi^2}\right]^{1/4} \exp\left[-\frac{w_k + w'_k}{2} b_k^2\right]$$
$$= \left[1 + \frac{(\sqrt{w_k} - \sqrt{w'_k})^2}{2\sqrt{w_k}\sqrt{w'_k}}\right]^{-1/2} . \tag{30}$$

Therefore the product, containing N such terms, will be

$$P = \sum_{k=1}^{N} A_k$$
$$= \exp\left[-\tfrac{1}{2}\sum_{k=1}^{N} \ln\left[1 + \frac{(\sqrt{w_k} - \sqrt{w'_k})^2}{2\sqrt{w_k}\sqrt{w'_k}}\right]\right]$$
$$= \exp\left[-\tfrac{1}{2}\sum_{k=1}^{N} p_k\right] . \tag{31}$$

We are now in a position to decide which spacetime geometries appear decoherently in the reduced density matrix. To each compact three-geometry $\mathcal{G}$ we associate a scale R ("size") and a set of eigenvalues of the Laplacian $[\nu_k]$ (which depends on the "shape"). Two macroscopically different geometries $\mathcal{G}$ and $\mathcal{G}'$ will have very different values for these parameters, and hence, very different set of w_k's. The sum $\sum p_k$ will be a large positive quantity for such geometries, making the exponential very small. Such off-diagonal elements in the space of three-geometries are strongly suppressed in the density matrix.

Note that the above analysis can be turned around to provide a rigorous definition of macroscopically distinguishable spacetimes: *Two spacetimes are macroscopically distinguishable if the sum* $\sum p_k$ *is large compared to unity for these two geometries.* Such a pair of spacetimes will exhibit negligible quantum interference. In fact, this sum provides an interesting "distance measure" in the space of three-geometries. The distance is clearly positive semidefinite and symmetric. [However it is not necessary that the set (w_k) uniquely specifies a three-geometry. This issue and further properties of $\sum p_k$ are under investigation.]

There is one particular case about which definite statements can be made. Consider two geometries which have the same intrinsic metric but differ in the overall scale. (For example, there could be two three-spheres of radii R_1 and R_2.) In this case the ν_k's are the same and the w_k's differ only because of R. Each term p_k in the sum $\sum p_k$ is a constant independent of k and is equal to

$$\frac{1}{2}\ln\left[1 + \frac{(R_1 - R_2)^2}{2R_1 R_2}\right] .$$

Clearly the sum increases monotonically with N. Thus the exponential factor suppresses off-diagonal elements differing in scale strongly. The width of the exponential decreases as $\sqrt{N}$. *We can therefore conclude that compact three-geometries differing in volume but having the same intrinsic geometry contribute decoherently to the density matrix.* This is an interesting generalization of previous results known in the case of Friedmann Universes.[8]

To obtain the above results it is not really necessary to assume that the modes b_k's are in the vacuum state. If the kth mode is in the state labeled by the integer n_k, then we will need to calculate, instead of (30), the integral

$$I_{nm} = \int_{-\infty}^{+\infty} dx\, \psi_n(w, x)\psi_n^*(w', x) , \tag{32}$$

where we have suppressed the subscript k for simplifying the notation. The form of such integrals can be determined using (28) and the generating function for Hermite polynomials. A tedious but straightforward calculation shows that A_k can be written in the form

$$A_k = \left[1 + \frac{(\sqrt{w_k} - \sqrt{w'_k})^2}{2\sqrt{w_k}\sqrt{w'_k}}\right]^{-1/2} F_k(\epsilon_k) , \tag{33}$$

where

$$\epsilon_k = \left[\frac{w_k - w'_k}{w_k + w'_k}\right] \tag{34}$$

and F_k's are a set of functions of the indicated argument ϵ_k. The detailed form of F_k is irrelevant except for the fact that $F_k(0)$ is unity. When we take the product of A_k's *we will recover a term identical to that in (31),* multiplied by the product of F_k's. Whenever the vacuum term suppresses the off-diagonal elements (with $w \neq w'$) the F_k's will be effectively reduced to unity and will not contribute anything significant to the product. Thus we see that the decoherence is essentially due to the factor in (31).

Since we assumed the quantum state of matter fields to be a state of fixed energy, we have used energy eigenfunctions of the harmonic oscillator in characterizing the wave functional. One can relax this assumption and redo the computation with an arbitrary superposition of energy eigenstates for each oscillator. The basic character of the results remain unaltered.

We shall now study the implications of the above result.

IV. INTERPRETATION, DISCUSSION, AND OUTLOOK

Let us examine the various assumptions and ingredients which have gone into the above result. This will allow us to determine the exact domain of validity and identify issues which are still open.

Our central philosophy regarding the emergence of classical spacetime from the wave function of the

Universe was the following. The wave function of the Universe, by its very definition, depends on all the degrees of freedom in the Universe; to determine this quantum state uniquely it is necessary to measure a complete set of commuting observables. In particular every free quantum field in the Universe will provide a large number of such modes. From a practical point of view it is impossible to observe all the operators which characterize this quantum state. It is necessary, therefore, to integrate over unobserved degrees of freedom and obtain a reduced density matrix. Classical behavior for the spacetime geometry will arise if the off-diagonal elements of this reduced density matrix are strongly suppressed.

The above analysis certainly seems to vindicate this basic philosophy. The crucial fact which goes into the analysis is that every degree of freedom in the Universe contributes to energy and hence couples to gravity. Thus integrating over any such modes leaves its mark on the gravitational sector of the density matrix.

The form of the density matrix which we obtain will, in general, depend on the form of the wave function we start with. In this paper we have considered only the wave functions which can be expressed approximately in the form

$$\Psi(\gamma_{\alpha\beta},f_k)\approx\Psi_0(\gamma_{\alpha\beta})\psi_{\rm adia}(\gamma_{\alpha\beta},f_k)\ , \tag{35}$$

where $\psi_{\rm adia}(\gamma_{\alpha\beta},f_k)$ is obtained in the adiabatic approximation. It is necessary to identify the role and relevance of this assumption.

The wave functions as the one in (35) have been studied by several people[10,11] and the prime candidates for describing the semiclassical limit. It is unlikely that any wave function that does not allow the separability assumed in (35) will be successful in describing the semiclassical limit. We may say that (35) is indeed a *necessary* condition for semiclassical behavior. This is the major motivation for starting our investigation with the wave functions of the form in (35).

There is, however, a lot more to the issue of semiclassical limit. The Ψ_0 in (35) will not usually be peaked around any single three-geometries. Hence it is difficult to interpret the wave function directly in the *configuration* space. Halliwell,[11] for example, has suggested that the semiclassical interpretation should be attempted in the phase space using the Wigner function. If Ψ_0 is a WKB solution, *then* the Wigner function will be peaked around a *set* of classical trajectories in the phase space.

Our analysis suggests another possible interpretation of the semiclassical limit for Ψ using the density matrix. The density matrix, unlike the Wigner function, can be defined in the configuration space itself. Further the decoherent contributions to ρ come from *specific* classical configuration rather than from *sets* of classical trajectories.

Another point worth noting about our analysis is the following. While (35) may be *necessary* for classical interpretation, it was not *a priori* clear whether it is also *sufficient*. Our analysis shows that it *is*, provided we follow the approach outlined in this paper to obtain the classical limit. Note that we made absolutely no assumption regarding the actual form of Ψ_0 in (35); it need not, for example, be a WKB solution.

A natural generalization of this work will be the computation of ρ for more general wave functions. It will probably be best if we write down the equation satisfied by ρ and look for its approximate solutions. This work—easier said that done—is in progress.

Let us next consider some details regarding the "unobserved modes" which were integrated out. To begin with, we have to settle whether the number of such modes is finite, countably infinite, or uncountably infinite. The total number of modes defined on a compact three-geometry will be, in general, countably infinite. (We have a natural upperbound on the wavelengths because of the finite size of the three-space, but we have no *a priori* cutoff at the lower end.) Among these modes we declare a set of N modes as unobserved. Whether this number N is finite or not is obviously linked to the reason why we do not observe these modes. One can take different stands in this matter.

The simplest assumption would be to say that these modes are not observed because of technological limitation. All the observations available today are confined to energies below, say, 100 GeV. We can therefore say that we have no information about the modes with frequencies higher than 100 GeV. These could, of course, be a countably infinite number of such modes. It is not very clear whether we should integrate over an infinite number of such modes. One may feel that the assumption of adiabaticity breaks down at Planck energies and that our analysis should not include modes above Planck frequencies. In such a case the quantity N which appears in Eq. (28) onward will be a finite number. (This point, however, is not quite settled. It is possible that the adiabatic approximation is actually valid for high frequencies but breaks down at lower end; see the work of Halliwell and Hawking in Ref. 10.) On the other hand we may not like to introduce such an artificial cutoff at Planck frequencies but instead consider all the infinite number of modes. In such a case our results depend crucially on the convergence properties of the sum $\sum p_k$ which occur in (31). Let us suppose the eigenfrequencies are arranged in ascending order. Depending on the nature of the three-geometries which are considered this sum may be finite or divergent. Any two three-geometries for which this sum diverges will be eliminated from the density matrix, i.e., off-diagonal elements which are "far away" are completely suppressed. It is possible that there exist some three-geometries which are sufficiently "close together" for this infinite sum to converge to a finite value. Such geometries will exist in the density matrix with coherent phase relations and hence will not be macroscopically distinguishable. In other words, they exhibit quantum interference.

There is another reason why we may have to integrate over certain modes. It is possible that there are regions of three-space which are unobservable because of the existence of horizons. Suppose a three-space $\mathcal{S}$ is the union of two regions $\mathcal{R}_1$ and $\mathcal{R}_2$. The mode functions defined globally in $\mathcal{S}$ may not be appropriate for observes confined to, say, $\mathcal{R}_1$. Such observers may use two separate sets of modes defined in the regions 1 and 2 and connected with the globally defined modes by some Bogoliubov coefficients. In such situations one will naturally integrate over one set of modes. The resulting density matrix, evaluated in a given, fixed, classical spacetime, is known to exhibit thermal character in some well-known situations. In the present context we are likely to obtain some decoherence of the wave function describing the spacetime as well. This idea needs to be worked out in detail and could provide an important link between the physics in the presence of horizons in a spacetime and quantum gravity.

Another interesting feature which arises from our analysis is the following. Consider a compact three-space with a positive definite metric $g_{\alpha\beta}$. We associate with this metric the set of eigenvalues $[\nu_k]$ and a length scale R which may be taken to be the cube root of the volume of the compact three-space. From R and ν_k we construct the set $[w_k]$ where each w_k is defined as $R^2\nu_k$. We can now define a metric in the space of three-geometries by the relation

$$l^2(g_{\alpha\beta},g'_{\alpha\beta}) = \frac{1}{2}\sum_{k=1}^{N} \ln\left[1+\frac{\left[\sqrt{w_k(g_{\alpha\beta})}-\sqrt{w'_k(g'_{\alpha\beta})}\right]^2}{2\sqrt{w_k}\sqrt{w'_k}}\right], \quad (36)$$

where N is some specified value which could even be infinite. (We are assuming that the eigenfrequencies are ordered in some meaningful way.) This metric is clearly symmetric, positive semidefinite, and vanishes only if all the eigenfrequencies match. This is clearly a metric in the space of eigenfrequencies for finite N and, if the sum converges for at least some sets of w's and w''s, possibly even for infinite N. But in order to qualify as a useful metric for three-geometries we would also like to establish some uniqueness relation between the three-geometries and their corresponding w's. It is not difficult to show that the two metrics which differ infinitesimally cannot have the same set of w's. To see this, consider the following functional integral over functions $f(\mathbf{x})$ defined in the three-space:

$$\mathcal{I}=\int \mathcal{D}f \exp\left[\frac{1}{2}\int d^3x\, f\nabla^2 f\right]=I([w_k]) .$$

In arriving at the last step we have used the well-known property that the functional integral can be expressed in terms of the determinant of the Laplacian operator and hence is only a function of the eigenfrequencies. Now consider the functional derivative of the above expression with respect to $g_{\alpha\beta}$. We get

$$\frac{\delta}{\delta g_{\alpha\beta}}\int \mathcal{D}f \exp\left[\frac{1}{2}\int d^3x\, f\nabla^2 f\right] = \int \mathcal{D}f\, T_{\alpha\beta}\exp\left[\frac{1}{2}\int d^3x\, f\nabla^2 f\right], \quad (37)$$

where $T_{\alpha\beta}$ is the "energy-momentum tensor" in three-space:

$$T_{\alpha\beta}=f_\alpha f_\beta-\tfrac{1}{2}g_{\alpha\beta}f^\mu f_\mu , \quad (38)$$

where f_α stands for $(\partial f/\partial x^\alpha)$, etc. Let us now suppose there exist two metrics which differ infinitesimally but have the same set of eigenfrequencies. If this is possible then the value of the functional integral $I([w_k])$ cannot change under the infinitesimal variation in (38). This, in turn, means that the functional average of $T_{\alpha\beta}$ in (38) should vanish. In particular the functional average of the trace of $T_{\alpha\beta}$ should vanish. But this is impossible because the trace is $(-\frac{1}{2}f_\alpha f^\alpha)$ which is negative definite. Therefore infinitesimal changes in the metric invariably leads to changes in the eigenfrequencies.

This shows that eigenfrequencies provide a good characterization of the metric "locally." The above proof, of course, does not rule out the possibility of two widely different geometries having the same set of eigenfrequencies. In fact, it is quite likely that the metric is not uniquely determined by the eigenfrequencies. This opens up the possibility that two widely different metrics could still contribute coherently to the density matrix if they have the same scale and eigenfrequencies. In the present context, however, this is to be expected since the scalar field couples only to these parameters of the geometry.

Another peculiar feature about the metric in (36) is that it may associate infinite "distance" between several geometries if we decide to sum over all eigenfrequencies. These are the geometries for which the sum in (36) diverges. Such divergence, of course, enhances the decoherence and reinforces classical behavior. Three geometries with the same intrinsic structure but differing in volume belong to this class; for such geometries, as we have seen before, the "distance" given by (36) increases monotonically with N and diverges for infinite N.

The metric in (36) also gives an operational criterion for deciding which three-geometries interfere in quantum gravity. These are precisely the ones for which the "distance" as defined in (36) is small compared to unity. As far as the author is aware of, this is the first time such a clear operational criterion has been established in quantum gravity.

Lastly, it is interesting to conjecture about the effect of our increasing knowledge on the wave function of the Universe. As more and more modes are observed we will be left with lesser and lesser modes to "trace out." This would reduce the decoherence and the quantum interference between geometries will become more and more apparent. In other words, the classical nature of the spacetime will tend to disappear as we observe more and more matter modes. Probably, ignorance *is* bliss.

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